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Sufficient conditions for pre-compactness of state trajectories

H. Zwart *,**

* *University of Twente, Faculty of Electrical Engineering, Mathematics and Computer Science, Department of Applied Mathematics, P.O. Box 217, 7500 AE Enschede, The Netherlands. (e-mail: h.j.zwart@utwente.nl)*
** *Technische Universiteit Eindhoven, Department of Mechanical Engineering, P.O. Box 513, 5600 MB Eindhoven, The Netherlands. (e-mail: h.j.zwart@tue.nl)*

Abstract: In this paper we give sufficient conditions on a semi-linear differential equation ensuring the pre-compactness of its solution. The result is illustrated by two examples of vibrating strings in a network with a static damper.

Keywords: Boundary control systems, infinite-dimensional systems, asymptotic stability, non-linear control.

1. INTRODUCTION

Proving the asymptotic stability of a non-linear differential equation follows often along the following lines. After existence is shown, a Lyapunov function V is constructed which along solutions satisfies $\dot{V} \leq 0$. Next the largest invariant subset of $\{x \mid \dot{V}(x) = 0\}$ is shown to consist of only the equilibrium point. Finally, LaSalle's invariance principle is used to conclude that the equilibrium point is asymptotically stable, see e.g. (Curtain & Zwart, 2020, Chapter 11).

This procedure is almost the same for systems described by ordinary- and by partial differential equations. The only difference is one but important condition in LaSalle's invariance principle. Namely the solution must be pre-compact. Meaning that its closure should be compact. For ordinary differential equations (ode's) this condition is satisfied whenever the solution stays bounded, because in $\mathbb{R}^n$ any bounded set is pre-compact. For systems described by partial differential equations (pde's) this needs not to hold, for a simple example see Zwart (2016). Hence it is important to know conditions which imply the pre-compactness.

In this paper we shall give some sufficient conditions. As shown in the examples, checking of these conditions can be done by looking at the expression for $\dot{V}$. The outline of the paper is as follows. In the next section we introduce our (abstract) differential equation, and the assumptions. We end that section with its main theorem, stating the pre-compactness of a trajectory. In Section 3 we shall give the proof of this theorem. Section 4 contains some examples showing how the result can be applied. We end the paper with the conclusion.

2. DIFFERENTIAL EQUATION AND PRE-COMPACTNESS

We consider the semi-linear differential equation

$$\dot{z}(t) = Az(t) + Bf(z(t)), \qquad z(0) = z_0, \tag{1}$$

where A generates a bounded semigroup on the Hilbert space Z, and B maps from the input (Hilbert) space U into the state space Z. We assume that B is an admissible operator for the semigroup.

Definition 1. Let A with domain $D(A)$ be the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on the state space Z, and let A^* be its dual. Furthermore, let B be a bounded linear operator from U to $D(A^*)'$ (the dual of $D(A^*)$). B is said to be infinite-time L^p-admissible, if for all $u \in L^p((0,\infty);U)$ and $t > 0$ there holds $\int_0^t T(\tau)Bu(\tau)d\tau \in Z$ and

$$\sup_{t>0} \| \int_0^t T(\tau)Bu(\tau)d\tau \| < \infty,$$

where the norm is the Z-norm. □

For more detail on admissible input operators, we refer the interested reader to Staffans (2000) or Tucsnak and Weiss (2009). Note that we will take $p \in [1,\infty)$. By the uniform boundedness theorem, we see that if B is infinite-time L^p-admissible, then there exists a M_p such that

$$\| \int_0^t T(\tau)Bu(\tau)d\tau \| \leq M_p \|u\|_{L^p}, \tag{2}$$

with M_p independent of u and t.

Although in this paper we shall not study the existence of solutions of the (abstract) differential equation we have to say what we mean with a solution of (1).

