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Neumann trace tracking of a constant reference input for 1-D boundary controlled heat-like equations with delay ⋆

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Abstract: This paper discusses the Proportional Integral (PI) regulation control of the left Neumann trace of a one-dimensional reaction-diffusion equation with a delayed right Dirichlet boundary control. Specifically, a PI controller is designed based on a finite-dimensional truncated model that captures the unstable dynamics of the original infinite-dimensional system. In this setting, the control input delay is handled by resorting to the Artstein transformation. The stability of the resulting infinite-dimensional system, as well as the tracking of a constant reference signal in the presence of a constant distributed perturbation, is assessed based on the introduction of an adequate Lyapunov function. The theoretical results are illustrated with numerical simulations.

Keywords: 1-D reaction-diffusion equation, PI regulation control, Neumann trace, Delay boundary control, Partial Differential Equations (PDEs).

1. INTRODUCTION

1.1 State of the art

Proportional-Integral (PI) regulation control of infinite-dimensional systems is an active topic of research. While early works in this area were reported in the 80's for bounded control operators (Pohjolainen, 1982, 1985; Xu and Jerbi, 1995), the PI boundary regulation of infinite-dimensional systems is more recent. This includes linear hyperbolic systems (Dos Santos et al., 2008; Xu and Sallet, 2014; Bastin et al., 2015; Lamare and Bekiaris-Liberis, 2015), 1-D nonlinear transport equation (Coron and Hayat, 2019; Trinh et al., 2017), regulation of the downside angular velocity of a drilling string (Terrand-Jeanne et al., 2018b), and the regulation of a drilling pipe under friction (Barreau et al., 2019). The PI regulation of open-loop exponentially stable semigroups with unbounded control operators were reported in (Terrand-Jeanne et al., 2018a, 2019) via a Lyapunov functional-based design procedure.

This paper is focused on the PI regulation control of the left Neumann trace of a one-dimensional reaction-diffusion equation, which might be either open-loop stable or unstable, with a delayed right Dirichlet boundary control. Specifically, we aim at achieving the Neumann trace tracking of a constant reference input in spite of the presence of an arbitrarily large constant input delay and a stationary distributed disturbance. It was shown in (Krstic, 2009) that backstepping-based control design can be used to achieve the feedback stabilization of a reaction-diffusion equation in the presence of an arbitrarily large input delay. In this paper, we adopt the approach reported in (Prieur and Trélat, 2019) which takes advantage of the following design procedure initially reported in (Russell, 1978): 1) design of the controller on a finite-dimensional model capturing the unstable modes of the original infinite-dimensional system; 2) use of a suitable Lyapunov functional to guarantee the stability of the resulting closed-loop infinite-dimensional system. This control design procedure, which was used in (Coron and Trélat, 2004, 2006; Schmidt and Trélat, 2006) to stabilize semilinear heat, wave or fluid equations via (undelayed) boundary feedback control, was extended in (Prieur and Trélat, 2019) to the case of delay boundary control of a one-dimensional reaction-diffusion equation in which the contribution of the input-delay was managed by the synthesis of a predictor feedback via the classical Artstein transformation (Artstein, 1982; Richard, 2003; Bresch-Pietri et al., 2018). This control strategy was first reused in (Guzmán et al., 2019) for the delay boundary feedback stabilization of a linear Kuramoto-Sivashinsky equation and then generalized to the delay boundary feedback stabilization of a class of diagonal infinite-dimensional systems for either a constant (Lhachemi and Prieur, 2020; Lhachemi et al., 2019c) or a time-varying (Lhachemi et al., 2019a, 2020) input delay.

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1.2 Investigated control problem

Let $L > 0$, let $c \in L^\infty(0, L)$, and let $D > 0$ be arbitrary. We consider the one-dimensional reaction-diffusion equation over $(0, L)$ with delayed Dirichlet boundary control:

$$y_t = y_{xx} + c(x)y + d(x), \qquad (t,x) \in \mathbb{R}_+^* \times (0,L) \quad \text{(1a)}$$
$$y(t,0) = 0, \qquad t \geq 0 \quad \text{(1b)}$$
$$y(t,L) = u_D(t) \triangleq u(t-D), \qquad t \geq 0 \quad \text{(1c)}$$
$$y(0,x) = y_0(x), \qquad x \in (0,L) \quad \text{(1d)}$$

where $y(t,\cdot) \in L^2(0,L)$ is the state at time t, $u(t) \in \mathbb{R}$ is the control input, $D > 0$ is the (constant) control input delay, $d \in L^2(0,L)$ is a stationary distributed disturbance, and $y_0 \in H^2(0,L)$ with $y_0(0) = 0$ and $y_0(L) = u(-D)$ is the initial condition.

