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Smaller and negative exponents in Lyapunov functions for interconnected iISS systems ⋆

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Abstract: For integral input-to-state stable (iISS) systems, stability of their interconnections can be established through a small-gain condition. Unlike input-to-state stable (ISS) systems, iISS systems admit gain functions only in limited ranges. Thus, composing a Lyapunov function which is valid globally is instrumental for addressing iISS. A Lyapunov function which is popular in the iISS framework proves the stability of interconnected systems whenever the small-gain condition is satisfied. However, it is hardly practical since its nonlinearities are often artificial and involve astonishingly large exponents. This paper drastically reduces the exponents analytically and numerically. This paper also extends the exponents to negative numbers, and demonstrates that two-sided exponents allow one to avoid unnecessary complicated Lyapunov functions.

Keywords: Nonlinear systems, Lyapunov methods; Integral input-to-state stability; Small gain theorem; Interconnected systems.

1. INTRODUCTION

This paper aims to directly and drastically improve a solution to a problem which is not merely technical but largely influential in analysis and design of various dynamical systems. Small-gain arguments form a major paradigm of building systems from components. The framework of input-to-state stability (ISS) offers the ISS small-gain theorem which effectively treats nonlinearities to which the L_p small-gain theorem cannot give meaningful answers (Sontag (1989); Jiang et al. (1994); Hill and Moylan (1977); van der Schaft (1999); Dashkovskiy et al. (2010); Karafyllis and Jiang (2012)). The broader notion of integral input-to-state stability (iISS) addresses saturation and bilinearity which are excluded by ISS (Sontag (1998); Arcak et al. (2002)). The target of this paper is the iISS small-gain theorem (Ito (2006); Angeli and Astolfi (2007)).

Unlike ISS systems, iISS systems admit nonlinear gain functions only in limited ranges. Thus, Lyapunov functions which are valid globally is basically the key to the small-gain argument for iISS systems (Ito and Jiang (2009)). Consider an interconnection of two systems. Suppose that V_i, $i = 1, 2$, are Lyapunov functions [1] of the two component systems when they are disconnected. A Lyapunov function verifying stability of the interconnection is

$$V_C(x) = \sum_{i=1}^{2} \int_0^{V_i(x_i)} \alpha_i(s)^\phi \sigma_{3-i}(s)^{\phi+1} ds, \qquad (1)$$

where $x = [x_1^T, x_2^T]^T$ is the state of the interconnected system, and x_i is the state of system i. The functions α_i and σ_i are the dissipation function and the supply function, respectively, of system i. The algebraic formula (1) is universal in the sense that it serves as a Lyapunov function whenever the dissipation and supply functions satisfy the iISS small-gain condition (Ito (2013)). Unfortunately, the formula (1) is often impractical. The existing theory requires the exponent $\phi \geq 0$ to be gigantically large, unless the margins of the small-gain condition are large (Dirr et al. (2015)). Other smooth Lyapunov functions proposed to cover iISS systems in the literature have essentially the same exponent (Ito (2006); Ito and Jiang (2009)).

The central information given by a Lyapunov function is its derivative or gradient representing behavior of the system. As $\phi \to \infty$, the function V_C in (1) becomes a discontinuous function which is almost flat near the origin, and suddenly soars vertically. Since this extreme shape hides the actual system behavior almost completely. aggregating such functions is useless in module-based strategies for systems design and analysis. When a Lyapunov-based controller is designed as in Freeman and Kokotović (1996) with a large exponent, the control signal hits actuator limitations immediately, and the bang-bang control loses theoretical guarantees. This motivated the development of new Lyapunov functions recently in Ito (2019c,b). However, they are not differentiable, which causes unnecessary discontinuities in analysis and design. Moreover, the not differentiable functions do not have the separation between V_1 and V_2 which (1) enjoys. The structure is referred to sum-separability in Dirr et al. (2015) [2]. The separability fits module-based approaches since, for instance, employing a sum-separable Lyapunov function in controller design results in a decentralized controller. In the absence of the separability, one needs to take multiple systems into account simultaneously. Algebraically, the formula (1) is simple enough. Only the gigantic exponent ϕ is problematic. This standpoint was initiated by the numerical approach in Ito and Kellett (2015).

When α_i and σ_i are monomial of the same order p for both $i = 1, 2$, the iISS small-gain theorem reduces to the L_p small-gain theorem (see, e.g., Hill and Moylan (1977)).

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[1] allowing inputs in view of iISS (Angeli et al. (2000)).

[2] Another popular type of separability is max-separability (Jiang et al. (1996)), but it cannot establish iISS small-gain theorem. See, e.g., Ito et al. (2012).

In the L_p case, it is known that linear combination of V_1 and V_2 is sufficient for composing a Lyapunov function. The formula (1) with $\phi \geq 0$ has nonlinearities disagreeing with the linear combination in the L_p case. It is natural that one does not want a Lyapunov function to involve unnecessary nonlinearities. Such nonlinearities not only distort the evaluation of system behavior, but also cause unnecessary dynamics in controller design.

To remove the drawbacks of the smooth Lyapunov function (1), first, this paper proposes an analytical formula yielding a much smaller exponent ϕ than the classical one. To achieve the significant reduction, a new technique replaces the iISS preservation which has been solely used to prove the iISS small-gain theorem. The small-gain condition is also reformulated. Second, a numerical method to reduce ϕ further is proposed. And third, the use of the exponent ϕ is extended to allow negative ϕ for removing artificial nonlinearities in the Lyapunov function.

