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Stability of Uniformly Attracting Sets for Impulsive-Perturbed Multi-Valued Semiflows ⋆

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Abstract: In this paper we investigate stability of uniformly attracting sets for semiflows generated by impulsive infinite-dimensional dynamical systems without uniqueness. Obtained abstract results are applied to weakly nonlinear parabolic system, whose trajectories have jumps at moments of intersection with certain surface in the phase space.

Keywords: Attractors, impulsive systems, stability, infinite dimensional dynamical systems, generalized semiflows, nonlinear parabolic systems.

1. INTRODUCTION

Existence of a globally attracting set of a dynamical system provides information about the long term behavior of the system which is rather valuable for design of practical systems in the framework of mathematical systems and control theory. The study of properties of this attracting set is of interest for applications and is a challenging problem from the mathematical viewpoint, especially in the case of nonlinear infinite dimensional systems. In particular its stability and robustness properties need to be studied for any practical system. It in known that in case of a finite dimensional system the dissipativity guarantees the existence of a global attractor, that is an invariant and globally attracting compact set. The existence of such a set in the infinite dimensional case needs some additional conditions Temam (1988), Chepyzhov et al. (2002). It is known that a dissipative continuous dynamical system possess a global attractor if it is asymptotically compact. The global attractor has such important for applications properties like stability and robustness Robinson (2001). These properties remain valid even in the case of systems with non-unique solutions of the Cauchy problem Kasyanov (2011), Kapustyan et al. (2014), Gorban et al. (2014), Kalita et al. (2014), da Costa et al. (2017).

However in some practical situations the dynamics is not always continuous due to such effects as triggered events, collisions and other instantaneous actions causing impulsive transitions of states. Qualitative analysis of systems with impulsive perturbations has been studied by many authors during last three decades (see, e.g., the monographs Lakshmikantham et al. (1989), Samoilenko et al. (1995), Akhmet (2005) and references therein). One of the most important in applications (and the least studied) class of such systems is impulsive dynamical systems (impulsive DS), i.e., autonomous systems whose trajectories undergo impulsive perturbations at the moments of intersection of the trajectories with a certain surface in the phase space. Stability questions for impulsive DS in finite-dimensional spaces were investigated in Kaul (1994), Bonotto (2007), Ciesielski (2004), Dashkovskiy et al. (2013), Perestyuk et al. (2016), Feketa et al. (2018).

The main problem we face when we try to expand the global attractor theory to impulsive DS is the lack of continuous dependence on the initial data. To overcome this difficulty two approaches exist in the literature. One approach has been proposed and developed in Bonotto et al. (2013)-Bonotto et al. (2019). The key idea of those papers is to keep the invariance property in the definition of attractor. This approach allows to construct theory of attractors for impulsive DS similar to the classical one, but it requires additional information about trajectories in a neighborhood of the impulsive set and causes significant restrictions on their behavior.

Another approach was developed in Perestyuk et al. (2016), Kapustyan et al. (2018) and uses the notion of uniform attractor, commonly used for non-autonomous problems Chepyzhov et al. (2002), in particular, for systems with impulses at fixed time instants. An advantage of this approach is that we work with a compact uniformly attracting set without any restrictive assumptions on the impulsive semiflow and, after that, consider the behavior of trajectories only in the neighborhood of the attractor. In Dashkovskiy et al. (2018) natural assumptions on dissi-

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pative multi-valued impulsive semiflow G were proposed, which guarantee invariance of non-impulsive part of the uniform attractor Θ, i.e.,

$$\forall\ t \geq 0 \quad G(t, \Theta \setminus M) = \Theta \setminus M,$$

where M is an impulsive set. It is also possible to prove that $\Theta \setminus M$ is a uniformly attracting set (see Theorem 7 below). Under another assumption, which is more suitable for linear systems, this fact was proved in Bonotto et al. (2019).

Having such properties the next question naturally arises: in what sense $\Theta \setminus M$ is stable with respect to the impulsive semiflow G. The initial discussion of this question started in Kapustyan et al. (2018), where for the single-valued impulsive semiflow G the following stability property was introduced

$$D^+(\Theta \setminus M) \subset \overline{\Theta \setminus M}, \tag{1}$$

where

$$D^+(A) := \bigcup_{x \in A} \{y \mid y = \lim G(t_n, x_n),\ x_n \to x,\ t_n \geq 0\}$$

is G-prolongation of A Bhatia et al. (2002).