Definition 2. Let A be the infinitesimal generator of a C_0-semigroup on Z, and let B be infinite-time L^p-admissible for some $p \in [1,\infty)$. We call $z(\cdot;z_0)$ a solution of (1) if

(1) The function $t \mapsto z(t;z_0)$ is continuous and $z(0;z_0) = z_0$;

(2) For every $t_1 > 0$, the function $t \in [0, t_1] \mapsto f(z(t; z_0))$ is in $L^p((0, t_1); U)$;
(3) For every $t_1 > 0$ the following equality holds on $[0, t_1]$

$$z(t; z_0) = T(t)z_0 + \int_0^t T(t-s)Bf(z(s; z_0))ds. \quad (3)$$

Now we have introduced all necessary concepts and can formulate the main theorem of this section.

Theorem 3. Consider the semi-linear differential equation (1) where A generates a bounded semigroup on Z, and A has compact resolvent. Furthermore, assume that B is infinite-time L^p-admissible for some $p \in [1, \infty)$.

Suppose that for a $z_0 \in Z$, the solution $z(t; z_0)$ exists for all $t \geq 0$ and this solution is such that $f(z(\cdot; z_0)) \in L^p((0, \infty); U)$, then the trajectory set $\{z(t; z_0), t \geq 0\}$ is bounded and pre-compact. □

Since the identity is an infinite-time L^1-admissible operator, provided the semigroup is uniformly bounded, we immediately obtain the following corollary, see Dafermos & Slemrod (1973).

Corollary 4. Consider the semi-linear differential equation (1) where A generates a bounded semigroup on Z, has compact resolvent, and $B = I$. Suppose that for a $z_0 \in Z$, the solution $z(t; z_0)$ exists for all $t \geq 0$ and this solution is such that $f(z(\cdot; z_0)) \in L^1((0, \infty); Z)$, then the trajectory set $\{z(t; z_0), t \geq 0\}$ is bounded and pre-compact.

3. PROOF OF THEOREM 3

The proof is divided into four steps. By Definition 2 we know that the solution can be written as

$$z(t; z_0) = T(t)z_0 + \int_0^t T(t-s)Bf(z(s; z_0))ds. \quad (4)$$

Since the semigroup $(T(t))_{t \geq 0}$ is bounded and B is infinite-time L^p-admissible, we see from (2) that the trajectory is uniformly bounded. So it remains to show that it is pre-compact.

So we have the sequence $\{z(t_n; z_0), n \in \mathbb{N}\}$. If the sequence $\{t_n, n \in \mathbb{N}\}$ has a converging subsequence, then the continuity of the solution implies the convergence of the state along these time instances. So it remains to prove the assertion when the sequence $\{t_n, n \in \mathbb{N}\}$ is unbounded, i.e., when $t_n \to \infty$. In part a. and b. we construct a subsequence such that $T(t_n)z_0$ converges along this subsequence. In part c. we show that for any given time-sequence there exists a subsequence, such that the integral part in (4) converges along this subsequence. In the final part we combine parts b. and c. to conclude that there exists a subsequence such that $z(t_n; z_0)$ converges along this subsequence as $n \to \infty$.

We denote by M the uniform bound of the semigroup, i.e., $M = \sup_{t \geq 0} \|T(t)\|$.

a. Let $\{t_n, n \in \mathbb{N}\}$ be a time sequence converging to infinity as $n \to \infty$. If $z_k \in D(A)$, then

$$\|AT(t_n)z_k\| = \|T(t_n)Az_k\| \leq M\|Az_k\|,$$

where we have used the uniform boundedness of the semigroup. Hence $T(t_n)z_k$ and $AT(t_n)z_k$ are bounded sequences, and since A has compact resolvent, there exists a subsequence $\{\tau_{n;k}, n \in \mathbb{N}\} \subset \{t_n\}$ such that $\{T(\tau_{n;k})z_k\}$ converges as $n \to \infty$. Note that the subsequence will in general depend on z_k. Furthermore, since the sequence is converging, we can always choose it such that

$$\|T(\tau_{n;k})z_k - T(\tau_{m;k})z_k\| \leq \frac{1}{\min\{n, m\}}. \quad (5)$$

b. Let $z_0 \in Z$ and the sequence $\{t_n, n \in \mathbb{N}\}$ be given. We begin by choosing a sequence $\{z_k, z_k \in D(A), k \in \mathbb{N}\}$ such that