Our objective is to achieve the PI regulation control of the left Neumann trace $y_x(t,0)$ to some prescribed constant reference input $r \in \mathbb{R}$ in spite of the stationary distributed disturbance d, i.e., $y_x(t,0) \to r$ as $t \to +\infty$. Note that an exponentially stabilizing controller for (1a-1d) was reported in (Prieur and Trélat, 2019) in the disturbance-free case ($d = 0$) for a system trajectory evaluated in H_0^1-norm. The control strategy that we develop in the present paper elaborates on the one of (Prieur and Trélat, 2019), adequately combined with a PI procedure.

2. CONTROL DESIGN STRATEGY

The sets of nonnegative integers, positive integers, real, nonnegative real, and positive real are denoted by $\mathbb{N}$, $\mathbb{N}^*$, $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_+^*$, respectively. All the finite-dimensional spaces $\mathbb{R}^p$ are endowed with the usual Euclidean inner product $\langle x, y\rangle = x^\top y$ and the associated 2-norm $\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{x^\top x}$. For any matrix $M \in \mathbb{R}^{p\times q}$, $\|M\|$ stands for the induced norm of M associated with the above 2-norms. For a given symmetric matrix $P \in \mathbb{R}^{p\times p}$, $\lambda_m(P)$ and $\lambda_M(P)$ denote its smallest and largest eigenvalues, respectively. In the sequel, the time derivative $\partial f/\partial t$ is either denoted by f_t or $\dot{f}$ while the spatial derivative $\partial f/\partial x$ is either denoted by f_x or f'.

2.1 Augmented system for PI feedback control

The control design objective is: 1) to stabilize the reaction-diffusion system (1a-1d); 2) to ensure the tracking of the constant reference input $r \in \mathbb{R}$ by the left Neumann trace $y_x(t,0)$. We address this problem by designing a PI controller. Following the general PI scheme, we introduce a new state $z(t) \in \mathbb{R}$ taking the form of the integral of the tracking error $y_x(t,0) - r$:

$$y_t = y_{xx} + c(x)y + d(x), \qquad (t,x) \in \mathbb{R}_+^* \times (0,L) \quad \text{(2a)}$$
$$\dot{z}(t) = y_x(t,0) - r, \qquad t \geq 0 \quad \text{(2b)}$$
$$y(t,0) = 0, \qquad t \geq 0 \quad \text{(2c)}$$
$$y(t,L) = u_D(t) \triangleq u(t-D), \qquad t \geq 0 \quad \text{(2d)}$$
$$y(0,x) = y_0(x), \qquad x \in (0,L) \quad \text{(2e)}$$
$$z(0) = z_0 \quad \text{(2f)}$$

with $z_0 \in \mathbb{R}$ the initial condition of the integral component. As we are only concerned in prescribing the future of the system, we assume that the system is uncontrolled for $t < 0$, i.e. $u(t) = 0$ for $t < 0$. Thus, we assume in the rest of the paper that $y_0 \in H^2(0,L) \cap H_0^1(0,L)$.

2.2 Spectral reduction

We rewrite (2) as an equivalent homogeneous Dirichlet problem. Assuming [1] that u is continuously differentiable and setting $w(t,x) = y(t,x) - \frac{x}{L}u_D(t)$, we have

$$w_t = w_{xx} + c(x)w + \frac{x}{L}c(x)u_D - \frac{x}{L}\dot{u}_D(t) + d(x) \quad \text{(3a)}$$
$$\dot{z}(t) = w_x(t,0) + \frac{1}{L}u_D(t) - r \quad \text{(3b)}$$
$$w(t,0) = w(t,L) = 0 \quad \text{(3c)}$$
$$w(0,x) = y_0(x) - \frac{x}{L}u_D(0) = y_0(x) \quad \text{(3d)}$$
$$z(0) = z_0 \quad \text{(3e)}$$

for $t > 0$ and $x \in (0,1)$. Introducing the real state-space $L^2(0,1)$ endowed with its usual inner product $\langle f, g\rangle = \int_0^L f(x)g(x)\,\mathrm{d}x$ and the operator $\mathcal{A} = \partial_{xx} + c\,\mathrm{id} : D(\mathcal{A}) \subset L^2(0,L) \to L^2(0,L)$ defined on the domain $D(\mathcal{A}) = H^2(0,L) \cap H_0^1(0,L)$, (3a-3c) can be rewritten as

$$w_t(t,\cdot) = \mathcal{A}w(t,\cdot) + a(\cdot)u_D(t) + b(\cdot)\dot{u}_D(t) + d(\cdot) \quad \text{(4a)}$$
$$\dot{z}(t) = w_x(t,0) + \frac{1}{L}u_D(t) - r \quad \text{(4b)}$$

with $a(x) = \frac{x}{L}c(x)$ and $b(x) = -\frac{x}{L}$ for every $x \in (0,L)$, with initial conditions (3d-3e). Since $\mathcal{A}$ is self-adjoint and has compact resolvent, we consider a Hilbert basis $(e_j)_{j\geq 1}$ of $L^2(0,L)$ consisting of eigenfunctions of $\mathcal{A}$ associated with the sequence of simple real eigenvalues

$$-\infty < \cdots < \lambda_j < \cdots < \lambda_1 \quad \text{with} \quad \lambda_j \underset{j\to+\infty}{\longrightarrow} -\infty.$$