Notation: This paper uses the symbol $\mathbb{R} = (-\infty, \infty)$. A function $\zeta : \mathbb{R}_+ := [0, \infty) \to \mathbb{R}_+$ is said to be of class $\mathcal{P}$ and written as $\zeta \in \mathcal{P}$ if ζ is continuous and satisfies $\zeta(0) = 0$ and $\zeta(s) > 0$ for all $s \in \mathbb{R}_+ \backslash \{0\}$. A function $\zeta \in \mathcal{P}$ is said to be of class $\mathcal{K}$ if ζ is strictly increasing. A class $\mathcal{K}$ function is said to be of class $\mathcal{K}_\infty$ if it is unbounded. For a continuous map $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$, the map $\zeta^\ominus$: $\overline{\mathbb{R}}_+ := [0, \infty] \to \overline{\mathbb{R}}_+$ is $\zeta^\ominus(s) = \sup\{v \in \mathbb{R}_+ : s \geq \zeta(v)\}$. Note that, given a function $\zeta \in \mathcal{K}$, $\zeta^\ominus(s) = \infty$ holds for all $s \geq \lim_{\tau\to\infty} \zeta(\tau)$, and $\zeta^\ominus(s) = \zeta^{-1}(s)$ elsewhere. For a continuous map $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\liminf_{l\to\infty} \zeta(l) = 0$, we have $\zeta^\ominus(s) = \infty$ for all $s \in \overline{\mathbb{R}}_+$. A non-decreasing map $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ is extended to a map $\overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ by $\zeta(s) := \sup_{\{r\in\mathbb{R}_+ : r\leq s\}} \zeta(r)$. For $p \in (0, \infty]$, $\zeta \in \mathcal{K}[0, p)$ if $\zeta : [0, p) \to \mathbb{R}_+$ is continuous, strictly increasing, and $\zeta(0) = 0$. A function $\zeta \in \mathcal{K}[0, p)$ is entitled to $\zeta \in \mathcal{K}_\infty[0, p)$ if $\lim_{s\to p^-} \zeta(s) = \infty$. Note that $\mathcal{K} = \mathcal{K}[0, \infty)$ and $\mathcal{K}_\infty = \mathcal{K}_\infty[0, \infty)$. For $u \in \mathbb{R}_+ \to \mathbb{R}^m$, we write $u = 0$ if $u(s) = 0$ for all $s \in \mathbb{R}_+$.

2. PROBLEM STATEMENT

Consider the interconnected system of the form

$$\dot{x} = f(x, u) := \begin{bmatrix} f_1(x_1, x_2, u_1) \\ f_2(x_1, x_2, u_2) \end{bmatrix}, \tag{2}$$

where $x(t) = [x_1(t)^T, x_2(t)^T] \in \mathbb{R}^N$, $u(t) = [u_1(t)^T, u_2(t)^T] \in \mathbb{R}^M$, $x_i(t) \in \mathbb{R}^{N_i}$ and $u_i(t) \in \mathbb{R}^{M_i}$ and $t \in \mathbb{R}_+$. We assume that the external signal $u : \mathbb{R}_+ \to \mathbb{R}^M$ is any measurable and locally essentially bonded function, which is denoted as $u \in \mathcal{U}$. Suppose that $f_i : \mathbb{R}^{N_1}\times\mathbb{R}^{N_2}\times\mathbb{R}^{M_i} \to \mathbb{R}^{N_i}$ is locally Lipschitz and satisfies $f_i(0, 0, 0) = 0$ for $i = 1, 2$. For $i = 1, 2$, instead of f_i, this paper uses

$$\frac{\partial V_i}{\partial x_i} f_i(x_1, x_2, u_i) \leq -\alpha_i(V_i(x_i)) + \sigma_i(V_{3-i}(x_{3-i})) + \kappa_i(|u_i|), \quad \forall x \in \mathbb{R}^N,\ u_i \in \mathbb{R}^{M_i}, \tag{3}$$

which is assumed to hold with some α_i, $\sigma_i \in \mathcal{K}$ and $\kappa_i \in \mathcal{K} \cup \{0\}$, where $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ is a continuously differentiable function which is positive definite and radially unbounded. System (2) is said to be 0-GAS if the equilibrium $x = 0$ is globally asymptotically stable for $u = 0$. Input-to-state stability (ISS) and integral input-to-state stability (iISS) are defined for system (2) in a standard way (Sontag (1989, 1998)). The objective of this paper is to construct a reasonable 0-GAS/iISS Lyapunov function $\hat{V}(x)$ defined as follows:

Definition 1. A continuously differentiable function $\hat{V}$: $\mathbb{R}^N \to \mathbb{R}_+$ is called an iISS Lyapunov function of system (2) if it is positive definite, radially unbounded and satisfies

$$\frac{\partial \hat{V}}{\partial x} f(x, u) \leq -\alpha(\hat{V}(x)) + \sigma(|u|),\ \forall x \in \mathbb{R}^N, u \in \mathbb{R}^N \tag{4}$$

for some $\alpha \in \mathcal{P}$ and $\sigma \in \mathcal{K}$. Furthermore, if

$$\lim_{s\to\infty} \alpha(s) = \infty \ \text{ or } \ \liminf_{s\to\infty} \alpha(s) \geq \lim_{s\to\infty} \sigma(s) \tag{5}$$

holds, then $\hat{V}$ is called an ISS Lyapunov function.

