

Insight into stability analysis of time-delay systems using Legendre polynomials

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Abstract: In this paper, a numerical analysis to assess stability of time-delay systems is investigated. The proposed approach is based on the design of a finite-dimensional approximation of the infinite-dimensional space of solutions of the system. Indeed, based on the dynamical coefficients on the sequence made of the first Legendre polynomials, the original time-delay system is modelled by a finite-dimensional model interconnected to a modelling error. Putting aside the interconnection, the resulting finite-dimensional system turns out to be a nice approximation of the time-delay system. Using Padé arguments, the eigenvalues of this finite-dimensional system are proven to converge towards a set of characteristic roots of the original time-delay system. Furthermore, considering now the whole interconnected system and having a deeper look at the interconnection, an enriched Lyapunov-Krasovskii functional is proposed to develop a sufficient condition expressed in terms of linear matrix inequalities for the stability of the time-delay system. Both results are illustrated on a toy example.

Keywords: Time-delay, Numerical analysis, Eigenvalues, Lyapunov stability, LMI.

1. INTRODUCTION

In several fields, delay phenomena appear while processing information or connecting different networked systems. These transmission delays have a significant impact on the behaviour of the state of the complete system and can even destabilise it. In consequence, taking into consideration these lag times is crucial (see Richard [2003]). Furthermore, from a theoretical point of view, the analysis of such systems is a difficult task since they belong to the wide class of infinite-dimensional systems. Hence, characterising the stability of time-delay systems (TDSs) is a current research purpose.

Several ways have been proposed to analyse its stability. Some of them are relying on the design of Lyapunov-Krasovskii functionals (LKF). Indeed, some necessary and sufficient conditions can be established by using the so-called complete LKF. Nevertheless, these conditions reveal complicated to fulfil relying on a solution of a second order ordinary differential equation with boundary conditions. That is the reason why, in the literature, many works have been conducted to find only sufficient conditions often expressed in the linear matrix inequality (LMI) framework (see Fridman [2014] or Gu et al. [2003]). Recently, some methods based on augmented systems have shown its efficiency even for non small delays (see Ariba et al. [2018]). They are all based on some inequalities (Jensen, Wirtinger, Bessel as presented in Seuret and Gouaisbaut [2015]) and require to extend the state with a finite-dimensional system. A second approach is based on the inspection of the characteristic roots of the linear TDS. To assess stability in a direct manner, a determination of the root crossing points through Routh criterion (see Olgac and Sipahi [2002]) or a formulation based on matrix pencils (see Louisell [2015]) can be implemented. However, to evaluate

each characteristic root, the infinite-dimensional system is often approximated, once again, by a finite-dimensional system. For example, a Padé approximant of the delay is largely implemented (see Golub and Van Loan [1989]). Otherwise, more recently, Breda et al. [2005] presents a method based on pseudospectral differentiation and different rough projections on Fourier, Chebychev or Legendre basis functions were also numerically investigated (see Pekar and Gao [2018]). All these numerical approaches can then characterise the root locus thanks to an approximate finite-dimensional model.

From comparative studies, both the best reduced LKF and the root approximation with the fastest convergence (see Vyasrayani et al. [2014]) are obtained using a decomposition on Legendre first polynomials. Based on these considerations, one proposes in this paper to get a deeper understanding of the equivalent model which includes the system satisfied by the first Legendre coefficients. The aim of this study is to highlight a link in between the reduced LKF and the finite-dimensional system, which approximates the characteristic roots of the TDS. Proving that the approximation is converging, this new link help to better understand the accurate underlying stability result using Legendre technique. First, the augmented system, which includes the dynamics satisfied by the $N + 1$ first Legendre coefficients, is presented. This resulting augmented system is made of an interconnection between a finite-dimensional model and an infinite element. Focusing on the finite-dimensional part, it is equivalent to a Padé approximant of the original system, which consists in approximating the transfer function of the delay with a rational fraction which numerator of order N and denominator of order $N + 1$ are given by Padé table. From this equivalence, the convergence of the eigenvalues of the finite-dimensional

model towards some of those of the infinite-dimensional system can be deduced and reinforces the choice of Legendre first polynomials. Based on the structure of the interconnection, the reduced LKF including Legendre coefficients provides a sufficient condition of stability thanks to Bessel inequality. This condition takes a very convenient form and is easy to express as an LMI. Knowing that the finite-dimensional part uniformly converges towards the original system, it gives now an understandable numerical stability condition with respect to the delay.

