Robustness of Multi-Robot Systems Controlling the Size of the Connected Component After Robot Failure

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Abstract: This study approaches a robustification method for a multi-robot network connectivity. Instead of the vertex connectivity, which is commonly used as a robustness index, here we consider the size of the connected component remaining after one robot has been removed from the network, and we propose a distributed control law for improvement and preservation of the remaining connected component size. Some conditions of a modified graph Laplacian eigenvalue are analyzed for the improvement and the preservation, and then the control strategy is composed using the Laplacian eigenvalue as an indicator of the remaining connected component size. From simulations, we observed that a multi-robot system with our control method achieves a convincing state regarding the trade-off between a network robustness and a coverage task performance.

Keywords: Distributed control, Network topologies, Fault tolerance

1. INTRODUCTION

A collection of autonomous mobile robots namely a multi-robot system, which is expected to achieve collective behaviours from inter-robot communications. An appropriate wireless communication technology (e.g. a mobile ad-hoc network) is provided on the robots to achieve such inter-robot communications, and information for the collective behaviours flows through the inter-robot network. Therefore the network topology should be connected during their task performances in spite of robots' individual autonomous actions. Studies of connectivity preserving consensus as Dai et al. (2011) tackle this problem. These literatures propose control laws to preserve all initial links since a consensus task can be achieved without link disconnections.

The network disconnection can also be caused by robot failures, and the possibility of occurring robot malfunctions increases as the number of robots increases. The multi-robot system is demanded to maintain its performances as much as possible even if such robot failures occur. From this point, some methods to robustify the network connectedness against a robot failure are studied, such as Ghedini et al. (2015); Zareh et al. (2016); Panerati et al. (2019), where the main concept is to let the network bi-connected, meaning that the network will remain connected if any one node is removed. Thanks to the robustification method, the multi-robot network stays connected even if any one robot fails.

Although vertex connectivity is a good indicator for the robustness to engage the entire network connectedness, it restricts a configuration space of the multi-robot sys-
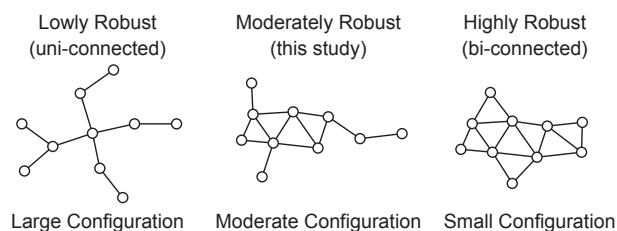

Fig. 1. Robustification concept we consider in this study. We handle a trade-off between the connected component size and the configuration space.

tem owing to limitations of wireless communication range. Thus the robustification reduces a feasible configuration space of the multi-robot system, and the restriction may disturb some configurations for task achievements. To relax this restriction, in Murayama (2018) we proposed an alternative concept of the multi-robot network robustness which is related to the giant component size after any one robot failed, in contrast to the conventional robustness in consideration of entire network connectedness. In other words, our concept pays attention only to connectedness between "centered" robots so that the restriction for "non-centered" robots will be relaxed. We conceive there is little difference between the removal of a single robot and a few number of robots from a sufficiently large multi-robot network. From this perspective, we proposed an algorithm to estimate the giant component size in Murayama (2018), and also proposed a robustness improvement control method based on a consensus control in Murayama and Sabattini (2019).

In this study, we develop a control method according to our robustness concept, and confirm validity of our

method from numerical examples. We theoretically remark some relations between our robustness and the algebraic connectivity, which is the second smallest eigenvalue of the graph Laplacian matrix, and then we construct the improvement and preserving control using the algebraic connectivity. Results of numerical simulations show that the control method tries to improve and preserve the connected component size. Additionally some coverage task simulations demonstrate that the system with our control method has a potential to achieve an optimal state, considering a the tradeoff between a network robustness and a configuration.