In the present paper we obtain the invariance and stability properties of the uniform attractor for a wider class of systems then in Bonotto et al. (2013)-Bonotto et al. (2019), Kapustyan et al. (2018)-Dashkovskiy et al. (2019) under verifiable forward in time conditions, that allows to avoid conditions used Bonotto et al. (2016), Bonotto et al. (2019) which seem to be more restrictive then ours. We also apply the obtained results to a weakly non-linear parabolic problem without uniqueness with impulsive perturbation characterized by semi-norms in the phase space.

We present our results without proofs, since they are rather lengthy and will not fit to the size limitations. Hence the complete proofs will be published elsewhere.

2. SETTING OF THE PROBLEM

We consider the following impulsive system

$$\frac{du}{dt} = Au + F(u),\ u \notin M, \tag{2}$$

$$u|_{t=0} = u_0 \in X, \tag{3}$$

$$\triangle u|_{u \in M} \in Iu - u, \tag{4}$$

where (2), (3) is an evolution system in the infinite-dimensional phase space X for which the uniqueness of solutions is not assumed, $\triangle u(t) = u(t+0) - u(t-0)$ denotes the instantaneous change of the state variable u, and M is some subset of the phase space X. The solution $u = u(t)$ to the problem (2)-(4) is right-continuous function satisfying (2) $\forall t \neq \tau$, where τ is defined by the equation $u(\tau - 0) \in M$, and jumps to the state $u(\tau) \in Iu(\tau - 0)$ at the moment of time τ, where $I : M \mapsto X$ is a given (maybe, multi-valued) map. The set M is called *impulsive set*. The map I is called *impulsive map*, points of the set IM are called *impulsive points*.

Assume that the problem (2)-(4) generates a multi-valued impulsive semiflow $G : R_+ \times X \mapsto P(X)$ which has a uniform attractor $\Theta \subset X$ (see Definitions below). It is known that in general case Θ is neither invariant nor stable with respect to impulsive semiflow G Kapustyan et al. (2018). In Dashkovskiy et al. (2018) it was proved that some continuous properties of the function $u \mapsto s(u)$, where $s(u) \in (0, +\infty]$ is the first impulsive moment for u, can guarantee invariance properties of $\Theta \setminus M$. In the first part of the present paper we show that the same properties guarantee that $\Theta \setminus M$ is stable in the sense (1). In the second part of the paper we apply the obtained results to the following weakly non-linear parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u - b\Delta v + \varepsilon f_1(u, v), \\ \dfrac{\partial v}{\partial t} = b\Delta u + a\Delta v + \varepsilon f_2(u, v), \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \tag{5}$$

on a bounded domain $\Omega \subset R^n$, where $\varepsilon > 0$ is a small parameter, $b \in R$, $a > 0$, nonlinear functions $f_i : R \times R \to R$, $i = 1, 2$ are continuous and bounded, but may be not smooth. Solutions of (5) have jumps in the state space $X = L^2(\Omega) \times L^2(\Omega)$ after reaching the impulsive set

$$M = \left\{ z = \begin{pmatrix} u \\ v \end{pmatrix} \in X \mid (u, \psi)^2 + (v, \psi)^2 = \gamma, \right\} \tag{6}$$

where $\gamma > 0$, ψ is an eigenvector of $-\Delta$ in $H_0^1(\Omega)$.

We will show that for a wide class of multi-valued impulsive maps $I : M \mapsto X$ the problem (5),(6) generates a multi-valued impulsive semiflow which possess a uniform attractor and that this attractor is stable in the sense of (1).

3. ATTRACTORS OF M-SEMIFLOWS AND THEIR STABILITY

Let (X, ρ) be a complete metric space, $P(X)$ ($\beta(X)$) be the set of all non-empty (non-empty bounded) subsets of X, and for any $A, B \in P(X)$ we denote

$$dist(A, B) = \sup_{x \in A} \inf_{y \in B} \rho(x, y),$$

$$O_\delta(A) = \{x \in X \mid dist(x, A) < \delta\}.$$

Definition. [Kapustyan et al. (1999)], [Valero et al. (2007)] A multi-valued map $G : R_+ \times X \to P(X)$ is called a multi-valued semiflow (m-semiflow) if
1) $\forall x \in X\ G(0, x) = x$;
2) $\forall x \in X\ \ \forall t, s \geq 0\ G(t + s, x) \subset G(t, G(s, x))$.