Note that $e_j(\cdot) \in H_0^1(0,L) \cap C^1([0,L])$ for every $j \geq 1$ and

$$e_j'(0) \sim \sqrt{\frac{2}{L}}\sqrt{|\lambda_j|}, \quad \lambda_j \sim -\frac{\pi^2 j^2}{L^2}, \quad (5)$$

when $j \to +\infty$. Since $w(0,\cdot) = y_0 \in H^2(0,L) \cap H_0^1(0,L)$, the classical solution $w(t,\cdot) \in H^2(0,L) \cap H_0^1(0,L)$ of (4a) can be expanded as a series in the eigenfunctions $e_j(\cdot)$, convergent in $H_0^1(0,L)$,

$$w(t,\cdot) = \sum_{j=1}^{+\infty} w_j(t)e_j(\cdot). \quad (6)$$

Thus (4) is equivalent to the infinite-dimensional control system:

$$\dot{w}_j(t) = \lambda_j w_j(t) + a_j u_D(t) + b_j \dot{u}_D(t) + d_j \quad \text{(7a)}$$
$$\dot{z}(t) = \sum_{j\geq 1} w_j(t)e_j'(0) + \frac{1}{L}u_D(t) - r \quad \text{(7b)}$$

for $j \in \mathbb{N}^*$, with $w_j(t) = \langle w(t,\cdot), e_j\rangle$, $a_j = \langle a, e_j\rangle$, $b_j = \langle b, e_j\rangle$, and $d_j = \langle d, e_j\rangle$. Introducing the auxiliary control input $v \triangleq \dot{u}$, and denoting $v_D(t) \triangleq v(t-D)$, (7) can be rewritten as

$$\dot{u}_D(t) = v_D(t) \quad \text{(8a)}$$
$$\dot{w}_j(t) = \lambda_j w_j(t) + a_j u_D(t) + b_j v_D(t) + d_j \quad \text{(8b)}$$
$$\dot{z}(t) = \sum_{j\geq 1} w_j(t)e_j'(0) + \frac{1}{L}u_D(t) - r \quad \text{(8c)}$$

for $j \in \mathbb{N}^*$. As $u(t) = 0$ for $t < 0$, (8a) yields $v(t) = 0$ for $t < 0$ and the initial condition $u_D(0) = 0$.

[1] This property will be ensured by the construction carried out in the sequel.

2.3 Finite-dimensional truncated model

In what follows, we fix the integer $n \in \mathbb{N}^*$ such that $\lambda_{n+1} < 0 \leq \lambda_n$. In particular, we have $\lambda_j \geq 0$ when $1 \leq j \leq n$ and $\lambda_j \leq \lambda_{n+1} < 0$ when $j \geq n+1$. Then, introducing as in (Prieur and Trélat, 2019):

$$X_1(t) = \begin{pmatrix} u_D(t) \\ w_1(t) \\ \vdots \\ w_n(t) \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_1 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & \lambda_n \end{pmatrix},$$

$$B_1 = (1 \ b_1 \ \ldots \ b_n)^\top,$$
$$D_1 = (0 \ d_1 \ \ldots \ d_n)^\top,$$

with $X_1(t) \in \mathbb{R}^{n+1}$, $A_1 \in \mathbb{R}^{(n+1)\times(n+1)}$, $B_1 \in \mathbb{R}^{n+1}$, $D_1 \in \mathbb{R}^{n+1}$, (8a) and the n first equations of (8b) yield the truncated model:

$$\dot{X}_1(t) = A_1 X_1(t) + B_1 v_D(t) + D_1. \tag{9}$$

Now, as $\left|\frac{e_j'(0)}{\lambda_j}\right|^2 \sim \frac{2L}{\pi^2 j^2}$ when $j \to +\infty$, we can introduce the change of variable

$$\zeta(t) \triangleq z(t) - \sum_{j \geq n+1} \frac{e_j'(0)}{\lambda_j} w_j(t), \tag{10}$$

whose time derivative is given by

$$\begin{aligned} \dot{\zeta}(t) &= \dot{z}(t) - \sum_{j \geq n+1} \frac{e_j'(0)}{\lambda_j} \dot{w}_j(t) \\ &= \alpha u_D(t) + \beta v_D(t) - \gamma + \sum_{j=1}^{n} w_j(t) e_j'(0), \end{aligned}$$