Notations : Throughout the paper, $\mathbb{R}, \mathbb{C}, \mathbb{R}^{n \times m}, \mathbb{S}^n$ denotes respectively the set of real numbers, complex numbers, $n \times m$ real matrices and $n \times n$ symmetric matrices. Furthermore, $|\cdot|$ is the modulus and $\|\cdot\|$ denotes some matrix norm. Then, for any square matrix A , $\mathcal{H}(A) = A + A^T$, $\text{adj}(A)$ the adjugate matrix (the transpose of its cofactor matrix), $\det(A)$ the determinant, $\det'(A)$ its derivatives given by Jacobi's formula, $\text{tril}(A)$ the lower triangular part of the matrix A and $A > 0$ means that A is symmetric positive definite. Moreover, I is the identity matrix, $\text{diag}(d_0, \dots, d_N)$ is the diagonal matrix defined by its diagonal coefficients (d_0, \dots, d_N) and the operation \otimes traduces a Kronecker product. The space $\mathcal{L}^2(-h, 0; \mathbb{R}^n)$ represents the set of square-integrable functions from $(-h, 0)$ to \mathbb{R}^n . The space $\mathcal{H}^1(-h, 0; \mathbb{R}^n)$ refers to absolutely continuous functions from $[-h, 0]$ to \mathbb{R}^n with derivative in $\mathcal{L}^2(-h, 0; \mathbb{R}^n)$. Lastly, for any function $x : (-h, +\infty) \rightarrow \mathbb{R}^n$, the notation $x_t(\tau)$ stands for $x(t + \tau)$, for all $t \geq 0$ and all $\tau \in (-h, 0)$.

2. MODELLING OF A TIME-DELAY SYSTEM ON LEGENDRE POLYNOMIALS BASIS

2.1 Definition of a time-delay system

Consider the linear TDS given by

$$\begin{cases} \dot{x}(t) = Ax(t) + B_d C_d x(t-h), \forall t \in \mathbb{R}^+, \\ (x(0), x_0) = (f(0), f), f \in \mathcal{H}^1(-h, 0; \mathbb{R}^n), \end{cases} \quad (1)$$

where matrices $A \in \mathbb{R}^{n \times n}$, $B_d \in \mathbb{R}^{n \times m}$, $C_d \in \mathbb{R}^{m \times n}$ and the single delay $h > 0$ are assumed to be known.

Remark 1. System (1) is well defined in the Hilbert space $\mathbb{R}^n \times \mathcal{L}^2(-h, 0; \mathbb{R}^n)$. For each $t \in \mathbb{R}^+$, the unique analytic solution $(x(t), x_t)$ belongs therefore to $\mathbb{R}^n \times \mathcal{L}^2(-h, 0; \mathbb{R}^n)$.

Since several years, one assists to a huge number of works dedicated to the stability analysis of TDSs based on an extended state space of a finite-dimensional system. These extension is related to the use of appropriate inequalities (Jensen, Wirtinger, Bessel) which needs extra-signals to be usefull. Usually, these extra-signals are based upon the projection of the state x_t on a basis of $\mathcal{L}^2(-h, 0; \mathbb{R}^n)$ such as the one generated by Legendre polynomials, which definition is recalled in the next subsection.

2.2 Definition of the Legendre polynomials basis

By definition, for all $\tau \in [-h, 0]$ and $k \in \mathbb{N}$, each k -order Legendre polynomial is written as

$$L_k(\tau) = (-1)^k \sum_{l=0}^k (-1)^l \binom{k}{l} \binom{k+l}{l} \left(\frac{\tau+h}{h}\right)^l. \quad (2)$$

As noted in Lagrange [1939], these polynomials form an orthogonal basis of $\mathcal{L}^2(-h, 0; \mathbb{R}^n)$. In addition, they have the following properties.

Lemma 2. For all $k \in \mathbb{N}$,

$$\begin{cases} \frac{d}{d\tau} L_k(\tau) = \sum_{l=0}^{k-1} \frac{2l+1}{h} (1 - (-1)^{k+l}) L_l(\tau) & k \geq 1, \\ \frac{d}{d\tau} L_0(\tau) = 0, L_k(-h) = (-1)^k, L_k(0) = 1. \end{cases} \quad (3)$$

Proof. The proof of (3), using Rodrigues formula, is given in Gautschi [2006].

2.3 Coefficients on the Legendre polynomials basis

Focusing on $C_d x_t$, which is the transported part of the state and can be seen as a function of $\mathcal{L}^2(-h, 0; \mathbb{R}^n)$, its $N + 1$ first components on Legendre polynomials orthogonal basis can be calculated. Let us define the vector X_N , which stores these Legendre coefficients.

$$X_N(t) = \begin{bmatrix} \int_{-h}^0 C_d x_t(\tau) L_0(\tau) d\tau \\ \dots \\ \int_{-h}^0 C_d x_t(\tau) L_N(\tau) d\tau \end{bmatrix}, \forall t \in \mathbb{R}^+. \quad (4)$$

These first Legendre coefficients represent the projection on a finite-dimensional basis of the retarded state. Hence, increasing N adds information on the functional state and the behaviour of $C_d x_t$.

2.4 Dynamics of the coefficients

In order to analyse the behaviour of X_N , one has to compute its dynamics. This is formulated in the next proposition.