$$\|z_k - z_0\| \leq \frac{1}{k}, \quad k \geq 1. \quad (6)$$

Since $D(A)$ is dense in Z this is possible. Now $z_1 \in D(A)$, and so by part a. there exists a subsequence $\{\tau_{n;1}, n \in \mathbb{N}\} \subset \{t_n, n \in \mathbb{N}\}$ such that the sequence $\{T(\tau_{n;1})z_1\}$ converges. For the second element, z_2, we consider the time sequence $\{\tau_{n;1}, n \in \mathbb{N}\}$. Again from part a. we find a subsequence $\{\tau_{n;2}, n \in \mathbb{N}\} \subset \{\tau_{n;1}, n \in \mathbb{N}\}$ such that the sequence $\{T(\tau_{n;2})z_2\}$ converges. As explained in part a. we can also adjust the subsequence such that (5) is satisfied. We repeat this process and find a sequence of subsequences, such that $\{\tau_{n;k}, n \in \mathbb{N}\} \subset \{\tau_{n;\ell}, n \in \mathbb{N}\}$ for all $k \geq \ell$ and (5) holds. Now we claim that $T(\tau_{n;\infty})z_0$ with $\tau_{n;\infty} = \tau_{n;n}$ is a Cauchy sequence. So we take the n'th element from the n'th subsequence. For $n < m$

$$\begin{aligned}
&\|T(\tau_{n;n})z_0 - T(\tau_{m;m})z_0\| \\
&\quad \leq \|T(\tau_{n;n})z_0 - T(\tau_{n;n})z_n\| + \\
&\quad\quad \|T(\tau_{n;n})z_n - T(\tau_{m;m})z_n\| + \\
&\quad\quad \|T(\tau_{m;m})z_n - T(\tau_{m;m})z_m\| + \\
&\quad\quad \|T(\tau_{m;m})z_m - T(\tau_{m;m})z_0\| \\
&\quad \leq \frac{M}{n} + \|T(\tau_{n;n})z_n - T(\tau_{m;m})z_n\| + \\
&\quad\quad \frac{2M}{n} + \frac{M}{m},
\end{aligned}$$

where we have used (6) combined with the bound on the semigroup. Since $\tau_{m;m}$ is an element of the n'th subsequence, we can estimate $\|T(\tau_{n;n})z_n - T(\tau_{m;m})z_n\|$ by $\frac{1}{n}$. Combining this with the above estimate, we find that $\{T(\tau_{n;\infty})z_0, n \in \mathbb{N}\}$ is a Cauchy sequence, and hence is converging.

c. In this part we consider the integral term of (4) and, for simplicity, we write $f(s)$ instead of $f(z(s; z_0))$. Furthermore, we define

$$\kappa := \max\{M, M_p\}.$$

So let $f \in L^p((0, \infty); U)$ and the sequence $\{t_n, n \in \mathbb{N}\}$ be given. Assume that we have a subsequence $\{t_n^{(k)}, n \in \mathbb{N}\} \subset \{t_n, n \in \mathbb{N}\}$ such that for $n, m \in \mathbb{N}$

$$\left\| \int_0^{t_n^{(k)}} T(t_n^{(k)} - s)Bf(s)ds - \int_0^{t_m^{(k)}} T(t_m^{(k)} - s)Bf(s)ds \right\| \leq \kappa \|f\| 2^{-k+1}, \quad (7)$$

where $\|f\|$ is the $L^p((0, \infty); U)$-norm of f. Now we shall construct a new subsequence $\{t_n^{(k+1)}, n \in \mathbb{N}\} \subset \{t_n^{(k)}, n \in \mathbb{N}\}$ such that for $n, m \in \mathbb{N}$

$$\left\| \int_0^{t_n^{(k+1)}} T(t_n^{(k+1)} - s)Bf(s)ds - \int_0^{t_m^{(k+1)}} T(t_m^{(k+1)} - s)Bf(s)ds \right\| \leq \kappa \|f\| 2^{-k}. \tag{8}$$