where we have used (8b-8c), with

$$\alpha = \frac{1}{L} - \sum_{j \geq n+1} \frac{e_j'(0)}{\lambda_j} a_j, \quad \beta = -\sum_{j \geq n+1} \frac{e_j'(0)}{\lambda_j} b_j, \tag{11a}$$

$$\gamma = r + \sum_{j \geq n+1} \frac{e_j'(0)}{\lambda_j} d_j. \tag{11b}$$

Then we have

$$\dot{\zeta}(t) = L_1 X_1(t) + \beta v_D(t) - \gamma \tag{12}$$

with $L_1 = \left(\alpha \ e_1'(0) \ \ldots \ e_n'(0)\right) \in \mathbb{R}^{1\times(n+1)}$. Now, defining the augmented state-vector $X(t) = \left[X_1(t)^\top \ \zeta(t)\right]^\top \in \mathbb{R}^{n+2}$, the exogenous input $\Gamma = \left[D_1^\top \ -\gamma\right]^\top \in \mathbb{R}^{n+2}$ and the matrices

$$A = \begin{pmatrix} A_1 & 0 \\ L_1 & 0 \end{pmatrix} \in \mathbb{R}^{(n+2)\times(n+2)}, \quad B = \begin{pmatrix} B_1 \\ \beta \end{pmatrix} \in \mathbb{R}^{n+2}, \tag{13}$$

we obtain from (9) and (12) the control system

$$\dot{X}(t) = AX(t) + Bv_D(t) + \Gamma \tag{14}$$

which is the finite-dimensional truncated model capturing the unstable part of the infinite-dimensional system along with the introduced integral component for PI regulation. Putting together the truncated model (14) along with (8b) for $j \geq n+1$, we get the final representation used for both control design and stability analyses:

$$\dot{X}(t) = AX(t) + Bv_D(t) + \Gamma \tag{15a}$$
$$\dot{w}_j(t) = \lambda_j w_j(t) + a_j u_D(t) + b_j v_D(t) + d_j \tag{15b}$$

with $j \geq n+1$.

2.4 Control design strategy

The adopted control design strategy relies on the use of the classical predictor feedback to stabilize the finite-dimensional truncated model (15a). Specifically, introducing the Artstein transformation (Artstein, 1982)

$$Z(t) = X(t) + \int_{t-D}^{t} e^{A(t-D-\tau)} Bv(\tau)\,\mathrm{d}\tau, \tag{16}$$

we have

$$\dot{Z}(t) = AZ(t) + e^{-DA}Bv(t) + \Gamma.$$

Since (A, B) satisfies the Kalman condition (the proof of this claim is omitted due to space limitation; details can be found in (Lhachemi et al., 2019b)), the pair $(A, e^{-DA}B)$ also satisfies the Kalman condition and we infer the existence of a feedback gain $K \in \mathbb{R}^{1\times(n+2)}$ such that $A_K \triangleq A + e^{-DA}BK$ is Hurwitz. Setting $v(t) = \chi_{[0,+\infty)}(t)KZ(t)$ where $\chi_{[0,+\infty)}$ denotes the characteristic function of the interval $[0, +\infty)$, we obtain the stable closed-loop dynamics

$$\dot{Z}(t) = A_K Z(t) + \Gamma.$$

Remark 1. In original coordinates, the control input v is solution of the fixed point implicit equation

$$v(t) = \chi_{[0,+\infty)}(t)KX(t) + K\int_{\max(t-D,0)}^{t} e^{A(t-D-\tau)}Bv(\tau)\,\mathrm{d}\tau.$$

Existence and uniqueness of the solution of the above equation are reported in (Bresch-Pietri et al., 2018). ◦

2.5 Characterization of the equilibrium point

We characterize the equilibrium point of the closed-loop system associated with a constant reference input $r \in \mathbb{R}$ and a stationary distributed disturbance $d \in L^2(0, L)$. In the sequel, we denote by a subscript "e" the equilibrium value of the different quantities. Noting that $u_{D,e} = u_e$ and $v_{D,e} = v_e$, we have

$$\begin{aligned} 0 &= A_K Z_e + \Gamma \\ 0 &= \lambda_j w_{j,e} + a_j u_e + b_j v_e + d_j, \qquad j \geq n+1 \end{aligned}$$

In particular, from $v_e = KZ_e$, we have $AZ_e + e^{-DA}Bv_e + \Gamma = 0$. As the first row of A and Γ are null and $[1 \ 0 \ \ldots \ 0]\, e^{-DA}B = 1$, we obtain that $v_e = 0$. Moreover we have $Z_e = -A_K^{-1}\Gamma$. Then we can set $X_e = Z_e$ because $AX_e + Bv_{D,e} + \Gamma = A_K Z_e + \Gamma = 0$, which is compatible with the Artstein transformation since $v_e = 0$ implies $Z_e = X_e + \int_{t-D}^{t} e^{(t-s-D)A}Bv_e\,\mathrm{d}s$. Then we have $u_e = u_{D,e} = [1 \ 0 \ \ldots \ 0]\, Z_e = -[1 \ 0 \ \ldots \ 0]\, A_K^{-1}\Gamma$, and $w_{j,e} = -\frac{a_j}{\lambda_j}u_e - \frac{d_j}{\lambda_j}$ for $j \geq n+1$. Now, noting that both $(w_{j,e})_j$ and $(\lambda_j w_{j,e})_j$ are square-summable sequences, we can define

$$w_e \triangleq \sum_{j \geq 1} w_{j,e} e_j \in D(\mathcal{A}) = H^2(0, L) \cap H_0^1(0, L).$$