The existence of an iISS (resp., ISS) Lyapunov function guarantees iISS (resp., ISS) of system (2) as proved in Sontag and Wang (1995) and Angeli et al. (2000). The converse also holds true in the sense of the existence of the functions $\hat{V}$, α and σ. By definition, an iISS system is 0-GAS. The assumption (3) implies that each x_i-system is iISS with respect to the inputs x_{3-i} and u_i jointly.

The iISS small-gain theorem guarantees that system (2) is iISS if there exist c_1, $c_2 > 1$ such that

$$\alpha_1^\ominus \circ c_1\sigma_1 \circ \alpha_2^\ominus \circ c_2\sigma_2(s) \leq s, \quad \forall s \in \mathbb{R}_+ \tag{6}$$

is satisfied (Ito (2006); Ito and Jiang (2009)). The property (6) posed with c_1, $c_2 > 1$ is called the (iISS) small-gain condition. Under the small-gain condition, the iISS is always established by (1). As demonstrated in Ito and Kellett (2016) comprehensively, the function $V_C(x)$ is an iISS Lyapunov function of system (2) for any $\phi \geq 0$ admitting the existence of τ such that

$$\tau - 1 > \left(\frac{\tau}{c_i}\right)^{\phi+1}, \quad c_i \geq \tau > 1, \quad i = 1, 2. \tag{7}$$

Clearly, the existence of such τ and ϕ is guaranteed. An estimate of ϕ by restricting τ to a choice is shown in Dirr et al. (2015), although the estimate is far larger than the infimum of ϕ among admissible τ in (7). This paper aims to replace (7) for allowing a smaller exponent ϕ in (1).

3. REDUCTION OF THE EXPONENT

The constant c_i in (6) describes the gain margin allotted to x_i-system. More precisely, c_i is the supremum of gain perturbation which x_i-system is allowed to have in guaranteeing the stability of system (2). The case of $c_1 = c_2 = 1$ is zero gain margins. It was shown by Ito and Jiang (2009) and Ito (2012) that (6) with $c_1 = c_2 = 1$ is a necessary condition for guaranteeing 0-GAS for all systems satisfying (3). The first idea this paper employs for replacing (7) is to lump the two margins c_1 and c_2 into a single parameter.

Proposition 2. The following three are equivalent:
(i) There exist $c_1 > 1$ and $c_2 > 1$ such that (6) holds.
(ii) There exist $\hat{c} > 1$ and $\hat{\tau} > 1$ such that

$$\alpha_i^\ominus \hat{c}\sigma_i \circ \alpha_{3-i}^\ominus \circ \hat{\tau}\sigma_{3-i}(s) \leq s, \quad \forall s \in \mathbb{R}_+,\ i = 1, 2. \tag{8}$$

(iii) There exists $\overline{c} > 1$ such that

$$\alpha_i^\ominus \overline{c}\sigma_i \circ \alpha_{3-i}^\ominus \circ \sigma_{3-i}(s) \leq s, \quad \forall s \in \mathbb{R}_+,\ i = 1, 2. \tag{9}$$

The next idea is to avoid the technique called the preservation of iISS. As shown in Ito and Kellett (2016), the establishment of the iISS small-gain theorem through Lyapunov functions has been relying on the process of preserving iISS of component systems. The process is applied to each x_i-system independently to transform its dissipation inequality (3) into another dissipation inequality before connecting the two component systems. Notice that (1) is the nonlinear scaled sum of V_is in the form of

$$V_C(x) = \sum_{i=1}^{2} W_i(V_i(x_i)) = \sum_{i=1}^{2} \int_0^{V_i(x_i)} \lambda_i(s)ds \tag{10}$$

$$\lambda_i(s) = \alpha_i(s)^{\phi}\sigma_{3-i}(s)^{\phi+1}. \tag{11}$$

For instance, to assess 0-GAS, property (3) gives the upper bound of the derivative of the scaled function $W_i(V_i)$ as

$$\frac{\partial W_i(V_i)}{\partial x_i} f_i \le -\lambda_i(V_i)\alpha_i(V_i) + \lambda_i(V_i)\sigma_i(V_{3-i}). \tag{12}$$

The process of preservation of iISS decouples $\lambda_i(V_i)$ from $\lambda_i(V_i)\sigma_i(V_{3-i})$ to obtain a dissipation inequality of x_i-system by looking for $\hat{\alpha}_i$, $\hat{\sigma}_i \in \mathcal{K}$ satisfying

$$-\lambda_i(V_i)\alpha_i(V_i) + \lambda_i(V_i)\sigma_i(V_{3-i}) \le -\hat{\alpha}_i(V_i) + \hat{\sigma}_i(V_{3-i}), \quad \forall [V_i, V_{3-i}]^T \in \mathbb{R}_+^2 \tag{13}$$

This paper pays attention to conservativeness arising in this independently-applied process since it ignores possible complementarity between component systems. In order to exploit the small-gain condition which is a cooperative property between component systems, this paper proposes to combine the two systems without the preprocessing (13), and divide the state space of the connected system into three regions so that the complementarity is exploited effectively in each region separately. This forms the basic idea of the proof to come at the following main theorem.