Choose N such that for all $n \geq N$

$$\sqrt[p]{\int_{t_n^{(k)}}^{\infty} \|f(s)\|^p ds} \leq \|f\| 2^{-k-2}. \tag{9}$$

Since $t_n^{(k)}$ is an unbounded sequence, $p < \infty$, and since $f \in L^p((0,\infty);U)$ this is possible. Next define $z_N \in Z$ by

$$z_N := \int_0^{t_N^{(k)}} T(t_N^{(k)} - s)Bf(s)ds.$$

The sequence $\{T(t_n^{(k)} - t_N^{(k)})z_N, n \geq N\}$ is a bounded sequence in Z, and by part b, there exists a subsequence $\{\tilde{t}_n^{(k+1)}, n \in \mathbb{N}\} \subset \{t_n^{(k)}, n \geq N\}$ such that $T(\tilde{t}_n^{(k+1)} - t_N^{(k)})z_N$ converges as $n \to \infty$. In particular, there exists a $N_1 > 0$ such that for $n, m \geq N_1$

$$\|T(\tilde{t}_n^{(k+1)} - t_N^{(k)})z_N - T(\tilde{t}_m^{(k+1)} - t_N^{(k)})z_N\| \leq \kappa \|f\| 2^{-k-1}. \tag{10}$$

Now we define $t_n^{(k+1)} := \tilde{t}_{n+N_1}^{(k+1)}$, $n \in \mathbb{N}$. We show that along this subsequence (8) holds. For this we also use the following equality

$$\int_0^t T(t-s)Bf(s)ds = T(t - t_N^{(k)})z_N + \int_{t_N^{(k)}}^t T(t-s)Bf(s)ds, \quad t \geq t_N^{(k)}. \tag{11}$$

Combining (9), (10) and (11), we see that

$$\begin{aligned}
&\left\| \int_0^{t_n^{(k+1)}} T(t_n^{(k+1)} - s)Bf(s)ds - \int_0^{t_m^{(k+1)}} T(t_m^{(k+1)} - s)Bf(s)ds \right\| \\
&= \Bigg\| T(t_n^{(k+1)} - t_N^{(k)}) \int_0^{t_N^{(k)}} T(t_N^{(k)} - s)Bf(s)ds + \int_{t_N^{(k)}}^{t_n^{(k+1)}} T(t_n^{(k+1)} - s)Bf(s)ds - \\
&\quad T(t_m^{(k+1)} - t_N^{(k)}) \int_0^{t_N^{(k)}} T(t_N^{(k)} - s)Bf(s)ds + \int_{t_N^{(k)}}^{t_m^{(k+1)}} T(t_m^{(k+1)} - s)Bf(s)ds \Bigg\| \\
&\leq \|T(t_n^{(k+1)} - t_N^{(k)})z_N - T(t_m^{(k+1)} - t_N^{(k)})z_N\| + \\
&\quad \left\| \int_{t_N^{(k)}}^{t_n^{(k+1)}} T(t_n^{(k+1)} - s)Bf(s)ds - \int_{t_N^{(k)}}^{t_m^{(k+1)}} T(t_m^{(k+1)} - s)Bf(s)ds \right\| \\
&\leq \kappa \|f\| 2^{-k-1} + M_p \sqrt[p]{\int_{t_N^{(k)}}^{t_n^{(k+1)}} \|f(s)\|^p ds} + M_p \sqrt[p]{\int_{t_N^{(k)}}^{t_m^{(k+1)}} \|f(s)\|^p ds} \\
&\leq \kappa \|f\| 2^{-k-1} + 2M_p \|f\| 2^{-k-2} \\
&\leq \kappa \|f\| 2^{-k},
\end{aligned}$$

where we have used (2). So we have proved (8).