Proposition 3. The vector X_N is solution of the dynamical model

$$\begin{cases} \dot{X}_N(t) = \mathcal{A}_N X_N(t) + \mathcal{B}_N C_d x(t) - \mathcal{B}_N^* \epsilon_N(t), \forall t \in \mathbb{R}^+, \\ \epsilon_N(t) = C_d x(t-h) - \mathcal{C}_N^* X_N(t) \end{cases} \quad (5)$$

with

$$\begin{cases} \mathbf{1}_N = [1 \dots 1]^T, \mathbf{1}_N^* = [(-1)^0 \dots (-1)^N]^T, \\ \mathbf{L}_N = \text{tril}(\mathbf{1}_N \mathbf{1}_N^T - \mathbf{1}_N^* \mathbf{1}_N^{*T}), \mathcal{L}_N = -(\mathbf{L}_N + \mathbf{1}_N^* \mathbf{1}_N^{*T}), \\ \mathcal{I}_N = \frac{1}{h} \text{diag}(1, \dots, 2N+1), \\ \mathcal{A}_N = (\mathcal{L}_N \mathcal{I}_N) \otimes I_m, \mathcal{B}_N = \mathbf{1}_N \otimes I_m, \mathcal{B}_N^* = \mathbf{1}_N^* \otimes I_m, \\ \mathcal{C}_N = (\mathbf{1}_N^T \mathcal{I}_N) \otimes I_m, \mathcal{C}_N^* = (\mathbf{1}_N^{*T} \mathcal{I}_N) \otimes I_m, \end{cases}$$

and satisfies an initial condition X_N^0 given by the coefficients of $C_d x_0$.

Proof. For all $k \in \llbracket 0, N \rrbracket$, thanks to Legendre basis properties (3), to an integration by parts the derivation of each coefficient gives, for all $t \in \mathbb{R}^+$,

$$\begin{aligned} \frac{d}{dt} \int_{-h}^0 C_d x(t+\tau) L_k(\tau) d\tau &= C_d x(t) - (-1)^k C_d x(t-h) \\ &\quad - \sum_{l=0}^{k-1} \frac{2l+1}{h} (1 - (-1)^{k+l}) \int_{-h}^0 C_d x(t+\tau) L_l(\tau) d\tau. \end{aligned}$$

Gathering all the components, a compact expression is obtained

$$\begin{aligned} \dot{X}_N(t) = & (\mathbf{1}_N \otimes I_m)C_d x(t) - (\mathbf{1}_N^* \otimes I_m)C_d x(t-h) \\ & - (\mathbf{L}_N \mathcal{I}_N \otimes I_m)X_N(t). \end{aligned}$$

Using the decomposition of $C_d x(t-h)$, for all $t \in \mathbb{R}^+$,

$$C_d x(t-h) = (\mathbf{1}_N^{*T} \mathcal{I}_N \otimes I_m)X_N(t) + \epsilon_N(t),$$

it gives

$$\begin{cases} \dot{X}_N(t) = (\mathbf{1}_N \otimes I_m)C_d x(t) - (\mathbf{1}_N^* \otimes I_m)\epsilon_N(t) \\ \quad - ((\mathbf{L}_N + \mathbf{1}_N^* \mathbf{1}_N^{*T})\mathcal{I}_N \otimes I_m)X_N(t), \\ \epsilon_N(t) = C_d x(t-h) - \mathcal{C}_N^* X_N(t). \end{cases}$$

The resulting non-autonomous dynamical system (5) is finally driven by two inputs, the current transported solution ($C_d x$) and the remainder of Legendre serie evaluated at $-h$ (ϵ_N). Notice that the proposed procedure is equivalent to decomposing the block $e^{-hs}I_m$ into a finite-dimensional system to which is added a structured disturbance ϵ_N .

2.5 Augmented time-delay system

Gathering the dynamics of x and X_N , one can construct an augmented TDS as described in this subsection. The new system of state x and X_N is build up an augmented finite-dimensional system which state error is related to the remainder ϵ_N . This remainder includes the infinite-dimensional part. To sum up, this new augmented system is an interconnection between a finite-dimensional and an infinite-dimensional model as it is proposed in Theorem 4 and represented by the block diagram on Fig. 1.

Theorem 4. The system (1) takes the following form

$$\begin{cases} \dot{\xi}_N(t) = \underbrace{\begin{bmatrix} A & B_d \mathcal{C}_N^* \\ \mathcal{B}_N C_d & \mathcal{A}_N \end{bmatrix}}_{\mathbb{A}_N} \xi_N(t) + \underbrace{\begin{bmatrix} B_d \\ -\mathcal{B}_N^* \end{bmatrix}}_{\mathbb{B}_N} \epsilon_N(t), \\ \epsilon_N(t) = [C_d \quad -\mathcal{C}_N^*] \begin{bmatrix} x(t-h) \\ X_N(t) \end{bmatrix} \end{cases}, \forall t \in \mathbb{R}^+, \quad (6)$$

with $\xi_N = [x^T \quad X_N^T]^T$ satisfying $\xi_N(0) = [x(0)^T \quad X_N^{0T}]^T$.