The m-semiflow is called strict if in 2) the equality takes place.

Definition. A non-empty compact subset $\Theta \subset X$ is called a uniform attractor of the m-semiflow G if Θ is uniformly attracting set, i.e.,

$$\forall\ B \in \beta(X)\ \ dist(G(t, B), \Theta) \to 0,\ t \to \infty$$

and Θ is minimal set among all closed uniformly attracting sets.

Lemma 1. [Dashkovskiy et al. (2017)] Assume that the m-semiflow G satisfies the dissipativity condition:

$$\begin{gathered} \exists B_0 \in \beta(X)\ \forall B \in \beta(X)\ \exists T = T(B) \geq 0 \\ \forall t \geq T\ G(t, B) \subset B_0. \end{gathered} \tag{7}$$

Then G has a uniform attractor Θ iff G is asymtotically compact, i.e., for every $t_n \nearrow \infty$, $B \in \beta(X)$
every sequence $\{\xi_n \in G(t_n, B)\}$ is precompact in X.(8)

Moreover, under the condition (7) it holds that

$$\Theta = \omega(B_0) := \bigcap_{\tau>0} \overline{\bigcup_{t\geq\tau} G(t, B_0)}. \quad (9)$$

Note that we do not assume any continuity properties for the map $G(t,\cdot)$. Therefore, in the definition of attractor we require minimality condition instead of the invariance property. On the other hand, if G possess a global attractor, i.e., if there exists a compact uniformly attracting set $\Theta_1 \subset X$ such that $\Theta_1 \subset G(t,\Theta_1)$ $\forall t \geq 0$ then, clearly, Θ_1 is minimal among all closed uniformly attracting sets. The converse statement is true under the following additional assumption

$$\forall\, t > 0 \text{ the map } x \mapsto G(t,x) \text{ has closed graph.} \quad (10)$$

Indeed, if Θ is a uniform attractor for G, then from Lemma 1 follows that $\Theta = \omega(B_0)$ with $B_0 = O_1(\Theta)$. Therefore, for any $\xi \in \Theta$ there is a sequence $\xi_n \in G(t_n, B_0)$ with $t_n \to \infty$, such that $\xi = \lim_{n\to\infty} \xi_n$. So, for every $t > 0$ for sufficiently large n we have $\xi_n \in G(t,\eta_n)$, where $\eta_n \in G(t_n - t, B_0)$ and due to Lemma 1 up to subsequence $\eta_n \to \eta \in \Theta$. Hence from (10) follows $\xi \in G(t,\eta)$ and, finally, $\Theta \subset G(t,\Theta)$.

The following result is a simple generalization of the well-known fact in the single-valued case (Bhatia et al., 2002, Chapter 5), Ciesielski (2004):

Lemma 2. Let $A \subset X$ be a compact set and let

$$\begin{gathered}\forall x_n \to x \in A,\ \forall t_n \geq 0 \\ \text{every sequence } \{\xi_n \in G(t_n, x_n)\} \text{ be precompact.}\end{gathered} \quad (11)$$

Then the following stability conditions are equivalent:

1) $\forall \varepsilon > 0\ \forall x \in A\ \exists \delta > 0\ \ \forall t \geq 0\ G(t, O_\delta(x)) \subset O_\varepsilon(A)$;

2) $\forall \varepsilon > 0\ \exists \delta > 0\ \ \forall t \geq 0\ G(t, O_\delta(A)) \subset O_\varepsilon(A)$;

3) $\forall x \in A\ \forall y \notin A\ \ \exists \delta > 0\ \forall t \geq 0\ G(t, O_\delta(x)) \cap O_\delta(y) = \emptyset$;

4) $A \supset D^+(A) := \{\xi \mid \xi = \lim \xi_n,\ \xi_n \in G(\tau_n, x_n),\ x_n \to x,\ \tau_n \geq 0,\ x \in A\}$.