2. DEFINITION AND FORMULATION

We consider a multi-robot system consisting of N robots, and we consider a simple connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as the network topology of the multi-robot system. Neighbor set for a robot i is defined by $\mathcal{N}_i = \{j|(i,j) \in \mathcal{E}\}$. Define a removed graph $\mathcal{G}_{-i} = (\mathcal{V} \setminus \{i\}, \mathcal{E} \setminus \mathcal{E}_i)$ as the graph obtained removing the robot i (and also the links $\mathcal{E}_i$ incident to the node i) from the graph $\mathcal{G}$. A node i is an articulation node if and only if the removed graph $\mathcal{G}_{-i}$ is disconnected. Define *remaining connected components* $\mathcal{C}_m^{(i)}$ as the connected components of a removed graph $\mathcal{G}_{-i}$, where m denotes the index of the components $m \in \{1, \ldots, M^{(i)}\}$, and $M^{(i)}$ is the number of the components. We express the node size of a remaining component $\mathcal{C}_m^{(i)}$ by $C_m^{(i)}$. Suppose the indices m are in ascending order of the size $C_m^{(i)}$, and call $\mathcal{C}_M^{(i)}$ the *remaining giant component* w.r.t. the node i removal.

We suppose the link set $\mathcal{E}$ of the graph $\mathcal{G}$ is depend on the distance $\|p_j - p_k\|$ between each pair of robots $j, k \in \mathcal{V}$, where $p_i \in \mathbb{R}^2$ denotes the position of the robot i. Although we assume the robots move on 2 dimensional plane in this study, our method is applicable in a 3 dimensional space in the same way. An element a_{jk} of the weighted adjacency matrix A of the graph $\mathcal{G}$ is defined by $a_{jk} = w(\|p_j - p_k\|)$, where the function $w(\cdot)$ satisfies $w(x) > 0$ if $x \le r_{\max}$ and $w(x) = 0$ if $x > r_{\max}$, considering a maximal communication distance $r_{\max}$.

Letting the weighted degree $d_i = \sum_j a_{ij}$, we define the Laplacian matrix $L = \mathrm{diag}(d_1, \ldots, d_N) - A$. As is well known, the second smallest eigenvalue λ_2 of the Laplacian matrix L is called the algebraic connectivity, and the eigenvector v corresponding to the eigenvalue λ_2 is called the Fiedler vector.

To estimate properties of the removed graph $\mathcal{G}_{-i}$, we introduce the perturbed adjacency matrix $A^{(i)}(\varepsilon)$ whose jk-th element $a_{jk}^{(i)}(\varepsilon)$ w.r.t. the node i removal is defined by

$$a_{jk}^{(i)}(\varepsilon) = \begin{cases} a_{jk} & \text{if } j \ne i \wedge k \ne i, \\ \varepsilon a_{jk} & \text{if } j = i \vee k = i, \end{cases}$$

where $\varepsilon > 0$ denotes a perturbation parameter. We suppose the perturbation parameter ε is sufficiently small in this study, because the perturbed adjacency matrix $A^{(i)}(\varepsilon)$ approximately represents the graph obtained removing node i from $\mathcal{G}$. From the perturbed adjacency matrix $A^{(i)}(\varepsilon)$, we also define the perturbed Laplacian matrix $L^\varepsilon(i)$. We define the second smallest eigenpair

$(\lambda_2^{(i)}(\varepsilon), v^{(i)}(\varepsilon))$ of the perturbed Laplacian matrix $L^{(i)}(\varepsilon)$ by a perturbed algebraic connectivity and a perturbed Fiedler vector respectively. We suppose the perturbed Fiedler vector $v^{(i)}(\varepsilon)$ in this study is normalized, i.e., $\lambda_2^{(i)}(\varepsilon) = v^{(i)T}(\varepsilon) L^{(i)}(\varepsilon) v^{(i)}(\varepsilon)$.