Proof. First, Proposition 3 can be rewritten as

$$\begin{cases} \dot{X}_N(t) = [\mathcal{B}_N C_d \quad \mathcal{A}_N] \xi_N(t) - \mathcal{B}_N^* \epsilon_N(t), \\ \epsilon_N(t) = C_d x(t-h) - \mathcal{C}_N^* X_N(t). \end{cases}$$

Then, equation (1) completes the dynamics. Using the previous equation, we have

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_d C_d x(t-h) \\ &= [A \quad B_d \mathcal{C}_N^*] \xi_N(t) + B_d \epsilon_N(t). \end{aligned}$$

Since, intuitively the additional error ϵ_N is expected to become small enough increasing the size N , the finite-dimension part can be investigated as an approximation of the TDS, which is the aim of the next section.

3. STABILITY ANALYSIS OF THE APPROXIMATE FINITE-DIMENSIONAL MODEL

3.1 Approximation by a finite-dimensional model

This part is dedicated to the stability analysis of the finite-dimensional system getting rid of the effect of the error

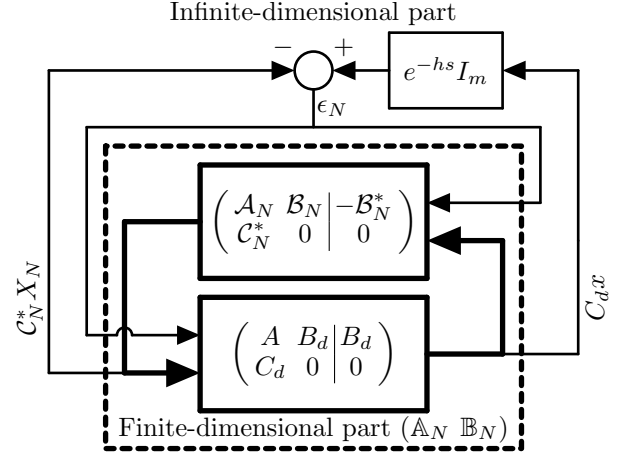


Fig. 1. Block diagram of augmented time-delay system (6).

ϵ_N , which is expected to be small when N is sufficiently large. In that case, the resulting system corresponds to the finite-dimensional part of Fig. 1.

The dynamical approximate model can be written as :

$$\dot{\hat{\xi}}_N(t) = \mathbb{A}_N \hat{\xi}_N(t), \forall t \in \mathbb{R}^+, \quad (7)$$

with $\hat{\xi}_N = [\hat{x}^T \quad \hat{X}_N^T]^T$ satisfying $\hat{\xi}_N(0) = \xi_N(0)$.

This model can then bring information on the locus of the eigenvalues and be used for the stability analysis of TDSs.

3.2 Link with the Padé approximant model

The aim of this subpart is to prove that system (7) described by the finite-dimensional part on Fig. 1 can also be interpreted as an approximation of the original TDS, where the time-delay element e^{-hs} has been replaced by its Padé approximant which transfer function is $H_N(s)$.

Proposition 5. For each $N \in \mathbb{N}$, the state representation $\begin{pmatrix} \mathcal{A}_N & \mathcal{B}_N \\ \mathcal{C}_N^* & 0 \end{pmatrix}$ is a realisation of $H_N = \frac{n_N(s)}{d_N(s)} I_m$, where

$$\begin{cases} n_N(s) = \sum_{j=0}^N \frac{N!(2N+1-j)!}{(N-j)!(2N+1)!} \frac{(-hs)^j}{j!}, \\ d_N(s) = \sum_{i=0}^{N+1} (-1)^i \frac{(N+1)!(2N+1-i)!}{(N+1-i)!(2N+1)!} \frac{(-hs)^i}{i!}, \end{cases} \quad (8)$$

are respectively the numerator and denominator of Padé approximant $(N, N+1)$ of the function e^{-hs} .

Proof. Consider, G_N the transfer function of the state space representation $\begin{pmatrix} \mathcal{A}_N & \mathcal{B}_N \\ \mathcal{C}_N^* & 0 \end{pmatrix}$. The objective is to show that $G_N = H_N$, for any value of N . Let first note that

$$G_N(s) = \mathcal{C}_N^* (sI_{m(N+1)} - \mathcal{A}_N)^{-1} \mathcal{B}_N,$$

Hence, in order to prove this result, one needs to show that each numerator and denominator of G_N are equal to $2^N n_N$ and $2^N d_N$ respectively. For any $s \in \mathbb{C}$, this means

$$\begin{cases} \mathbf{1}_N^{*T} \text{adj}(s\mathcal{I}_N^{-1} - \mathcal{L}_N) \mathbf{1}_N = 2^N n_N(s), \\ \det(s\mathcal{I}_N^{-1} - \mathcal{L}_N) = 2^N d_N(s), \end{cases} \forall N \in \mathbb{N}.$$