Expending the last row of $AX_e + \Gamma = 0$, we obtain that $w_e'(0) + \frac{1}{L}u_e = r$. Consequently, the introduction of $y_e \triangleq w_e + \frac{x}{L}u_e$ yields $y_e'(0) = r$. Finally, from $\mathcal{A}w_e = \sum_{j \geq 1} \lambda_j w_{j,e} e_j$ we deduce that $\mathcal{A}w_e + au_{D,e} + bv_{D,e} + d = 0$.

2.6 Dynamics of deviations

We now define the deviations of the various quantities with respect to their equilibrium value: $\Delta X = X - X_e$, $\Delta Z = Z - Z_e$, $\Delta w = w - w_e$, $\Delta w_j = w_j - w_{j,e}$, $\Delta\zeta = \zeta - \zeta_e$, $\Delta u = u - u_e$, $\Delta u_D = u_D - u_e$, $\Delta v = v - v_e$, and $\Delta v_D = v_D - v_{D,e}$. Then, in original coordinates:

$$\Delta w_t = \mathcal{A}\Delta w + a\Delta u_D + b\Delta v_D \tag{17}$$

and

$$\Delta\dot{X}(t) = A\Delta X(t) + B\Delta v_D(t)$$
$$\Delta\dot{w}_j(t) = \lambda_j \Delta w_j(t) + a_j\Delta u_D(t) + b_j\Delta v_D(t)$$

for $j \geq n+1$ with the auxiliary control input $\Delta v(t) = \chi_{[0,+\infty)}(t)K\Delta Z(t)$ (because $v_e = KZ_e = 0$) where

$$\Delta Z(t) = \Delta X(t) + \int_{t-D}^{t} e^{(t-s-D)A}B\Delta v(s)\,\mathrm{d}s. \tag{18}$$

In Z coordinates, the closed-loop dynamics is given by

$$\Delta\dot{Z}(t) = A_K\Delta Z(t) \tag{19a}$$
$$\Delta\dot{w}_j(t) = \lambda_j\Delta w_j(t) + a_j\Delta u_D(t) + b_j\Delta v_D(t) \tag{19b}$$

for $j \geq n+1$.

3. STABILITY ANALYSIS

The stability of the closed-loop infinite-dimensional system is assessed by the following theorem.

Theorem 1. *There exist $\kappa, \overline{C}_1 > 0$ such that*

$$\Delta u_D(t)^2 + \Delta\zeta(t)^2 + \|\Delta w(t)\|^2_{H_0^1(0,L)} \leq \overline{C}_1 e^{-2\kappa t}\left(\Delta u_D(0)^2 + \Delta\zeta(0)^2 + \|\Delta w(0)\|^2_{H_0^1(0,L)}\right). \tag{20}$$

The proof of Theorem 1 relies on the following Lyapunov function:

$$V(t) = \frac{M}{2}\Delta Z(t)^\top P\Delta Z(t) + \frac{M}{2}\int_{\max(t-D,0)}^{t}\Delta Z(s)^\top P\Delta Z(s)\,\mathrm{d}s - \frac{1}{2}\sum_{j\geq 1}\lambda_j\Delta w_j(t)^2, \tag{21}$$

where $P \in \mathbb{R}^{(n+2)\times(n+2)}$ is the solution of the Lyapunov equation $A_K^\top P + PA_K = -I$ and $M > 0$ is chosen such that

$$M > \max\left(\frac{\gamma_1\lambda_1}{\lambda_m(P)}, 2\left(\gamma_1\|a\|^2 + \|b\|^2\|e^{-DA_K}\|^2\|K\|^2\right)\right)$$

with $\gamma_1 \triangleq 2\max\left(1, De^{2D\|A\|}\|BK\|^2\right)$. Useful properties of V are stated in the three following lemmas. Due to space limitation, only a sketch of proof is provided.

Lemma 1. *There exists a constant $C_1 > 0$ such that*

$$V(t) \geq C_1\sum_{j\geq 1}(1+|\lambda_j|)\Delta w_j(t)^2, \tag{22a}$$
$$V(t) \geq C_1\left(\Delta u_D(t)^2 + \Delta\zeta(t)^2 + \|\Delta w(t)\|^2_{H_0^1(0,L)}\right), \tag{22b}$$
$$V(t) \geq C_1\|\Delta Z(t)\|^2, \tag{22c}$$

for every $t \geq 0$.