Lemma 3. Let Θ be uniform attractor of a strict m-semiflow G and

$$\begin{gathered}\forall x_n \to x \in \Theta\ \forall t_n \to t \geq 0\ \ \forall\, \xi_n \in G(t_n, x_n) \\ \text{it holds up to subsequence } \xi_n \to \xi \in G(t,x).\end{gathered} \quad (12)$$

Then Θ is stable in the sense of 1)–4).

Property (12) is crucial for the classical stability. As it was shown in Kapustyan et al. (2018) that even for a simple impulsive semiflow, when (12) does not hold, the uniform attractor does not satisfy any of the properties 1)–4).

However, many examples prompt that under reasonable assumptions the embedding

$$D^+(\Theta \setminus M) \subset \overline{\Theta \setminus M}, \quad (13)$$

which is very close to the classical stability property 4) can be expected. In the next section this property will be proved for general classes of impulsive m-semiflows.

4. ATTRACTORS OF IMPULSIVE M-SEMIFLOWS

Now we briefly describe a special subclass of m-semiflows called impulsive m-semiflows. The most of facts and constructions are taken from Dashkovskiy et al. (2018).

Let K be some family of maps $\varphi : [0,+\infty) \to X$ and the following properties hold:

K0) every $\varphi \in K$ is continuous on $[0,+\infty)$;

K1) $\forall x \in X\ \exists \varphi \in K : \varphi(0) = x$;

K2) $\forall \varphi \in K\ \forall s \geq 0\ \varphi(\cdot + s) \in K$.

Our impulsive m-semiflow G will be constructed with the help of the family of maps K, a given impulsive set $M \subset X$ and a given impulsive map $I : M \to P(X)$. We denote it by $G = (K, M, I)$. It means that a phase point moves along trajectories of K and when it meets the set M, it jumps onto a new position in the set IM.

We denote

$$K_x = \{\varphi \in K | \varphi(0) = x\}.$$

For $x \in M$ we denote by x^+ some element of Ix.

For $\varphi \in K$ we denote $M^+(\varphi) = \bigcup_{t>0} \varphi(t) \cap M$.

For "well-posedness" of impulsive problem we require the following conditions:

$$M \subset X \text{ is a closed set;} \quad (14)$$

$$I : M \mapsto P(X) \text{ is a closed-valued map;} \quad (15)$$

$$M \cap IM = \emptyset; \quad (16)$$

$$\begin{gathered}\forall x \in M\ \forall \varphi \in K_x\ \exists \tau = \tau(\varphi) > 0 \\ \forall t \in (0,\tau)\ \varphi(t) \notin M.\end{gathered} \quad (17)$$

Remark. Unlike conditions in Definition 2.7 from Bonotto et al. (2019) we do not suppose any conditions on φ before its intersection with M. It allows us to consider a wider class of applications including non-linear evolution problems.

Lemma 4. (Dashkovskiy et al. (2018)). If conditions (14)-(17) hold, then for every $\varphi \in K$ either $M^+(\varphi) = \emptyset$ or $\exists\, s^* > 0$ such that

$$\varphi(s^*) \in M \text{ and } \varphi(t) \notin M\ \forall t \in (0, s^*). \quad (18)$$

According to (18) we can define the following function $s : K \to (0,+\infty]$

$$s(\varphi) = \begin{cases} s^*, & \text{if } M^+(\varphi) \neq \emptyset, \\ +\infty, & \text{if } M^+(\varphi) = \emptyset. \end{cases} \quad (19)$$

Let us construct an impulsive trajectory $\tilde{\varphi}$ starting from $x_0 \in X$. Let $\varphi_0 \in K_{x_0}$. If $s(\varphi_0) = \infty$, then we have non-impulsive case:

$$\tilde{\varphi}(t) = \varphi_0(t)\ \forall t \geq 0.$$

Otherwise we set $s_0 := s(\varphi_0) > 0$ and

$$\tilde{\varphi}(t) = \begin{cases} \varphi_0(t), & t \in [0, s_0), \\ x_1^+ \in I\varphi_0(s_0), & t = s_0. \end{cases}$$