Theorem 3. Suppose that there exists $\overline{c} > 1$ such that (9) holds. Let $\phi \in \mathbb{R}_+$ be such that

$$(\overline{c} - 1)\overline{c}^{\phi} > 1, \quad i = 1, 2 \tag{14}$$

is satisfied. Then the function $V_C(x)$ in (1) is a 0-GAS Lyapunov function of system (2). Moreover, the function $V_C(x)$ is an iISS Lyapunov function of system (2) if

$$\left\{ \lim_{s\to\infty} \alpha_i(s) < \infty \Rightarrow \lim_{s\to\infty} \kappa_i(1)\sigma_{3-i}(s) < \infty \right\}, i = 1, 2. \tag{15}$$

Proof: Due to (9), (14) and Proposition 2, there exist $\hat{c} \in (1, \overline{c})$ and $\hat{\tau} > 1$ such that (8) and

$$(\hat{c} - 1)\hat{c}^{\phi} > 1, \quad i = 1, 2 \tag{16}$$

hold. Property (16) with $\phi \in \mathbb{R}_+$ implies the existence of $\tau \in (1, \min\{\hat{\tau}, \hat{c}\}]$ satisfying

$$\hat{c} - 1 > \left(\frac{\tau}{\hat{c}}\right)^{\phi}. \tag{17}$$

This inequality ensures $\epsilon := 1 - 1/\hat{c} - \tau^{\phi}/\hat{c}^{\phi+1} > 0$. Let $V = [V_1, V_2]^T$. Define

$$\Omega := \left\{ V \in \mathbb{R}_+^2 : \alpha_j(V_j) > \tau\sigma_j(V_{3-j}),\ j = 1, 2 \right\} \tag{18}$$

$$A_i := \left\{ z \in \mathbb{R}_+^2 : \alpha_{3-i}(V_{3-i}) \le \tau\sigma_{3-i}(V_i) \right\} \tag{19}$$

for $i = 1, 2$. By virtue of (8) and $\tau \in (1, \hat{\tau}]$, we have

$$\hat{c}\sigma_i(V_{3-i}) \le \alpha_i(V_i), \quad \forall V \in A_i \tag{20}$$

for $i = 1, 2$. Property (8) with $\tau \in (1, \min\{\hat{\tau}, \hat{c}\}]$ also gives $\mathbb{R}_+^2 = A_1 \cup \Omega \cup A_2$. Let λ_i be given as in (11). We have

$$\begin{aligned} &\lambda_i(V_i)\{-\alpha_i(V_i) + \sigma_i(V_{3-i})\} \\ &\le -\alpha_i(V_i)^{\phi+1}\sigma_{3-i}(V_i)^{\phi+1} + \alpha_i(V_i)^{\phi}\sigma_{3-i}(V_i)^{\phi+1}\frac{1}{\hat{c}}\alpha_i(V_i). \end{aligned} \tag{21}$$

for all $V \in A_i$, due to (20). For all $V \in A_i$, we also obtain

$$\begin{aligned} &\lambda_{3-i}(V_{3-i})\{-\alpha_{3-i}(V_{3-i}) + \sigma_{3-i}(V_i)\} \\ &\le -\alpha_{3-i}(V_{3-i})^{\phi+1}\sigma_i(V_{3-i})^{\phi+1} \\ &\quad + [\tau\sigma_{3-i}(V_i)]^{\phi}[\sigma_i \circ \alpha_{3-i}^{\ominus} \circ \tau\sigma_{3-i}(V_i)]^{\phi+1}\sigma_{3-i}(V_i) \\ &\le -\alpha_{3-i}(V_{3-i})^{\phi+1}\sigma_i(V_{3-i})^{\phi+1} \\ &\quad + \frac{\tau^{\phi}}{\hat{c}^{\phi+1}}\alpha_i(V_i)^{\phi+1}\sigma_{3-i}(V_i)^{\phi+1} \end{aligned} \tag{22}$$

for all $V \in A_i$ from (8). Thus for each $i = 1, 2$, combining (21) and (22) yields

$$\sum_{i=1}^{2} \lambda_i(V_i)\{-\alpha_i(V_i) + \sigma_i(V_{3-i})\} \le -\epsilon \sum_{i=1}^{2} \alpha_i(V_i)^{\phi+1}\sigma_{3-i}(V_i)^{\phi+1}. \tag{23}$$

if $V \in A_i$. In the case of $V \in \Omega$, the definition of Ω gives

$$\lambda_i(V_i)\{-\alpha_i(V_i) + \sigma_i(V_{3-i})\} \le -\left(1 - \frac{1}{\tau}\right)\lambda_i(V_i)\alpha_i(V_i). \tag{24}$$

Hence, using $\delta = \min\{\epsilon, 1 - 1/\tau\}$, we arrive at

$$\frac{\partial V_C}{\partial x} f(x, 0) \le -\delta \sum_{i=1}^{2} \alpha_i(V_i)^{\phi+1}\sigma_{3-i}(V_i)^{\phi+1} \tag{25}$$

for all $V \in \mathbb{R}_+^2$ with respect to (1) and (3), In the case of $u \neq 0$, the non-decreasing function λ_i given in (11) satisfy