Sketch of proof Using $M > \frac{\gamma_1\lambda_1}{\lambda_m(P)}$, the claimed estimates are obtained similarly to the ones reported in (Prieur and Trélat, 2019). □

Lemma 2. *There exist $\kappa, C_2 > 0$ such that*

$$V(t) \leq e^{-2\kappa(t-D)}V(D)$$

for every $t \geq D$.

Sketch of proof As $\mathcal{A}$ is self-adjoint, we have for $t > D$,

$$\dot{V}(t) = -\frac{M}{2}\|\Delta Z(t)\|^2 - \frac{M}{2}\int_{t-D}^{t}\|\Delta Z(s)\|^2\,\mathrm{d}s - \|\mathcal{A}\Delta w(t)\|^2 - \langle\mathcal{A}\Delta w(t), a\rangle\Delta u_D(t) - \langle\mathcal{A}\Delta w(t), b\rangle\Delta v_D(t).$$

The use of Cauchy-Schwarz and Young inequalities show that, for all $t > D$,

$$\dot{V}(t) \leq -\frac{1}{2}\|A\Delta w(t)\|^2 - \frac{\gamma_2}{\lambda_M(P)}\left(\Delta Z(t)^\top P\Delta Z(t) + \int_{t-D}^{t}\Delta Z(s)^\top P\Delta Z(s)\,\mathrm{d}s\right)$$

with $\gamma_2 = M/2 - \left(\gamma_1\|a\|^2 + \|b\|^2\|e^{-DA_K}\|^2\|K\|^2\right) > 0$. Similarly to (Prieur and Trélat, 2019), we infer the existence of a constant $\gamma_3 > 0$ such that, for all $t \geq 0$,

$$-\langle\mathcal{A}\Delta w(t), \Delta w(t)\rangle \leq \gamma_3\|\mathcal{A}\Delta w(t)\|^2.$$

Consequently, we obtain that $\dot{V}(t) \leq -2\kappa V(t)$ for all $t > D$ with $\kappa = \frac{1}{2}\min\left(\frac{2\gamma_2}{M\lambda_M(P)}, \frac{1}{\gamma_3}\right) > 0$. □

Lemma 3. *There exists $C_2 > 0$ such that*

$$V(t) \leq C_2\left(\Delta u_D(0)^2 + \Delta\zeta(0)^2 + \|\Delta w(0)\|^2_{H_0^1(0,L)}\right)$$

for all $0 \leq t \leq D$ with $\Delta u_D(0) = -u_e$.

Sketch of proof Estimations similar to the ones reported in the proof of Lemma 2 show the existence of $\gamma_4 > 0$ such that

$$\dot{V}(t) \leq \gamma_4\|\Delta X(0)\|^2$$

for all $0 \leq t < D$. Therefore, $V(t) \leq V(0) + D\gamma_4\|\Delta X(0)\|^2$ for all $0 \leq t \leq D$. The estimation of $V(0)$ from (21) and a direct integration with $t \leq D$ show the claimed result. □

The proof of Theorem 1 is now a straightforward combination of the results reported in Lemmas 1, 2 and 3. Recalling that $\Delta v(t) = K\Delta Z(t)$ for $t \geq 0$ and $\Delta v(t) = 0$ for $t < 0$, we also obtain that

$$\Delta v_D(t)^2 \leq \hat{C}_1 e^{-2\kappa t}\left(\Delta u_D(0)^2 + \Delta\zeta(0)^2 + \|\Delta w(0)\|^2_{H_0^1(0,L)}\right) \tag{23}$$

for $t \geq 0$ with $\hat{C}_1 = \|K\|^2\overline{C}_1 e^{2\kappa D}$.

4. SETPOINT REFERENCE TRACKING ANALYSIS

We assess that the tracking of the constant reference input r is achieved in spite of the stationary distributed disturbance d.

Theorem 2. *Let $\kappa > 0$ be provided by Theorem 1. There exists $\overline{C}_2 > 0$ such that*

$$|y_x(t,0) - r| \leq \overline{C}_2 e^{-\kappa t}\left(|\Delta u_D(0)| + |\Delta\zeta(0)| + \|\Delta w(0)\|_{H_0^1(0,L)} + \|\mathcal{A}\Delta w(0)\|_{L^2(0,L)}\right). \tag{24}$$

Sketch of proof Recalling that $w_{e,x}(0) + \frac{1}{L}u_e = r$, we have

$$\begin{aligned}|y_x(t,0)-r| &= \left| w_x(t,0) + \frac{1}{L}u_D(t) - r \right| \\ &\leq |w_x(t,0) - w_{e,x}(0)| + \frac{1}{L}|\Delta u_D(t)|. \quad (25)\end{aligned}$$