This result is obtained recursively. The complete proof is given in Bajodek et al. [2020], but the initialization part is provided here to highlight the main features of this proof. For $N \in \{0, 1\}$, we easily find that

$$\begin{cases} \mathbf{1}_0^{*T} \text{adj}(s\mathcal{I}_0^{-1} - \mathcal{L}_0)\mathbf{1}_0 = 1 = n_0(s), \\ \det(s\mathcal{I}_0^{-1} - \mathcal{L}_0) = sh + 1 = d_0(s), \\ \mathbf{1}_1^{*T} \text{adj}(s\mathcal{I}_1^{-1} - \mathcal{L}_1)\mathbf{1}_1 = 2\left(1 - \frac{sh}{3}\right) = 2n_1(s), \\ \det(s\mathcal{I}_1^{-1} - \mathcal{L}_1) = 2\left(1 + \frac{2sh}{3} + \frac{(sh)^2}{6}\right) = 2d_1(s). \end{cases}$$

Then, to give an idea of the induction, let express the result at the order $N = 2$ relying on the two previous ones.

To begin with, we know that

$$\begin{cases} \mathbf{1}_2^{*T} \text{adj}(s\mathcal{I}_2^{-1} - \mathcal{L}_2)\mathbf{1}_2 = \mathbf{1}_2^{*T} E_0 \text{adj}(F_0(s\mathcal{I}_2^{-1} - \mathcal{L}_2)E_0)F_0\mathbf{1}_2, \\ \det(s\mathcal{I}_2^{-1} - \mathcal{L}_2) = \det(F_0(s\mathcal{I}_2^{-1} - \mathcal{L}_2)E_0), \end{cases}$$

where E_0 and F_0 are nonsingular matrices given by

$$E_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Indeed, we have

$$\begin{cases} \mathbf{1}_2^{*T} \text{adj}(s\mathcal{I}_2^{-1} - \mathcal{L}_2)\mathbf{1}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}^T \text{adj} \begin{bmatrix} sh+1 & -1 & 0 \\ 1 & \frac{sh}{3} & \frac{sh}{3} \\ 0 & -\frac{sh}{3} & 2(1-\frac{sh}{15}) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \\ \det(s\mathcal{I}_2^{-1} - \mathcal{L}_2) = \det \begin{bmatrix} sh+1 & -1 & 0 \\ 1 & \frac{sh}{3} & \frac{sh}{3} \\ 0 & -\frac{sh}{3} & 2(1-\frac{sh}{15}) \end{bmatrix}. \end{cases}$$

From one side, we note that

$$\begin{aligned} \mathbf{1}_2^{*T} \text{adj}(s\mathcal{I}_2^{-1} - \mathcal{L}_2)\mathbf{1}_2 &= 4\left(\left(1 - \frac{sh}{15}\right)n_1(s) + \left(\frac{sh}{6}\right)^2 n_0(s)\right), \\ &= 4\left(1 - \frac{2sh}{5} + \frac{(sh)^2}{20}\right) = 4n_2(s). \end{aligned}$$

From the other side, we obtain

$$\begin{aligned} \det(s\mathcal{I}_2^{-1} - \mathcal{L}_2) &= 4\left(\left(1 - \frac{sh}{15}\right)d_1(s) + \left(\frac{sh}{6}\right)^2 d_0(s)\right), \\ &= 4\left(1 + \frac{3sh}{5} + \frac{3(sh)^2}{20} + \frac{(sh)^3}{60}\right) = 4d_2(s), \end{aligned}$$

which completes the proof for $N = 2$. The proof for any $N \geq 3$ follows the same process relied on three-term Pade identity and can be found in Bajodek et al. [2020].

From this result given by induction, we obtain the following transfer function

$$G_N(s) = \frac{2^N n_N(s)}{2^N d_N(s)} I_m = \frac{n_N(s)}{d_N(s)} I_m = H_N(s), \quad \forall N \in \mathbb{N}.$$

The previous calculations and statement allow us to state the main result of this paper.

Theorem 6. Approximate model (7) is a Padé approximant of time-delay system (1).

Proof. Identifying the transfer function H_N given in Proposition 5, one can recognise a Padé approximant of the exponential function e^{-hs} repeated m times. That directly gives Theorem 6.

Then, the uniform convergence result on open ball of the Padé approximant towards the exponential function e^{-hs} could be used on our finite-dimensional model.

3.3 Convergence of the characteristic roots of the model towards some of the time-delay system

The finite-dimensional model studied is equivalent to a Padé approximant. Hence, the convergence results issued

from Padé approximant theory (see Baker [1975]) can be used to link the characteristic roots of the TDS (1) and the eigenvalues of \mathbb{A}_N , state matrix of approximate model (7). More precisely, one proposes Theorem 8. But, before, a first technical lemma is recalled, showing that, on a compact set, the Padé approximant converges to the delay transfer function e^{-hs} .

Lemma 7. Let $R > 0$. On a compact set $\mathcal{B}(0, R)$, $n_N(s)$ and $d_N(s)$ uniformly converge when $N \rightarrow \infty$ towards $n(s) = e^{-\frac{hs}{2}}$ and $d(s) = e^{-\frac{hs}{2}}$ respectively. In other words,

$$\forall \epsilon > 0 \exists N^*; \forall N \geq N^*, \forall s \in \mathcal{B}(0, R), \begin{cases} |n_N(s) - n(s)| \leq \epsilon \\ |d_N(s) - d(s)| \leq \epsilon \end{cases}$$

Proof. The proof of this convergence result is given in Baker [1975].