In previous study in Murayama (2018), we analyzed the fact that any two elements $v_j^{(i)}(\varepsilon)$ and $v_k^{(i)}(\varepsilon)$ of the perturbed Fiedler vector converge to the same value as $\varepsilon \to +0$ if both the node j and k are in the same remaining component $\mathcal{C}_m^{(i)}$ of the removed graph $\mathcal{G}_{-i}$. Moreover, under the assumption

$$\frac{D_{m_1}^{(i)}}{C_{m_1}^{(i)}} \ne \frac{D_{m_2}^{(i)}}{C_{m_2}^{(i)}}, \ ^\forall m_1, m_2 \in \{1, \ldots, M^{(i)}\}, \tag{1}$$

where $D_m^{(i)} = \sum_{j \in \mathcal{N}_i \cap \mathcal{C}_m^{(i)}} a_{ij}$, we showed the below properties.

Lemma 1. $v_j^{(i)}(0) = v_k^{(i)}(0)$ holds for all $j, k \in \mathcal{C}_m^{(i)}$. If the assumption in (1) is satisfied, then $v_j^{(i)}(0) \ne v_k^{(i)}(0)$ holds for all $j \in \mathcal{C}_m^{(i)} \wedge k \notin \mathcal{C}_m^{(i)}$ and $v_i^{(i)}(0) \ne 0$. ◇

Lemma 2. For the perturbed eigenpair $(\lambda_2^{(i)}(\varepsilon), v^{(i)}(\varepsilon))$, the following property holds:

$$\dot{\lambda}_2^{(i)}(0) v_j^{(i)}(0) = \frac{D_m^{(i)}}{C_m^{(i)}}(v_j^{(i)}(0) - v_i^{(i)}(0)), \ ^\forall j \in \mathcal{C}_m^{(i)}. \tag{2}$$

where $\dot{\lambda}_2^{(i)}(0) = \lim_{\varepsilon \to +0} \lambda_2^{(i)}(\varepsilon)/\varepsilon$. ◇

Corollary 3. If the node i is not an articulation node, then the value $\dot{\lambda}_2^{(i)}(0)$ is given by

$$\dot{\lambda}_2^{(i)}(0) = D^{(i)} \frac{N}{N-1}, \tag{3}$$

where $D^{(i)} = \sum_{j \in \mathcal{N}_i} a_{ij}$. ◇

The assumption in (1) practically holds in many cases since the value $D_m^{(i)}$ continuously changes depending on the robots distances $\|p_i - p_j\|$, even though the component size $C_m^{(i)}$ is an integer.

The locally available values for each robot k are defined as below. In this study, we assume the robot k knows its position p_k and the perturbed link weights $a_{kj}^{(i)}(\varepsilon)$ for all time. Because the eigenpair $(\lambda_2^{(i)}(\varepsilon), v^{(i)}(\varepsilon))$ can be computed by a distributed algorithm introduced in Yang et al. (2010), we assume that the perturbed algebraic connectivities $\lambda_2^{(i)}(\varepsilon)$ and the elements $v_k^{(i)}(\varepsilon)$ are available for the robot k. Assuming all above values of neighbor $j \in \mathcal{N}_k$ are also available for the robot k, the remaining component sizes $C_m^{(k)}$ and the remaining component link weight $D_m^{(k)}$ are locally computable using Lemma 1 and Lemma 2.

The objective of this study is to control the size of the remaining giant component size $C_M^{(i)}$ not to be less than a given threshold $\bar{C}$, in a distributed fashion. We consider two control strategy: (1) improvement of the remaining giant component sizes $C_M^{(i)}$ when $C_M^{(i)} < \bar{C}$, and (2) preservation of it when $C_M^{(i)} \ge \bar{C}$.