Let $\varphi_1 \in K_{x_1^+}$. If $s(\varphi_1) = \infty$, then $\tilde{\varphi}(t) = \varphi_1(t - s_0)$ $\forall t \geq s_0$. In the other case for $s_1 := s(\varphi_1) > 0$ we define

$$\tilde{\varphi}(t) = \begin{cases} \varphi_1(t - s_0), & t \in [s_0, s_0 + s_1), \\ x_2^+ \in I\varphi_1(s_1), & t = s_0 + s_1. \end{cases}$$

Continuing this process we obtain an impulsive trajectory $\tilde{\varphi}$ with finite or infinite number of impulsive points

$\{x_n^+\}_{n\geq 1} \subset IM$, corresponding durations between impulses $\{s_n\}_{n\geq 0} \subset (0,\infty)$ and functions $\{\varphi_n\}_{n\geq 0} \subset K$.

If $\tilde{\varphi}$ has infinite number of jumps, then it is defined for $n \geq 0$ and $t \geq 0$ by the formula

$$\tilde{\varphi}(t) = \begin{cases} \varphi_n(t - t_n), \ t \in [t_n, t_{n+1}), \\ x_{n+1}^+ \in I\varphi_n(t_{n+1} - t_n), \ t = t_{n+1}, \end{cases} \tag{20}$$

where $t_0 = 0, t_{n+1} := \sum_{k=0}^{n} s_k$.

By $\tilde{K}_x$ we denote the set of all impulsive trajectories starting from $x \in X$ and assume that every impulsive trajectory is defined on $[0, +\infty)$, i.e.,

$$\forall x \in X \text{ every } \tilde{\varphi} \in \tilde{K}_x \text{ is defined on } [0, +\infty). \tag{21}$$

Assumption (21) means that for every impulsive trajectory $\tilde{\varphi}$ either the number of its impulsive points is finite (including the non-impulsive case) or $\sum\limits_{n=0}^{\infty} s_n = \infty$. This excludes Zeno type solutions (see Dashkovskiy and Feketa (2018) for stability investigations of such solutions).

Remark. According to our construction we have

$$\forall x \in X \ \forall \tilde{\varphi} \in \tilde{K}_x \ \forall t > 0 \ \tilde{\varphi}(t) \notin M. \tag{22}$$

Lemma 5. (Dashkovskiy et al. (2018)). Under conditions K0)- K2), (14)-(17), (21) the following property holds:

$$\forall x \in X \ \forall \varphi \in \tilde{K}_x \ \forall s \geq 0 \ \varphi(\cdot + s) \in \tilde{K}_{\varphi(s)}. \tag{23}$$

In particular, a multi-valued map $G : R_+ \times X \to P(X)$ defined by the formula

$$G(t, x) = \{\tilde{\varphi}(t) | \tilde{\varphi} \in \tilde{K}_x\} \tag{24}$$

is an m-semiflow. If, additionally, $\forall \varphi, \psi \in K$, $\forall s > 0$ such that $\varphi(0) = \psi(s)$ we have

$$\theta(p) := \begin{cases} \psi(p), \ p \in [0, s), \\ \varphi(p - s), \ p \geq s \end{cases} \in K, \tag{25}$$

then the m-semiflow G is strict.

Remark. In the sequel we will say that the problem (2)-(4) *generates an impulsive m-semiflow* (by formula (24)) if solutions of (2) generate a set K of maps $\varphi : [0, +\infty) \mapsto X$ satisfying $K0) - K2)$ and for a given set $M \subset X$ and map $I : M \mapsto P(X)$ the conditions (14)-(17), (21) are satisfied.

Let us additionally assume that

K3) $\forall x_n \to x \ \forall \varphi_n \in K_{x_n} \ \exists \varphi \in K_x$ such that up to subsequence $\varphi_n \to \varphi$ uniformly on every $[a, b] \subset R_+$.