$$\lambda_i(V_i)\kappa_i(|u_i|) \le \begin{cases} \hat{\epsilon}\alpha_i(V_i)^{\phi+1}\sigma_{3-i}(V_i)^{\phi+1} & \text{if } \hat{\epsilon}\alpha_i(V_i) > \kappa_i(V_{3-i}) \\ \left[\alpha_i \circ \alpha_i^{\ominus} \circ \frac{1}{\hat{\epsilon}}\kappa_i(|u_i|)\right]^{\phi} \left[\sigma_{3-i} \circ \alpha_i^{\ominus} \circ \frac{1}{\hat{\epsilon}}\kappa_i(|u_i|)\right]^{\phi+1} \cdot \kappa_i(|u_i|), & \text{otherwise} \end{cases}$$

for any $\hat{\epsilon} > 0$. Due to (25), assumption (15) guarantees

$$\frac{\partial V_C}{\partial x} f(x, u) \le -\alpha_C(V_C(x)) + \sigma_C(|u|),$$

for $\hat{\epsilon} \in (0, \delta)$, with some $\alpha_C \in \mathcal{K}$ and $\sigma_C \in \mathcal{K} \cup \{0\}$. □

The above proof shows how to avoid the process (13) of preservation of iISS by dividing the space $\mathbb{R}_+^2$ of the interconnection $[V_1, V_2]$ into the three regions as $\mathbb{R}_+^2 = A_1 \cup \Omega \cup A_2$, and exploiting the cooperative property the small-gain condition offers in each region separately as (21)-(24).

Due to this cooperative division technique (21)-(24), Theorem 3 demonstrates that replacing the classical small-gain condition (6) with the equivalent expression (9) allows one to explicitly determine the exponent ϕ in the Lyapunov function V_C with (14). Due to α_i and $\sigma_i \in \mathcal{K}$, the left hand side of (9) is an increasing continuous function of $\overline{c}$ as long as it is finite. For each $\overline{c} > 1$, the left side of (14) is an increasing continuous function of $\phi \in \mathbb{R}_+$. For each $\phi \in \mathbb{R}_+$, the left side is also increasing in $\overline{c} > 1$. Hence, the larger $\overline{c}$ we pick in (9), the smaller ϕ the condition (14) can accept. Clearly, the choice $\hat{c} = \min\{c_1, c_2\}$ for c_1, $c_2 > 1$ satisfying (6) achieves (8). Thus, property (9) holds for $\overline{c} = \min\{c_1, c_2\}$. Nevertheless, compared with this choice $\overline{c} = \min\{c_1, c_2\}$, the direct use of $\overline{c}$ solving (9) reduces the exponent ϕ significantly in (14). Indeed, (9) is guaranteed to be satisfied with $\overline{c} > \min\{c_1, c_2\}$. For example, if a constant k_i fulfills $k_i\alpha_{3-i}(s) = \sigma_i(s)$ for some $i = 1, 2$, the choice $\overline{c} = c_1c_2$ achieves (9). Thus, the infimum of ϕ achieving (14) decreases by more than half.

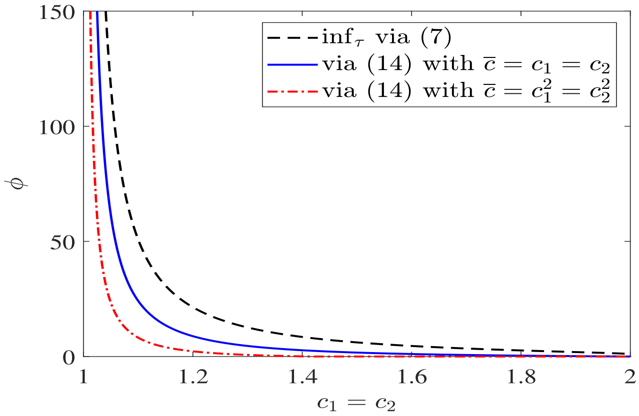

Fig. 1. Drastic reduction of the exponent ϕ in the Lyapunov function $V_C(x)$ with (14) from (7).

As recapitulated in Ito and Kellett (2016), the classical iISS small-gain theorem uses (7) to construct the Lyapunov function V_C in (1) instead of using (14). The classical condition (7) not only involves the intermediate variable τ which does not appear in V_C, but also is far restrictive than (14). Indeed, property (7) implies

$$c_i - 1 \geq \tau - 1 > \left(\frac{\tau}{c_i}\right)^{\phi+1} > \frac{\tau^{\phi+1}}{c_i} \cdot \frac{1}{\bar{c}^{\phi}}. \tag{26}$$

Recall that $\bar{c} > \min\{c_1, c_2\}$ is guaranteed in achieving (9). For any $\tau > 1$, we have

$$\lim_{\phi\to\infty} \frac{\tau^{\phi+1}}{c_i} = \infty. \tag{27}$$

Since $(c_i - 1)c_i^{\phi}$ is an increasing function of $c_i \in (1, \infty)$ properties (26) and (27) imply that the classical condition (7) produces much larger ϕ than (14) by the factor of infinity as the stability margins approach zero. The exponents ϕ computed via (14) with $\bar{c} = c_1 = c_2$ and $\bar{c} = c_1^2 = c_2^2$ are plotted in Fig.1. The exponent ϕ is also plotted for (7). For example, in the case of $c_1 = c_2 = 1.1$, the infimum of ϕ satisfying (7) over admissible τ is 50.81, while the infimum of ϕ satisfying (14) with $\bar{c} = c_1 = c_2$ is 24.16. The infimum computed with (14) for $\bar{c} = c_1^2 = c_2^2$ is 8.19. The vertical distance between the curves in Fig.1 shows tremendous reduction of ϕ as $\bar{c} \to 1+$.