For all $s \in \mathbb{C}$, let matrices $\Delta_N(s)$ and $\Delta(s)$ in $\mathbb{R}^{n \times n}$ be

$$\begin{cases} \Delta_N(s) = (sI_n - A)d_N(s) - A_d n_N(s), \\ \Delta(s) = (sI_n - A)d(s) - A_d n(s), \end{cases}$$

with $A_d = B_d C_d$, $d(s) = e^{-\frac{hs}{2}}$ and $n(s) = e^{-\frac{hs}{2}}$.

Now, the aim is to prove that, for N sufficiently large, the characteristic roots of model (7), i.e. zeros of $\chi_N(s) = \det(\Delta_N(s))$, are close enough to some of those of the TDS (1), i.e. zeros of $\chi(s) = \det(\Delta(s))$.

Theorem 8. For all $R > 0$, if the time-delay system (1) contains K characteristic roots with multiplicities $\nu_{k \in [1, K]}^*$

into the open ball $\mathcal{B}(0, R)$, then $\sum_{k=1}^K \nu_k^*$ eigenvalues of \mathbb{A}_N converges towards them. More precisely,

$$\forall r \in (0, r^*), \exists N^*; \forall N \geq N^*, \max_{\substack{k \in [1, K] \\ i \in [1, \nu_k^*]}} |s_{k,i}^N - s_k^*| \leq r. \quad (9)$$

Proof. Starting from the uniform convergence of $\Delta_N(s)$ towards $\Delta(s)$ obtained by application of Lemma 7, the proof follows the one provided by Breda et al. [2015] in the case of the uniform convergence of the eigenvalues given by the pseudospectral differentiation method towards the characteristic roots directly. Calculations can be found in the complete version (see Bajodek et al. [2020]).

Thus, increasing N , some eigenvalues of \mathbb{A}_N can approximate as close as desired the roots of system (1). Especially, for unstable TDSs, one can find a value N^* such that matrix \mathbb{A}_N has at least one eigenvalue with positive real parts for each $N \geq N^*$. From these promising properties of this finite-dimensional model, a Lyapunov-Krasovskii stability analysis is proposed going back on the interconnected system (6) to take in account the infinite-dimensional part which have been neglected in this section.

4. STABILITY ANALYSIS OF THE INTERCONNECTED SYSTEM

The aim of this part is to analyse the stability of the whole system depicted in Fig. 1. One proposes to design an LKF, highly related to system (6).

4.1 A Lyapunov-Krasovskii functional

To be consistent with augmented TDS (6), let define the LKF enriched by Legendre coefficients,

$$V_N(x(t), x_t) = V_{\mathbb{P}_N}(\xi_N(t)) + V_S(x_t) + V_R(x_t), \quad (10)$$

with

$$\begin{cases} V_{\mathbb{P}_N}(\xi_N(t)) = \xi_N^T(t) \mathbb{P}_N \xi_N(t), \\ V_S(x_t) = \int_{-h}^0 (C_d x_t(\tau))^T S (C_d x_t(\tau)) d\tau - X_N^T(t) \mathcal{S}_N X_N(t), \\ V_R(x_t) = \int_{-h}^0 (h + \tau) (C_d x_t(\tau))^T R (C_d x_t(\tau)) d\tau. \end{cases}$$

Matrices $\mathbb{P}_N \in \mathbb{S}^{n+m(N+1)}$ and $S, R \in \mathbb{S}^m$ are assumed to be symmetric positive definite and \mathcal{S}_N stands for $\mathcal{I}_N \otimes S$.

4.2 Bessel-Legendre inequality

Bessel inequality, applied to $C_d x_t$ and its $N + 1$ first components X_N on Legendre polynomial basis, is a tool allowing to bound the integral terms which appear in V_S or in the derivative of V_R .

Lemma 9. For any positive definite matrix $M \in \mathbb{S}^m$, Bessel-Legendre inequality provides

$$\int_{-h}^0 (C_d x_t(\tau))^T M (C_d x_t(\tau)) d\tau \geq X_N^T(t) (\mathcal{I}_N \otimes M) X_N(t).$$

This inequality leads to the following stability condition.

4.3 Sufficient condition of stability

The LKF defined previously combined with Lemma 9 provides Theorem 10, a rewrite of the LMI condition given by Seuret and Gouaisbaut [2015].