From the exponential convergence of $\Delta u_D(t)$ to zero provided by (20), it is sufficient to study the term $w_x(t,0) - w_{e,x}(0) = \sum_{j \geq 1} \Delta w_j(t) e_j'(0)$. As $e_j'(0) \sim \sqrt{\frac{2}{L}}\sqrt{|\lambda_j|}$, there exists a constant $\gamma_7 > 0$ such that $|e_j'(0)| \leq \gamma_7 \sqrt{|\lambda_j|}$ for all $j \geq n+1$. Let $m \geq n+1$ be such that $\eta \triangleq -\lambda_m > \kappa > 0$. Thus $\lambda_j \leq -\eta < -\kappa < 0$ for all $j \geq m$. We have:

$$\begin{aligned}&|w_x(t,0) - w_{e,x}(0)| \\ &\leq \sum_{j=1}^{m-1} |\Delta w_j(t)||e_j'(0)| + \gamma_7 \sum_{j \geq m} \sqrt{|\lambda_j|}|\Delta w_j(t)| \\ &\leq \sqrt{\sum_{j=1}^{m-1} e_j'(0)^2}\sqrt{\sum_{j=1}^{m-1} \Delta w_j(t)^2} \quad (26) \\ &\quad + \gamma_7 \sqrt{\sum_{j \geq m} \frac{1}{|\lambda_j|}}\sqrt{\sum_{j \geq m} \lambda_j^2 \Delta w_j(t)^2}\end{aligned}$$

where $\sum_{j \geq m} \frac{1}{|\lambda_j|} < +\infty$ because $|\lambda_j| \sim \pi^2 j^2 / L^2$. Based on (20) and Poincaré inequality, it is sufficient to study the term $\sqrt{\sum_{j \geq m} \lambda_j^2 \Delta w_j(t)^2}$. To do so, we integrate (19b) for $j \geq m$ and we use estimates (20) and (23), yielding

$$\begin{aligned}&|\lambda_j \Delta w_j(t)| \\ &\leq e^{\lambda_j t}|\lambda_j \Delta w_j(0)| \\ &\quad + \int_0^t (-\lambda_j) e^{\lambda_j (t-\tau)} \left\{ |a_j||\Delta u_D(\tau)| + |b_j||\Delta v_D(\tau)| \right\} \mathrm{d}\tau \\ &\leq e^{-\eta t}|\lambda_j \Delta w_j(0)| + C_{3,j} e^{\lambda_j t} \int_0^t (-\lambda_j) e^{-\lambda_j \tau} e^{-\kappa \tau} \,\mathrm{d}\tau \,\Delta\mathrm{CI}\end{aligned}$$

with $\Delta\mathrm{CI} = \sqrt{\Delta u_D(0)^2 + \Delta\zeta(0)^2 + \|\Delta w(0)\|^2_{H_0^1(0,L)}}$ and constant $C_{3,j} = |a_j|\sqrt{\overline{C}_1} + |b_j|\sqrt{\hat{C}_1}$. As $\lambda_j \leq -\eta < -\kappa$ for all $j \geq m$, we obtain that $e^{\lambda_j t} \int_0^t (-\lambda_j) e^{-\lambda_j \tau} e^{-\kappa \tau} \,\mathrm{d}\tau = \frac{\lambda_j}{\lambda_j + \kappa}(e^{-\kappa t} - e^{\lambda_j \tau}) \leq \frac{\lambda_j}{\lambda_j + \kappa} e^{-\kappa t} \leq \frac{\eta}{\eta - \kappa} e^{-\kappa t}$, hence

$$|\lambda_j \Delta w_j(t)| \leq e^{-\kappa t}|\lambda_j \Delta w_j(0)| + \frac{C_{3,j}\eta}{\eta - \kappa} e^{-\kappa t} \Delta\mathrm{CI}$$