By the bound on the semigroup, we see that if we take $t_n^{(0)} = t_n$, $n \in \mathbb{N}$, then (7) is satisfied. Hence we have made a nested sequence of subsequences. We now define $t_n^{(\infty)} := t_n^{(n)}, n \in \mathbb{N}$. Given an $\varepsilon > 0$, there exists a $k \in \mathbb{N}$ such that $\kappa \|f\| 2^{-k+1} \leq \varepsilon$. By construction, we have that for $n, m \geq k$

$$\left\| \int_0^{t_n^{\infty}} T(t_n^{\infty} - s)Bf(s)ds - \int_0^{t_m^{\infty}} T(t_m^{\infty} - s)Bf(s)ds \right\| \leq \kappa \|f\| 2^{-k+1} \leq \varepsilon.$$

Hence $\{\int_0^{t_n^{\infty}} T(t_n^{\infty} - s)Bf(s)ds, n \in \mathbb{N}\}$ is a Cauchy sequence.

d. Now let $\{t_n, n \in \mathbb{N}\}$ be an unbounded sequence. By part b. we can construct a subsequence $\{\tau_{n;\infty}, n \in \mathbb{N}\}$ such that $T(\tau_{n;\infty})z_0$ converges. In part c. we showed that for any sequence there exists a subsequence such that the integral part in (4) converges. We take as the initial sequence $\{\tau_{n;\infty}, n \in \mathbb{N}\}$, and we construct a subsequence $\{t_n^{\infty}, n \in \mathbb{N}\}$ such that $\int_0^{t_n^{\infty}} T(t_n^{\infty} - s)Bf(s)ds$ converges as $n \to \infty$. Combining the above two facts, we see that $z(t_n^{\infty}; z_0)$ converges as $n \to \infty$. Hence for every sequence $\{t_n, n \in \mathbb{N}\}$ we can construct a subsequence such that $z(t_n; z_0)$ converges along this subsequence.

4. EXAMPLES

Consider the coupled (undamped) strings as shown in Figure 1. Every (undamped) string is modelled using the state variables $z_1(\zeta,t) = \rho \frac{\partial w}{\partial t}$ (the momentum) and $z_2(\zeta,t) = \frac{\partial w}{\partial \zeta}$ (the strain). The port-Hamiltonian representation of the model is given by

$$\frac{\partial z}{\partial t}(\zeta,t) = P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}(\zeta)z(\zeta,t)), \tag{12}$$

where $z(\zeta,t) = [z_1(\zeta,t) \ z_2(\zeta,t)]^T$, $P_1 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and $\mathcal{H}(\zeta) = \left[\begin{smallmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{smallmatrix}\right]$. So the total string modelled using three copies of the single model. We introduce the state $z(\zeta,t) = [z_{1,I}(\zeta,t), \cdots, z_{2,III}(\zeta,t)]^T$, and the total model is again of the form (12). From the picture we see that the following static boundary conditions hold

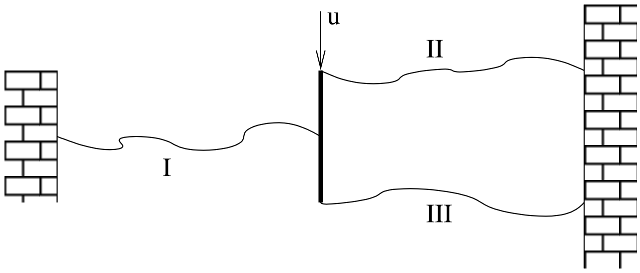

Fig. 1. Three controlled strings

$$\begin{aligned}&z_{1,I}(a,t)=0, z_{1,II}(b,t)=0, z_{1,III}(b,t)=0,\\&z_{1,I}(b,t)=z_{1,II}(a,t),\\&z_{1,I}(b,t)=z_{1,III}(a,t),\end{aligned}$$

where the index I denote the most left string. Index II and III denote the lower and upper right string, respectively. The final boundary condition comes from the force applied in the middle

$$\begin{aligned}u(t)=T_I(b)\frac{\partial w_I}{\partial\zeta}(b,t)-T_{II}(a)\frac{\partial w_{II}}{\partial\zeta}(a,t)-\\T_{III}(a)\frac{\partial w_{III}}{\partial\zeta}(a,t).\end{aligned}\tag{13}$$