4. NEGATIVE EXPONENTS

This section investigates the possibility and utility of using a negative exponent ϕ, which have not been explored in the literature (e.g. Ito (2006); Ito and Jiang (2009); Ito and Kellett (2016)). A motivation for this extension is to discover a direct relationship between the algebraic formula (1) and the popular L_p-gain analysis. The following achieves the extension to negative exponents.

Theorem 4. Suppose that there exist $\hat{c} > 1$ and $\hat{\tau} > 1$ such that (8) holds. Assume that there exist $\phi \in (-1, 0)$ and $\tau > 1$ satisfying

$$\hat{c} - 1 > \left(\frac{\tau}{\hat{c}}\right)^{\phi}, \quad \min\{\hat{\tau}, \hat{c}\} \geq \tau. \tag{28}$$

If λ_1 and λ_2 given in (11) are non-decreasing and satisfy

$$\lambda_i(s) = \lim_{v\to\infty} \lambda_i(v), \quad \forall s \in [b_i, \infty) \tag{29}$$

for $i = 1, 2$, where

$$b_i = \lim_{s\to\infty} \sigma_i^{\ominus} \circ \frac{1}{\tau}\alpha_i(s), \tag{30}$$

then the function $V_C(x)$ given in (1) is a 0-GAS Lyapunov function of system (2), Moreover, the function V_C is an iISS Lyapunov function of system (2) if (15) holds.

The selection of ϕ can be simplified by removing the involvement of $\hat{\tau}$ and τ in Theorem 4 at the cost of some conservativeness. The assumption (29) can be removed if x_i-system is ISS. The following demonstrates these facts.

Corollary 5. Suppose that there exist $\bar{c} > 1$ and $\phi \in (-1, 0)$ such that (9) and (14) are satisfied. If λ_1 and λ_2 given by (11) are non-decreasing and satisfy (29) with

$$b_i = \lim_{s\to\infty} \sigma_i^{\ominus} \circ \frac{1}{\bar{c}}\alpha_i(s) \tag{31}$$

for $i = 1, 2$, then the function $V_C(x)$ given in (1) is a 0-GAS Lyapunov function of system (2). Moreover, the function V_C is an iISS Lyapunov function of system (2) if (15) holds. Furthermore, if $i \in \{1, 2\}$ satisfies

$$\lim_{s\to\infty} \alpha_i(s) = \infty \text{ or } \lim_{s\to\infty} \alpha_i(s) > \lim_{v\to\infty} \sigma_i(s), \tag{32}$$

the assumption (29) is not required for that i.

For $\phi = -1/2$, (14) is satisfied if $\bar{c} > (3 + \sqrt{5})/2$. Hence, the following is a direct consequence of Corollary 5.

Corollary 6. Suppose that there exist $\ell_1, \ell_2 > 0$ such that

$$\sigma_{3-i}(s) \leq \ell_i \alpha_i(s), \quad \forall s \in \mathbb{R}_+, \ i = 1, 2 \tag{33}$$

$$\frac{3 + \sqrt{5}}{2}\ell_1\ell_2 < 1 \tag{34}$$

holds. Then the function $V_C(x)$ given by (10) and

$$\lambda_i(s) = \sqrt{\ell_i}, \quad i = 1, 2 \tag{35}$$

is an iISS Lyapunov function of system (2),

If the lumped expression (9) were not employed in Corollary 6, the inequality (34) could become as conservative as $4\ell_1\ell_2 < 1$. When the two component systems are L_p-gain finite, property (33) are achieved with equality signs. For systems satisfying the matching domination (33), it is not hard to prove the following for (1) without relying on (14).

Proposition 7. Suppose that there exist $\ell_1, \ell_2 > 0$ such that (33) and $\ell_1\ell_2 < 1$ hold. Then the function V_C given by (1) with $\phi = -1/2$ is an iISS Lyapunov function of system (2) if V_C radially unbounded.

The radially unboundedness of V_C defined with $\phi = -1/2$ can be satisfied whenever (33) holds. In fact, for each $i = 1, 2$, one can replace σ_i so that λ_{3-i} is not non-decreasing and (3) is satisfied. The simplest replacement is $\ell_{3-i}\alpha_{3-i}$ producing (35). Proposition 7 and Corollary 6 demonstrate the universality of V_C in the form of (1) which naturally extends the popular linear construction of a Lyapunov function used for finite L_p-gain systems. The conservativeness of the factor 2.618 arises in (34) if one uses (14) independently of the sign of ϕ. The next section proposes an approach to numerical removal of possible conservativeness of the analytically derived ϕ.