Now we are ready to discuss the invariance property of uniform attractors for impulsive m-semiflows. Such results has firstly appeared in Bonotto et al. (2015) for single-valued impulsive semiflows with "tube conditions". For multi-valued case it was proved in Dashkovskiy et al. (2018), but under rather restrictive assumptions about non-impulsiveness of the limit of non-impulsive trajectories (see Bonotto et al. (2019) for detailed comparison analysis and examples). Here we use another conditions which can be easily verified in applications. More precisely, according to K3) let us consider

$$x \in \Theta \setminus M, \ x_n \to x, \ \varphi_n \in K_{x_n} \text{ and } \varphi \in K_x \text{ such that } \forall t \geq 0 \ \varphi_n(t) \to \varphi(t).$$

In Dashkovskiy et al. (2018) the following assumption was used

$$s(\varphi) = \infty, \text{ if } s(\varphi_n) = \infty \text{ for infinitely many } n \geq 1. \tag{26}$$

Now we will use the following assumption

$$\text{if } \ s(\varphi_n) = \infty, \ t_n \to \infty, \text{ and } \psi_n(t) = \varphi_n(t + t_n), \ t \geq 0, \\ \text{then for } \psi(t) = \lim_{n\to\infty} \psi_n(t) \text{ we have } s(\psi) = \infty. \tag{27}$$

Lemma 6. Assumption (26) implies (27).

Lemma 6 shows that the older assumption (26) is stronger then the new one (27). Let us consider an example illustrating the difference between these two assumptions

Example.[Dashkovskiy et al. (2018),Bonotto et al. (2019)] For $\lambda > 0$ let us consider the impulsive system

$$\begin{cases} \dot{x} = -\lambda x, \\ \dot{y} = -\lambda y, \end{cases} \tag{28}$$

$$M = \{x \geq 0, \ y \geq 0, \ x + y = 1\}, \tag{29}$$

$$\text{for } z = (x, y) \in M \text{ we define } Iz = 2z. \tag{30}$$

Such a system generates an impulsive semiflow which has uniform attractor

$$\Theta = \{e^{-\lambda\tau}(c_1, c_2) \mid c_1 \geq 0, \ c_2 \geq 0, \ c_1 + c_2 = 2, \\ \tau \in [0, \frac{1}{\lambda} \ln 2]\} \cup (0, 0)$$

Let us consider $(-\frac{1}{n}, 2) \to (0, 2) \in \Theta \setminus M$. Then

$$\varphi_n(t) = (-\frac{1}{n} e^{-\lambda t}, 2e^{-\lambda t}) \to \varphi(t) = (0, 2e^{-\lambda t}),$$

$$s(\varphi_n) = \infty, \ s(\varphi) = \frac{1}{\lambda} \ln 2$$

So, condition (26) does not take place. On the other hand, for $t_n \to \infty$

$$\varphi_n(t_n) \to (0, 0) \in \Theta, \ \psi(t) \equiv (0, 0), \ s(\psi) = \infty$$

Theorem 7. Let conditions K0)-K3), (14)-(17), (21) be satisfied and the impulsive map $I : M \to P(X)$ be compact-valued and upper-semicontinuous. Let Θ be a uniform attractor of a strict impulsive m-semiflow $G = (K, M, I)$. Let the function $s : K \to (0, \infty]$ defined in (19) satisfy the following property:

$$\forall \ x \in IM \ \forall \ \varphi \in K_x \text{ we have } s(\varphi) < \infty, \tag{31}$$

and let the following implications hold true:

(i) For $x_n \to x \in \Theta \setminus M$, $\varphi_n \in K_{x_n}$ and $\varphi \in K_x$ with $\varphi_n(t) \to \varphi(t) \ \forall t \geq 0$,

$$s(\varphi_n) < \infty, \ n \in N \ \Rightarrow \ s(\varphi) < \infty, \ s(\varphi_n) \to s(\varphi); \tag{32}$$

$$\left.\begin{array}{l} s(\varphi_n) = \infty, \ n \in N, \ t_n \to \infty, \\ \psi_n(t) := \varphi_n(t + t_n), \ t \geq 0, \\ \psi(t) := \lim \psi_n(t) \end{array}\right\} \Rightarrow \ s(\psi) = \infty. \tag{33}$$

(ii) For $x_n \to x \in \Theta \cap M$, $\varphi_n \in K_{x_n}$ and $\varphi \in K_x$ with $\varphi_n(t) \to \varphi(t) \ \forall t \geq 0$, follows that up to subsequence

$$\text{either } \forall \ n \ \ s(\varphi_n) = \infty \text{ or } s(\varphi_n) \to 0. \tag{34}$$

Then Θ is invariant in the sense that the following properties hold:

$$\forall \ t \geq 0 \ \Theta \setminus M \subset G(t, \Theta); \tag{35}$$

$$\forall\ t \geq 0\ G(t, \Theta \setminus M) \subset \Theta \setminus M. \tag{36}$$