For the total energy, we find that (see e.g. Jacob and Zwart (2012))

$$\begin{aligned}\dot{E}(t)=&\frac{\partial w_I}{\partial t}(b,t)\left[T_I(b)\frac{\partial w_I}{\partial\zeta}(b,t)-T_{II}(a)\frac{\partial w_{II}}{\partial\zeta}(a,t)\right.\\&\left.-T_{III}(a)\frac{\partial w_{III}}{\partial\zeta}(a,t)\right]\end{aligned}\tag{14}$$

$$=\frac{\partial w_I}{\partial t}(b,t)\cdot u(t).\tag{15}$$

We define the velocity of the connecting bar as our output, i.e., $y(t)=\frac{\partial w_I}{\partial t}(b,t)$. Introduce $v(t)=u(t)+y(t)$, then by (15) we see that

$$\dot{E}(t)=-y(t)^2+v(t)y(t)\le-\frac{3}{4}y^2(t)+v(t)^2.$$

Thus for $t>0$

$$E(t)-E(0)\le-\frac{3}{4}\int_0^t y^2(\tau)d\tau+\int_0^t v(\tau)^2d\tau.$$

So if $v\in L^2(0,\infty)$, then the state stays bounded. This combined with the fact that the system is a boundary control system, see Jacob and Zwart (2012), shows that the system with input v is infinite-time L^2-admissible.

As it is clear from (15) when the input u is identically zero, the system does not loose energy, and so no non-zero state will converge to zero. Thus to stabilise the system damping must be applied. This damping force we choose

$$u(t)=-f(y(t)),\ \text{or}\ v(t)=-f(y(t))+y(t).$$

Here we assume that f is a non-decreasing continuous function from $\mathbb{R}$ to $\mathbb{R}$ satisfying $\alpha y^2\le yf(y)\le\beta y^2$ for some $\alpha,\beta>0$.

Applying this damping to our system, we find by equation (15) that

$$\dot{E}(t)=-y(t)f(y(t)).\tag{16}$$

Thus we have the following energy balance for $t>0$

$$E(0)=E(t)+\int_0^t y(\tau)f(y(\tau))d\tau.\tag{17}$$

In particular, this implies that

$$\int_0^\infty y(\tau)f(y(\tau))d\tau<\infty\tag{18}$$

Using the bounds on f we find that

$$\int_0^\infty \alpha y(\tau)^2d\tau\le\int_0^\infty y(\tau)f(y(\tau))d\tau\tag{19}$$

$$\int_0^\infty f(y(\tau))^2d\tau\le\beta^2\int_0^\infty y(\tau)^2d\tau.\tag{20}$$

Combining these three inequalities we find that y and $f(y$ are in $L^2(0,\infty)$. In particular, we find that $v\in L^2(0,\infty)$. By Hastir et al. (2019) we know that the closed loop system possesses a unique solution, in the sense of Definition 2. Thus applying Theorem 3 gives that the closed loop trajectory is pre-compact. Applying LaSalle's invariance principle, we conclude that every solution converges to the largest invariant subset in which $\dot{E}=0$. In other words, to a solution of (12), satisfying the original boundary conditions,

$$\begin{aligned}&z_{1,I}(a,t)=0, z_{1,II}(b,t)=0, z_{1,III}(b,t)=0,\\&z_{1,I}(b,t)=z_{1,II}(a,t),\\&z_{1,I}(b,t)=z_{1,III}(a,t).\end{aligned}$$

and the extra "zero" boundary conditions

$$\begin{aligned}0=&T_I(b)\frac{\partial w_I}{\partial\zeta}(b,t)-T_{II}(a)\frac{\partial w_{II}}{\partial\zeta}(a,t)-\\&T_{III}(a)\frac{\partial w_{III}}{\partial\zeta}(a,t)\\0=&z_{1,I}(b,t).\end{aligned}$$

These two are the consequence of $\dot{E}=0$. Although there is one boundary condition too many, this does not necessarily imply that the state must be zero. For instance, if the strings II and III are identical, and move anti-phase, whereas string I is standing still, all seven boundary conditions are satisfied. So without extra information on the system, we cannot conclude stability.