5. NUMERICAL REDUCTION OF THE EXPONENT

To decrease the magnitude of the exponent ϕ in the Lyapunov function $V_C(x)$ of the form (1), the Legendre-Fenchel (LF) transformation was first introduced to the preservation of iISS (13) by Ito and Kellett (2015). The typical LF transformation is defined as

$$\ell\kappa(s) := \int_0^s (\kappa')^{-1}(l)dl, \quad \forall s \in \mathbb{R}_+ \tag{36}$$

for some continuously differentiable function $\kappa \in \mathcal{K}_\infty$ satisfying $\kappa' \in \mathcal{K}_\infty$ (e.g. Praly and Wang (1996); Krstić and Li (1998); Kellett (2014); Kellett and Wirth (2016)). Integrating by parts verifies that

$$\ell\kappa(s) = s(\kappa')^{-1}(s) - \kappa \circ (\kappa')^{-1}(s), \quad \forall s \in \mathbb{R}_+ \tag{37}$$

is equivalent to (36). A general version of Young's inequality (Hardy et al., 1989, Theorem 156) can be given in terms of the LF transformation as

$$ts \leq \kappa(t) + \ell\kappa(s), \quad \forall(t, s) \in \mathbb{R}_+ \times \mathbb{R}_+. \tag{38}$$

In applying (38) to the decoupling in (13) for the preservation of iISS, Ito (2017) removed the assumption restricting κ and κ' to class $\mathcal{K}_\infty$ functions. This paper replaced the preservation of iISS by the cooperative division technique with which Section 3 derived the analytical criterion (14) producing a smaller ϕ. The next theorem proposes a numerical framework to reduce the exponent ϕ further in the cooperative division technique.

Theorem 8. Suppose that there exist $\hat{c} > 1$ and $\hat{\tau} > 1$ such that (8) holds. Let $\phi \in (0, \infty)$ be such that (16) is satisfied. For $\psi \in \mathbb{R}_+$, define

$$\lambda_{i,\psi}(s) = \alpha_i(s)^\psi \sigma_{3-i}(s)^{\psi+1} \tag{39}$$

$$L_{i,\psi} = \lim_{s\to\infty} \lambda_{i,\psi}(s) \tag{40}$$

$$\tilde{P}_{i,\psi} = \begin{cases} L_{i,0} & \text{if } \psi=0 \text{ and } L_{i,0} < \lim_{s\to\infty} \alpha_i(s)=\infty \\ \infty & \text{otherwise} \end{cases} \tag{41}$$

for $i = 1, 2$. Suppose that $\alpha_i \circ \lambda_{i,\psi}^{\ominus}$ is locally Lipschitz on $[0, L_{i,\psi})$ for each $i = 1, 2$. Then there exist $\tau \in (1, \min\{\hat{\tau}, \hat{c}\}]$ and $\kappa_{i,\psi} \in \mathcal{K}_\infty[0, \tilde{P}_{i,\psi})$ such that (17) and

$$\frac{1}{\tau}\alpha_i \circ \lambda_{i,\psi}^{\ominus}(s) \leq \kappa'_{i,\psi}(s) \leq \frac{1}{\tau}\alpha_i \circ \lambda_{i,\psi}^{\ominus}(s) + \frac{s}{\tau}(\alpha_i \circ \lambda_{i,\psi}^{\ominus})'(s), \quad \text{a.e. } s \in [0, L_{i,\psi}) \tag{42}$$

$$\lim_{s\to\tilde{P}_{i,\psi}^-} \kappa'_{i,\psi}(s) \geq \lim_{s\to\infty} \sigma_i(s) \tag{43}$$

hold for $i = 1, 2$. Define

$$\hat{\alpha}_{i,\psi}(s) = \left(1 - \frac{1}{\hat{c}}\right) \lambda_{i,\psi}(s)\alpha_i(s) \tag{44}$$

$$\hat{\sigma}_{i,\psi}(s) = \min\{\ell\kappa_{i,\psi}\circ\sigma_i(s), L_{i,\psi}\sigma_i(s)\} \tag{45}$$

for $i = 1, 2$. If there exists $\epsilon > 0$ such that

$$\hat{\alpha}_{i,\psi}(s) \geq \epsilon\lambda_{i,\psi}(s)\alpha_i(s) + \hat{\sigma}_{3-i,\psi}(s), \ \forall s \in \mathbb{R}_+ \tag{46}$$

holds for $i = 1, 2$, the function $V_C(x)$ given in (1) is a 0-GAS Lyapunov function of system (2). Moreover, the function $V_C(x)$ is an iISS Lyapunov function of system (2) if (15) holds. Furthermore, (46) is guaranteed to hold for for $i = 1, 2$ with some $\epsilon > 0$ if $\psi \geq \phi$.

The criterion (46) for $i = 1$ is decoupled from that for $i = 2$. Therefore, based on Theorem 8, one can numerically perform the two independent searches on the two lines of $V_1 \in \mathbb{R}_+$ and $V_2 \in \mathbb{R}_+$ to check (46) for a given ψ. In contrast, the original problem of constructing V_C requires a coupled search on the two dimensional space $\mathbb{R}_+^2$ of $V = [V_1, V_2]^T$ for each ϕ. The last statement in Theorem 8 confirms that the numerical search based on (46) never produces a larger ψ than the analytical ϕ. Theorem 8 always accepts a smaller ϕ than the ones proposed in Ito and Kellett (2015) and Ito (2017) since (44) replaces τ and (7) of the previous methods with $\hat{c}$ and (16), respectively. Note that $\hat{c}$ can be arbitrarily close to $\overline{c}$ solving (9) employed by the cooperative division technique in Section 3.