Remark. Condition (31) means that if an impulsive trajectory has one impulsive point, then it has infinite number of impulsive points. In particular, in condition (33) we have

$$s(\varphi_n) = \infty \ \Leftrightarrow\ s(\psi_n) = \infty.$$

Remark. If we additionally assume that

$$\begin{array}{l}\text{for arbitrary } \{x_n\} \subset B_0,\ \varphi_n \in K_{x_n}\ t_n \to \infty \\ \text{on some subsequence } \varphi_n(t_n) \to \xi \notin M,\end{array} \tag{37}$$

then we can state the equality

$$\forall\ t \geq 0\ G(t, \Theta \setminus M) = \Theta \setminus M. \tag{38}$$

Remark. Unfortunately, only under conditions of Theorem 7, even with additional assumption (37), we cannot expect stability property (13). Indeed, in the example, given above, all conditions of Theorem 7 are fulfilled and, moreover, every non-impulsive trajectory tends to $(0, 0) \notin M$ as $t \to \infty$, so (37) is also fulfilled. But we can see that for $z_n := (-\frac{1}{n}, 2) \to z := (0, 2) \in \Theta \setminus M$ we have

$$G(t, z_n) = (-\frac{1}{n}e^{-\lambda t}, 2e^{-\lambda t}) \to (0, 2e^{-\lambda t}).$$

Now taking $t > \frac{1}{\lambda}\ln 2$, we conclude that

$$D^+(\Theta \setminus M) \not\subset \Theta.$$

To guarantee the stability property, we have to impose the condition (26).

Theorem 8. Assume that all conditions of Theorem 7 are satisfied. Let the following implication be satisfied

$$\left.\begin{array}{l} x_n \to x \in \Theta \setminus M,\ \varphi_n \in K_{x_n} \\ \varphi \in K_x \text{ with } \varphi_n(t) \to \varphi(t)\ \forall t \geq 0 \\ s(\varphi_n) = \infty \text{ for infinitely many } n \end{array}\right\} \Rightarrow s(\varphi) = \infty. \tag{39}$$

Then the following properties hold:

$$\forall\ t \geq 0\ G(t, \Theta \setminus M) = \Theta \setminus M, \tag{40}$$

$$\Theta = \overline{\Theta \setminus M}, \tag{41}$$

$$D^+(\Theta \setminus M) \subset \overline{\Theta \setminus M}. \tag{42}$$

Remark. The equality (41) means in fact that $\Theta \setminus M$ is uniformly attracting set. It should be mentioned that this property was firstly proved in Bonotto et al. (2019) under some different conditions.

5. APPLICATION TO IMPULSIVE PARABOLIC SYSTEM WITHOUT UNIQUENESS

Let $\Omega \subset R^n, n \geq 1$ be a bounded domain. For unknown scalar valued functions $u(t, x), v(t, x)$ defined on $(0, +\infty) \times \Omega$ we consider the following nonlinear parabolic problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u - b\Delta v + \varepsilon f_1(u, v), \\ \dfrac{\partial v}{\partial t} = b\Delta u + a\Delta v + \varepsilon f_2(u, v), \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \tag{43}$$

where $\varepsilon > 0$ is a small parameter, $a > 0$, $b \in R$, nonlinear functions $f_i, i = 1, 2$ are continuous (smoothness is not required) and boundeed, that is

$$\exists C > 0\ \forall u, v \in R\ |f_1(u, v)| + |f_2(u, v)| \leq C. \tag{44}$$

We consider (43) in the distributional sense in the following phase space $X = L^2(\Omega) \times L^2(\Omega)$, where for $z = \begin{pmatrix} u \\ v \end{pmatrix}$ we put $\|z\| = \sqrt{\|u\|^2 + \|v\|^2}$, where $\|\cdot\|$ and $(\cdot, \cdot)$ denote the usual norm and scalar product in $L^2(\Omega)$. It is known, (see for example Chepyzhov et al. (2002)), that for every $\varepsilon > 0$ and $z_0 \in X$ there exists at least one (weak) solution $z \in C([0, +\infty); X)$ to the problem (43) satisfying $z(0) = z_0$. Thus, problem (43) generates a family of continuous maps

$$K^\varepsilon = \{z : [0, +\infty) \to X \mid z \text{ is a solution of (43)}\},$$

which satisfies K0)–K3).