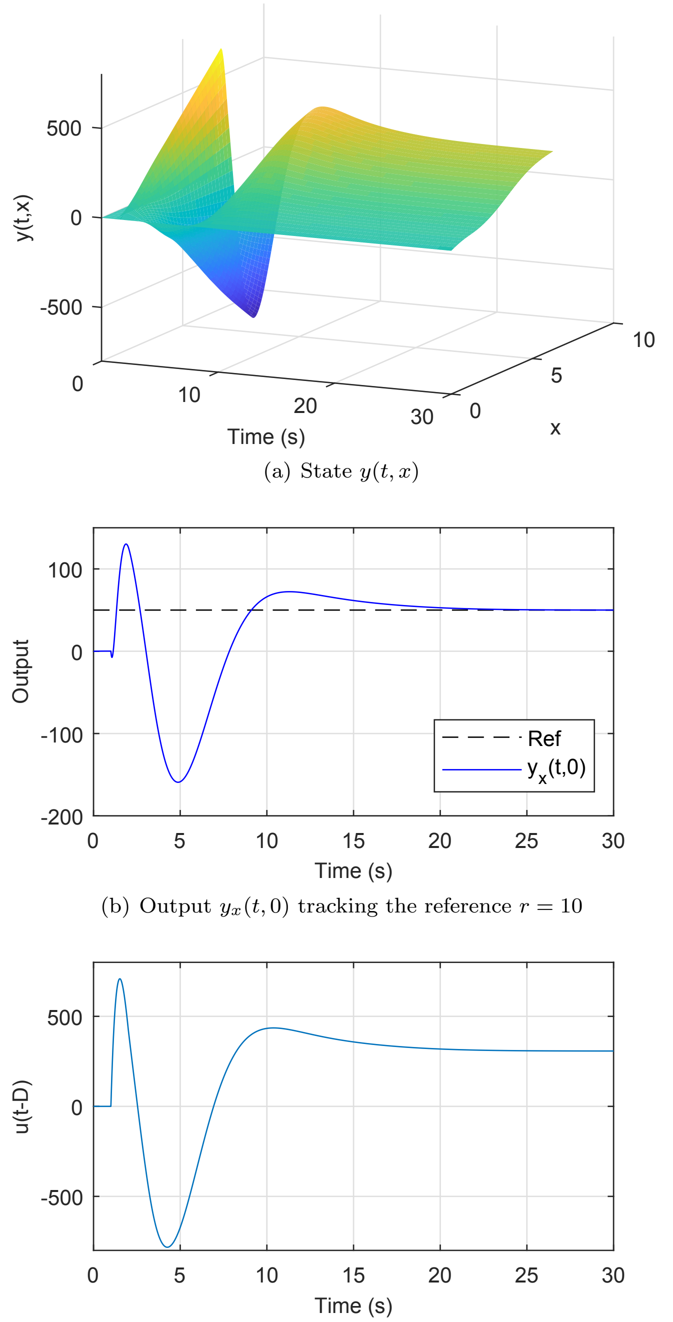
and thus

$$\sum_{j \geq m} \lambda_j^2 \Delta w_j(t)^2 \leq 2e^{-2\kappa t}\|\mathcal{A}\Delta w(0)\|^2 + \frac{2C_3^2\eta^2}{(\eta - \kappa)^2} e^{-2\kappa t} \Delta\mathrm{CI}^2. \quad (27)$$

with $C_3 > 0$ defined by $C_3^2 = \sum_{j \geq m} C_{3,j}^2 \leq 2\overline{C}_1\|a\|^2 + 2\hat{C}_1\|b\|^2$. Using now (25) along with (26) and estimates (20) and (27), we obtain the existence of a constant $\overline{C}_2 > 0$ such that the claimed estimate (24) holds for all $t \geq 0$. $\square$

5. NUMERICAL ILLUSTRATION

We take $c = 1.25$, $L = 2\pi$, and $D = 1\,\mathrm{s}$. The three first eigenvalues of the open-loop system are $\lambda_1 = 1$, $\lambda_2 = 0.25$, and $\lambda_2 = -1$. Only the two first modes need to be stabilized. Thus we have $n = 2$ and we compute the feedback gain $K \in \mathbb{R}^{1 \times 4}$ such that the poles of the closed-loop truncated model (capturing the two unstable modes of the infinite-dimensional system plus two integral components, one for the control input and one for reference tracking) are given by -0.5, -0.6, -0.7, and -0.8. The adopted numerical scheme is the modal approximation of the infinite-dimensional system using its first 10 modes. The initial condition is set as $y_0(x) = -\frac{x}{L}\left(1 - \frac{x}{L}\right)$.



(a) State $y(t,x)$

(b) Output $y_x(t,0)$ tracking the reference $r = 10$

(c) Delayed command input $u(t-D)$

Fig. 1. Time evolution of the closed-loop system

The obtained simulation results with $r = 50$ and $d(x) = x$ are depicted in Fig. 1. As expected from the theoretical analysis, the PI controller achieves the stabilization of the reaction-diffusion equation and ensures that the Neumann trace $y_x(t,0)$ tracks the constant reference input r.

6. CONCLUSION

This paper discussed the PI regulation control of the left Neumann trace of a one-dimensional linear reaction-diffusion equation with delayed right Dirichlet boundary control. The proposed strategy extends to PI control a recently proposed approach for the delay boundary feedback stabilization of infinite-dimensional systems combining spectral reduction and the use of the classical Artstein transformation for handling the delay in the control input. The validity of this control strategy for the tracking of a constant reference input and in the presence of a stationary perturbation was assessed via a Lyapunov-based argument. The extension of these results to the set-point regulation control of a time-varying reference input $r(t)$ and in the presence of a time-varying distributed perturbation $d(t,x)$ can be found in (Lhachemi et al., 2019b).

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