In the above we assumed that the connecting bar had no mass. In the following we make the more realistic assumption that it has a mass, which we denote by m. Let us denote the velocity of the bar by $\nu(t)$, then Newton's law gives

$$\begin{aligned}m\dot{\nu}(t)=&F_{tot}(t)\\=&u(t)-T_I(b)\frac{\partial w_I}{\partial\zeta}(b,t)+T_{II}(a)\frac{\partial w_{II}}{\partial\zeta}(a,t)+\\&T_{III}(a)\frac{\partial w_{III}}{\partial\zeta}(a,t).\end{aligned}\tag{21}$$

We have that $\nu(t)=\frac{\partial w_I}{\partial t}(b,t)$, which still equals the output $y(t)$.

Since we have added a mass, the energy get an extra term, i.e., the total energy equals

$$E_{tot}(t)=E(t)+\frac{1}{2}m\nu(t)^2.\tag{22}$$

Using (14) and (21), a simple calculation gives

$$\dot{E}_{tot}(t)=\frac{\partial w_I}{\partial t}(b,t)u(t)=y(t)u(t).\tag{23}$$

By a similar argument as in the first case, this equality shows that $v(t) := u(t) + y(t)$ is an infinite-time L^2-admissible input.

We again try to stabilise the system by applying a damping force, i.e.,

$$u(t) = -f(y(t)), \text{ or } v(t) = -f(y(t)) + y(t).$$

However, now we will weaken the conditions on f. We assume the following

- f is a (locally) Lipschitz continuous non-decreasing function with $f(0) = 0$;
- There exists a $\alpha, \delta > 0$ such that $yf(y) \geq \alpha y^2$ whenever $|y| \leq \delta$.

Note that the above assumptions allow for the damping force to saturate.

By (23) we find, see also (17) and (18),

$$E_{tot}(0) = E_{tot}(t) + \int_0^t y(\tau)f(y(\tau))d\tau. \tag{24}$$

This implies that

$$\int_0^\infty y(\tau)f(y(\tau))d\tau < \infty \text{ and } E_{tot}(t) \leq E_{tot}(0). \tag{25}$$

Define the following subsets of $[0, \infty)$

$$\Theta = \{t \in [0, \infty) \mid |y(t)| \leq \delta\}$$
$$\Omega = \{t \in [0, \infty) \mid |y(t)| > \delta\}.$$

On these sets we find

$$\int_\Theta y(t)^2 dt \leq \int_\Theta \frac{1}{\alpha} y(t)f(y(t))dt < \infty, \tag{26}$$

where we have used the second assumption, and (25). Furthermore, using the fact that f is non-decreasing,

$$\int_\Omega \alpha\delta^2 dt \leq \int_\Omega y(t)f(y(t))dt < \infty.$$

This implies that Ω has finite measure.

From the second inequality in (25) and the expression of E_{tot}, we see that $\nu^2(t)$ is bounded on $[0, \infty)$. Since ν equals the output and Ω has finite measure, we find that

$$\int_\Omega y(t)^2 dt < \infty. \tag{27}$$

Combining (26) and (27) gives that $y \in L^2(0, \infty)$.

Since f is locally Lipschitz continuous and $f(0) = 0$, there exists a $\beta > 0$ such that

$$|f(y)| \leq \beta|y|$$

whenever $|y| < \delta$. This implies that $\int_\Theta f(y(t))^2 dt < \infty$. Using once more the continuity of f, the boundedness of y, and the fact that Ω has finite measure, we find that $\int_\Omega f(y(t))^2 dt < \infty$. So we conclude that y and $f(y)$ are elements of $L^2(0, \infty)$. Using a similar argument as in the first case, we have that every solution converges to the largest invariant subset in which $\dot{E}_{tot} = 0$. This leads to the same ω-limit set as in the previous case.

5. CONCLUSION

We have presented a sufficient condition for the pre-compactness of a trajectory, and we illustrated our result by two examples of a simple network of vibrating strings with damping at the boundary. We remark that these models falls in the well-established class of linear port-Hamiltonian systems on 1-D spatial domain, and so our results are easily applicable for these systems. The examples show how the energy, together with its derivative can help to prove pre-compactness of the trajectories. It is clear that this technique is not restricted to pde's in a one-dimensional spatial domain.

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