Theorem 10. If it exists symmetric positive definite matrices $\mathbb{P}_N > 0$, $S > 0$ and $R > 0$ such that

$$\begin{bmatrix} \mathcal{H}(\mathbb{P}_N \mathbb{A}_N) + \mathbb{C}_N^T S \mathbb{C}_N + \begin{bmatrix} h C_d^T R C_d & 0 \\ * & -\mathcal{R}_N \end{bmatrix} & \mathbb{P}_N \mathbb{B}_N \\ * & -S \end{bmatrix} < 0, \quad (11)$$

where $\mathcal{R}_N = \mathcal{I}_N \otimes R$ and with

$$\mathbb{A}_N = \begin{bmatrix} A & B_d \mathbb{C}_N^* \\ \mathcal{B}_N C_d & \mathcal{A}_N \end{bmatrix}, \quad \mathbb{B}_N = \begin{bmatrix} B_d \\ -\mathcal{B}_N^* \end{bmatrix}, \quad \mathbb{C}_N = [C_d \quad -C_N],$$

then system (1) is asymptotically stable, for the delay h .

Proof. Consider the LKF candidate V_N given by (10). Indeed, the positivity of V_N is ensured by the positive definiteness of \mathbb{P}_N , S and R . More precisely, we show that V_S is positive by application of Lemma 9.

Then, the derivative of V_N along the trajectories of system (6) is composed of the derivative of the finite-dimensional part,

$$\dot{V}_{\mathbb{P}_N}(\xi) = \xi_N^T \mathcal{H}(\mathbb{P}_N \mathbb{A}_N) \xi_N + \xi_N^T \mathbb{P}_N \mathbb{B}_N \epsilon_N + \epsilon_N^T \mathbb{B}_N^T \mathbb{P}_N \xi_N,$$

to which is added the derivative of V_S ,

$$\dot{V}_S(x_t) = (C_d x(t))^T S (C_d x(t)) - (C_d x(t-h))^T S (C_d x(t-h)) - 2X_N^T(t) \mathcal{S}_N (\mathbf{L}_N \mathcal{I}_N X_N(t) + \mathbf{1}_N C_d x(t) - \mathbf{1}_N^* C_d x(t-h)),$$

which, reorganising the terms, is equal to

$$\begin{aligned} \dot{V}_S(x_t) &= (C_d x(t) - C_N X_N(t))^T S (C_d x(t) - C_N X_N(t)) \\ &\quad - (C_d x(t-h) - C_N^* X_N(t))^T S (C_d x(t-h) - C_N^* X_N(t)), \\ &= \xi_N^T(t) \mathbb{C}_N^T S \mathbb{C}_N \xi_N(t) - \epsilon_N^T(t) S \epsilon_N(t), \end{aligned}$$

and, lastly, the derivative of V_R ,

$$\begin{aligned} \dot{V}_R(x_t) &= h (C_d x(t))^T R (C_d x(t)) \\ &\quad - \int_{-h}^0 (C_d x_t(\tau))^T R (C_d x_t(\tau)) d\tau. \end{aligned}$$

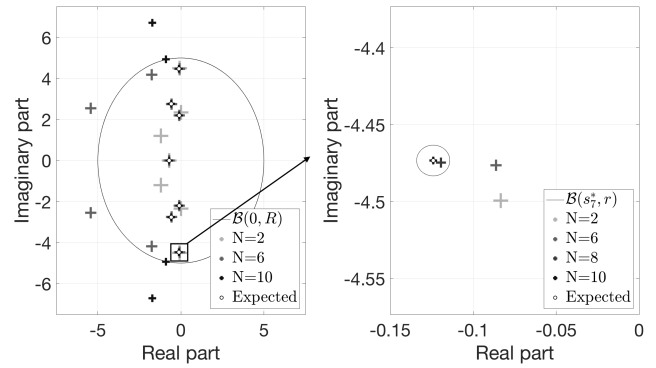


Fig. 2. Example 11 for $h = 3$: Convergence of the eigenvalues with $(R, r) = (5, 10^{-2})$.

Table 1. Example 11 : Eigenvalues for $h = 3$.

Method	$N = 2$	$N = 6$
Legendre	-0.6955	-0.7026
Theorem 8	$+0.0058 \pm 2.3377j$	$-0.1007 \pm 2.1920j$
	$-1.1997 \pm 1.2033j$	$-0.5656 \pm 2.7489j$
	$-0.0834 \pm 4.4994j$	$-0.0863 \pm 4.4764j$
Collocation Breda et al. [2005]	-0.6542	-0.7026
	$-0.0249 \pm 2.3553j$	$-0.1007 \pm 2.1938j$
	$-0.5895 \pm 0.9612j$	$-0.6155 \pm 2.7404j$
Legendre-Tau Ito and Teglás [1986]	-0.7018	-0.7026
	$+0.0185 \pm 2.2573j$	$-0.1007 \pm 2.1919j$
	$-1.2259 \pm 1.1696j$	$-0.5712 \pm 2.7535j$
	$-0.1210 \pm 4.5012j$	$-0.0987 \pm 4.4641j$

Putting all the terms together, according to Lemma 9,

$$\begin{aligned} \dot{V}_N(x(t), x_t) &\leq \xi_N^T \mathcal{H}(\mathbb{P}_N \mathbb{A}_N) \xi_N + \begin{bmatrix} \xi_N \\ \epsilon_N \end{bmatrix}^T \begin{bmatrix} \mathbb{C}_N^T S \mathbb{C}_N & \mathbb{P}_N \mathbb{B}_N \\ * & -S \end{bmatrix} \begin{bmatrix} \xi_N \\ \epsilon_N \end{bmatrix} \\ &\quad + \xi_N^T \begin{bmatrix} h C_d^T R C_d & 0 \\ * & -\mathcal{R}_N \end{bmatrix} \xi_N. \end{aligned}$$

Therefore, if the LMI (11) is satisfied, system (1) is asymptotically stable by application of the Lyapunov-Krasovskii theorem.