3. PROPERTIES OF PERTURBED ALGEBRAIC CONNECTIVITY

This section analyzes and details some properties of the derivative of the perturbed connectivity $\dot{\lambda}_2^{(i)}(0) = \lim_{\varepsilon \to +0} \lambda_2^{(i)}(\varepsilon)/\varepsilon$ defined in the above section, for the sake of designing a distributed controller. Since we discuss about graph theoretical properties here, a robotic node is simply called a node in this section. Additionally, we handle only cases when node i is removed for simplicity, thus we describe $\dot{\lambda}_2 = \lambda_2^{(i)}(0)$ and $v = v^{(i)}(0)$. Since $v_j = v_k$ for all $j, k \in \mathcal{C}_m$ from Lemma 1, we define a notation $w_m = v_j, \forall j \in \mathcal{C}_m^{(i)}$ in this section.

Our goal in this section is to derive conditions of the value $\dot{\lambda}_2$ to satisfy the control objective $C_M \geq \bar{C}$. First we present an order of elements of the perturbed eigenvector v.

Lemma 4. Consider a perturbed Laplacian matrix $L(0)$ w.r.t removal of a node i. Then,

(1) If $v_i > 0$ then exactly one component $\mathcal{C}_m$ has negative value $w_m < 0$ and all the other components $\mathcal{C}_s$ has the value $w_s > v_i$.
(2) If $v_i = 0$ then each remaining component $\mathcal{C}_m$ has either $w_m > 0$, $w_m < 0$, or $w_m = 0$.

$\diamond$

Proof. It is directly obtained from Fiedler (1975). $\square$

We recall that the case of $v_i = 0$ is rare from Lemma 1 since the component link weight D_m is continuous and the component size C_m is discrete. Therefore, there exists only one component $\mathcal{C}_{m^-}$ such that $v_i w_{m^-} < 0$ in many cases. Moreover, we can express a bound of the value $\dot{\lambda}_2$ using the values of the component $\mathcal{C}_{m^-}$ as below.

Lemma 5. Suppose an articulation node i and a remaining component $\mathcal{C}_{m^-}$ such that $v_i w_{m^-} < 0$. Then, the following relation holds:

$$\frac{D_{m^-}}{C_{m^-}} \leq \dot{\lambda}_2 \leq \frac{D - D_{m^-}}{N - 1 - C_{m^-}}. \tag{4}$$

The equality holds if and only if $v_i = 0$. $\diamond$

Proof. We easily see $v_j \geq v_j - v_i$ for both cases m^- and m^+. Thus,

$$\dot{\lambda}_2 v_j = \frac{D_m}{C_m}(v_j - v_i) \leq \frac{D_m}{C_m} v_j, \tag{5}$$

from Lemma 2. From monotonicity of the function $\dot{\lambda}_2$ about link addition/derision, we get

$$\dot{\lambda}_2 \leq \frac{\sum_{m^+} D_{m^+}}{\sum_{m^+} C_{m^+}} = \frac{D - D_{m^-}}{N - 1 - C_{m^-}},$$

where m^+ denotes an index of the remaining component such that $v_i w_{m^+} > 0$. $\square$

From Lemma 5, we can get a sufficient condition of the value $\dot{\lambda}_2$ to satisfy the control objective $C_{m^-} \geq \bar{C}$ as below.

Theorem 6. Assume the value $\dot{\lambda}_2$ of an articulation node $i \in \mathcal{V}$ satisfies the inequality

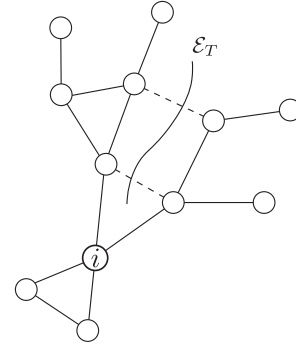$$\dot{\lambda}_2 \geq \frac{D - D_{m^-}}{N - 1 - \bar{C}}, \tag{6}$$

Fig. 2. Example of tearing $\mathcal{E}_T$. Tearing exists only when there are two or more links from node i to giant component.

where m^- denotes labels of remaining component such that $v_i w_{m^-} < 0$, and the component threshold $\bar{C} < N - 1$, then the remaining giant component is characterized $C_M \geq \bar{C}$. $\diamond$

Proof. It is directly obtained from Lemma 5. $\square$

To satisfy the condition in (6), increasing the value $\dot{\lambda}_2$ will be a reasonable strategy for improving the remaining giant component size C_M. Because Theorem 6 indicates the sufficient condition, the component size may satisfy $C_M \geq \bar{C}$ even if the value $\dot{\lambda}_2$ does not satisfy the condition in (6). Actually, the node i becomes a non-articulation node (that is $C_M = N - 1$) when the value $\dot{\lambda}_2$ approaches its upper-bound given in (3).