Let $\{\psi_i\}_{i=1}^\infty$ be an orthonormal basis in $L^2(\Omega)$ such that

$$-\Delta\psi_i = \lambda_i\psi_i,\ \psi_i \in H_0^1(\Omega). \tag{45}$$

We consider the following impulsive problem for fixed numbers $\mu > 0$, $\gamma > 0$:

$$M = \left\{z = \begin{pmatrix} u \\ v \end{pmatrix} \in X \mid (u, \psi_1)^2 + (v, \psi_1)^2 = \gamma\right\}; \tag{46}$$

$$M' = \left\{z = \begin{pmatrix} u \\ v \end{pmatrix} \in X | (u, \psi_1)^2 + (v, \psi_1)^2 = \gamma(1 + \mu)\right\}; \tag{47}$$

$I : M \mapsto P(M')$ is compact-valued upper semicontinuous map, such that for $z = \sum_{i=1}^\infty \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i \in M$

$$Iz \subseteq \left\{ \begin{pmatrix} \bar{c}_1 \\ \bar{d}_1 \end{pmatrix} \psi_1 + \sum_{i=2}^\infty \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i \;\middle|\; \bar{c}_1^2 + \bar{d}_1^2 = \gamma(1 + \mu)\right\}. \tag{48}$$

Remark. As a particular example we can consider the following continuous single-valued map $I : M \mapsto M'$ defined by

$$I\left(\sum_{i=1}^\infty \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i\right) = \begin{pmatrix} \sqrt{1 + \mu}c_1 \\ \sqrt{1 + \mu}d_1 \end{pmatrix} \psi_1 + \sum_{i=2}^\infty \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i.$$

Theorem 9. For sufficiently small ε the problem (43), (46), (48) generates an impulsive m-semiflow $G_\varepsilon : R_+ \times X \to P(X)$, which possess a uniform attractor Θ_ε satisfying the following properties

$$\forall\ t \geq 0\ G_\varepsilon(t, \Theta_\varepsilon \setminus M) = \Theta_\varepsilon \setminus M, \tag{49}$$

$$\Theta_\varepsilon = \overline{\Theta_\varepsilon \setminus M}, \tag{50}$$

$$D^+(\Theta_\varepsilon \setminus M) \subset \overline{\Theta_\varepsilon \setminus M}. \tag{51}$$

Remark. Statement of Theorem 9 remains true for a more general class of impulsive parameters

$$M = \left\{z = \begin{pmatrix} u \\ v \end{pmatrix} \in X | \sum_{i=1}^p \Big(\alpha_i(u, \psi_i)^2 + \beta_i(v, \psi_i)^2\Big) = \gamma\right\},$$

where $\alpha_i > 0, \beta_i > 0$, $p \in N$ are arbitrary numbers, $I : M \mapsto P(X)$ is compact-valued upper semicontinuous impulsive map, such that for $z = \sum_{i=1}^\infty \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i \in M$

$$Iz \subseteq \left\{ \sum_{i=1}^p \begin{pmatrix} \bar{c}_i \\ \bar{d}_i \end{pmatrix} \psi_i + \sum_{i=p+1}^\infty \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i\right\},$$

where $\sum_{i=1}^{p}(\alpha_i\bar{c}_i^2+\beta_i\bar{d}_i^2)=\gamma(1+\mu)$.

6. CONCLUSION

Our work contributes to the development of the theory of global attractors for infinite dimensional impulsive dynamical systems. This requires additional restrictions on the systems properties compared with the case of continuous dynamics, since for impulsive systems a global attractor in the usual sense does not exist in many interesting cases. In contrary to rather restrictive conditions imposed in Bonotto et al. (2013)-Bonotto et al. (2019) our approach leads to weaker restrictions that are easier to be verified. The idea of this approach is to identify, whether a system possess a weaker type of an attracting set (uniform attractor) and to use its existence as an additional property of the studied system. This allows us to obtain result similar to Bonotto et al. (2013)-Bonotto et al. (2019) and additionally to study the stability properties of the attracting set.

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