As the eigenvalues of \mathbb{A}_N approximate the characteristic roots of the original TDS, one expects that the stability condition proposed in Theorem 10 can approximate the entire stability chart with respect to h .

5. EXAMPLES

An example was studied to illustrate our results.

Example 11. $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -11 & 10 & 0 & 0 \\ 5 & -15 & 0 & -\frac{1}{4} \end{bmatrix}$, $B_d = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $C_d = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$.

5.1 Analysis of the eigenvalues

For a given value h , the eigenvalues of \mathbb{A}_N are depicted in Figure 2 for Example 11, where there are materialised, increasing N , by increasingly dark and small crosses. Theorem 8 ensures the convergence of some of them towards the expected ones contained in a ball $\mathcal{B}(0, R)$ with $R = 5$. These first expected eigenvalues $s^* \in \{-0.7026; -0.1007 \pm 2.1919j; -0.5712 \pm 2.7559j; -0.1241 \pm 4.4733j\}$ were calculated with a precision 10^{-4} following Breda et al. [2005] and are materialised by white points on Figure 2.

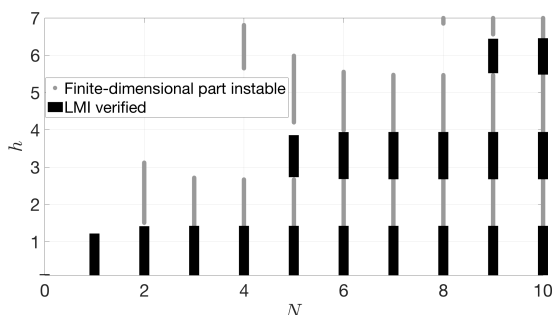


Fig. 3. Example 11 : Instability of \mathbb{A}_N versus Lyapunov-Krasovskii stability.

The convergence of some of the eigenvalues is confirmed by zooming on expected characteristic roots s^* contained in $\mathcal{B}(0, R)$ and finding a value $N^* = 8$ from which the computed ones are inside a ball $\mathcal{B}(s^*, r)$ with $r = 10^{-2}$.

To see how fast the proposed computation is converging, a comparison with collocation (pseudospectral discretization given by Breda et al. [2005] here) and Tau (Ito and Teglás [1986] on Legendre basis too) methods are performed. The closest calculated eigenvalues of those expected are given in Table 1 for Example 11.

From this table, one can conclude that the proposed approximate model seems to give a better approximation than collocation technique. Even though the Tau method seems to converge faster, our method has the advantage to bring, in addition, sufficient stability results.

5.2 Lyapunov-Krasovskii stability analysis

The sufficient stability condition given by the LMI (11) can be easily implemented and ensures pointwise stability with respect to the delay. On Example 11, a numerical test was done with a precision of 10^{-3} and for $N \in \llbracket 0, 10 \rrbracket$. The first analytical bound of stability $h = 1.142$ is reached from $N = 3$. As expected, these numerical results are equivalent to those of Seuret and Gouaisbaut [2015].

On Figure 3, for Example 11, the intervals of stability with respect to the delay given by Theorem 10 are represented with thick dark lines and the instability of \mathbb{A}_N with respect to the delay with thin gray lines.

First, by increasing N , the set of instability of \mathbb{A}_N with respect to h converges as expected towards the entire set of instability of the original TDS. Likewise, the intervals of stability given by Theorem 10 appear to slightly grow until to fill in the set of stability of the TDS. Then, as aforementioned, LMI (11) based on the finite-dimensional model also seems to converge to the entire stability region with respect to h . Lastly, the intervals of stability of the LMI at order N and those of instability of \mathbb{A}_N are disjoint. In other words, the stability of \mathbb{A}_N could be a necessary condition for the LMI at order N .

6. CONCLUSIONS

This work proposes some new insights for the stability analysis of TDSs using the first projections on Legendre polynomials. Taking into account these coefficients and its dynamics, an interconnection scheme between a finite-dimensional part and an infinite-dimensional error part

was designed to model such systems. By getting rid of the error, the finite-dimensional system turns out to be a Padé approximant which eigenvalues converges therefore towards the expected characteristic roots. From the whole augmented system, a sufficient stability condition of TDSs expressed in terms of LMIs is also proposed. Thus, the new model proposed in this paper seems to be really useful to yield numerical accurate stability conditions. Therefore, keeping this same framework, future work focused on control and observation of TDSs can provide interesting new numerical solutions.

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