6. ILLUSTRATIVE EXAMPLES

To illustrate significant reduction of ϕ, consider

$$\alpha_1(s) = \frac{s}{1+s}, \ \sigma_1(s) = \frac{3s}{1+s}, \ \kappa_1(s) = s^2 \tag{47a}$$

$$\alpha_2(s) = \frac{6s}{1+3s}, \ \sigma_2(s) = \frac{s}{1+s}, \ \kappa_2(s) = s^2, \tag{47b}$$

used in Ito (2019a). The small-gain condition (6) is satisfied with $c_1 = c_2 = 1.1$. Thus, the iISS small-gain theorem guarantees that any system admitting (47) is iISS with respect to the input u. It is also known that the satisfaction of (6) entitles $V_C(x)$ of the form (1) as an iISS Lyapunov function of the system. As shown in Fig.1, the classical construction which is based on (7) gives the infimum $\phi = 50.81$. The choice $\phi = 51$ achieves (7) in the interval of $\tau \in [1.017, 1.023]$. The exponent $\phi \geq 50.81$ in (1) is astonishingly large and makes V_C absolutely impractical for the use in further analysis and controller design.

Firstly, Theorem 3 with the choice $\overline{c} = c_1 = c_2 = 1.1$ in (14) gives $\phi = 24.16$. Secondly, ϕ in (14) can be reduced further to $\phi = 8.83$ since the functions in (47) satisfy (9) for $\overline{c} = 1.2$. Thirdly, property (17) with $\hat{c} = 1.2$ and $\phi = 25$ is fulfilled for $\tau \in (1, 1.0032)$, while (17) with $\hat{c} = 1.2$ and $\phi = 9$ is satisfied for $\tau \in (1, 1.0035)$. Take $\tau = 1.001$. Numerical computation confirms that (46) is satisfied with $\psi = 1$, $\hat{c} = 1.2$ and $\epsilon = 0.001$ by $\kappa_{i,1}(s) = (s/\tau)\alpha_i \circ \lambda_{i,\psi}^{\ominus}(s)$, $i = 1, 2$. Hence, Theorem 8 reduces ϕ to further $\phi = 1$ in (1). The exponent $\phi = 1$ is a significant decrease from 50.81. It is noteworthy that the functions in (47) prevents W_1 and W_2 from being linear to guarantee $V_C(x)$ to be a 0-GAS Lyapunov function.

Next, to illustrate how a negative exponent ϕ is entitle to give a Lyapunov function, consider

$$\alpha_1(s) = \frac{s}{1+s}, \ \sigma_1(s) = 2s, \ \alpha_2(s) = \frac{6s}{1+Ts}, \ \sigma_2(s) = \frac{s}{1+s}. \tag{48}$$

The small-gain condition (6) is obtained as

$$c_2 \mathrm{sat}(2c_1 s) \leq \frac{6s}{1+Ts}, \quad \forall s \in \mathbb{R}_+. \tag{49}$$

First, suppose that $T = 2$. Since the small-gain condition (49) holds with $c_1 = c_2 = 1.302$, any system (2) described by (48) is guaranteed to be 0-GAS. The classical estimation (7) of the exponent in the 0-GAS Lyapunov function (1) gives the infimum $\phi = 12.469$, while (14) in Theorem 3 gives $\phi = 1.710$ for $\overline{c} = 3/2$ satisfying (9). Next, in the case of $T = 0.3$, the small-gain condition (49) is satisfied with $c_1 = c_2 = 1.658$. The classical condition (7) gives $\phi = 3.900$, while the new condition (14) gives $\phi = 0$ for $\overline{c} = 6/2.3$ achieving (9). Finally, let $T = 0$. Then the small-gain condition (49) is satisfied with $c_1 = c_2 = \sqrt{3}$. Inequalities (33) and (34) is satisfied with $\ell_1 = 1$ and $\ell_2 = 1/3$. Corollary 6 allows one to use $\phi = -1/2$ in (11) and establishes that the linear combination $V_c(x) = V_1(x_1) + \sqrt{1/3}V_2(x_2)$ of V_i is an 0-GAS Lyapunov function of any system admitting (48). Hence, the matching domination (33) takes effect, and the negative exponent $\phi = -1/2$ is allowed to yield a Lyapunov function V_C in the linear combination form as T in (48) decreases to zero.

Ito (2019c) and Ito (2019b) proposed non-differentiable Lyapunov functions by throwing away the separation between V_1 and V_2 in (1). Although they do not involve the exponent ϕ any more, the artificial switchings caused by the non-differentiability need careful attention in system analysis and controller design. In contrast, as illustrated by the above two examples, this paper keeps the handy and separable structure of the smooth function (1), and reduces ϕ.

7. CONCLUSIONS

The algebraic formula of a smooth Lyapunov function that has been popular for interconnected iISS systems involves nonlinearities accompanied by an exponent which explodes extremely rapidly as gain margins decrease. The nonlinearities do not appear in the L_p-gain theory. This gap between the general iISS theory and the special case has been eliminated in this paper by allowing a negative exponent. For iISS systems which are not L_p-gain finite, this paper has achieved immense reduction of the exponent. As demonstrated in Fig. 1, the reduction reaches the factor of tenth, hundredth and thousandth as gains margins decrease closer to zero. For this achievement, the small-gain condition that lumps stability margins is employed, and the cooperative division technique has been developed to replace the classical technique of the iISS preservation. By combining the cooperative division technique with the Legendre-Fenchel transformation, this paper has proposed a formulation to decrease the exponent further numerically. It decouples an n-dimensional problem into n one-dimensional problems. These developments drastically facilitate the use of the smooth Lyapunov function through the general iISS small-gain argument.

For time-delay systems, separability in bounding derivatives of Lyapunov functionals is the key to the iISS small-gain theorem in Ito et al. (2010). A topic of future research is to extend the developed techniques for maintaining or bypassing the separability to address time delays.

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