The condition in (6) is also useful for preservation of the giant component size. But it depends on component link weights of the non-giant components, and therefore it has a risk to be too conservative. To relax the conservativeness, we introduce the following definition and theorems.

Definition 7. (*Tearing*) Assume a connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ which has the remaining giant component $\mathcal{C}_M(\mathcal{G})$. Define a link set $\mathcal{E}_T \subseteq \mathcal{E} \setminus \mathcal{E}_i$ such that the connected graph $\mathcal{H} = (\mathcal{V}, \mathcal{E} \setminus \mathcal{E}_T)$ with the giant component $\mathcal{C}_M(\mathcal{H}) < \mathcal{C}_M(\mathcal{G})$. The operation to make the connected graph $\mathcal{H}$ with the cutting link set $\mathcal{E}_T$ is called *tearing*. We also call the cutting link set $\mathcal{E}_T$ *tearing* of the component $\mathcal{C}_M$ (see Fig. 2). $\diamond$

Theorem 8. Assume the threshold satisfies $\bar{C} > N/2$, a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{C}_M(\mathcal{G}) \geq \bar{C}$, and the graph $\mathcal{H} = (\mathcal{V}, \mathcal{E} \setminus \mathcal{E}_T)$. Also assume $v_i(\mathcal{G})w_M(\mathcal{G}) < 0$ and $v_i(\mathcal{H})w_M(\mathcal{H}) < 0$. If the value $\dot{\lambda}_2(\mathcal{H})$ satisfies the inequality

$$\dot{\lambda}_2(\mathcal{H}) \geq \frac{D_M(\mathcal{G})}{\bar{C}} \frac{N}{N - \bar{C}}, \tag{7}$$

then the remaining giant component holds $C_M(\mathcal{H}) \geq \bar{C}$. $\diamond$

Proof. See Appendix A. $\square$

Theorem 9. Assume the giant remaining component $\mathcal{C}_M(\mathcal{G})$ of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $C_M(\mathcal{G}) > \bar{C}$, the graph $\mathcal{H} = (\mathcal{V}, \mathcal{E} \setminus \mathcal{E}_T)$. And also assume the component $\mathcal{C}_{m^-}$ such that $v_i(\mathcal{G})w_{m^-}(\mathcal{G}) < 0$ and $v_i(\mathcal{H})w_{m^-}(\mathcal{H}) < 0$ is not the giant component. If the graph $\mathcal{H}$ satisfies the inequality

$$\dot{\lambda}_2(\mathcal{H}) \geq \frac{D_M(\mathcal{G}) - d_*}{C_M(\mathcal{G}) - \bar{C}}, \tag{8}$$

where $d_* = D_{m^-}\bar{C}/C_{m^-}$, then $C_M(\mathcal{H}) \geq \bar{C}$. ◇

Proof. See Appendix A. □

From above discussions, the giant component size $C_M \geq \bar{C}$ will be preserved when either the conditions of Theorem 6, Theorem 8, or Theorem 9 are satisfied, and we employ them for the preservation control law. On the other hand, the preservation control may be conservative and then configuration of the system may be restricted, since the above all theorems indicate sufficient conditions to satisfy $C_M \geq \bar{C}$. Moreover, because Theorem 8 and Theorem 9 assume the case when the sign of $v_i w_M$ is invariant, they may not ensure the preservation when a tearing changes the sign of $v_i w_M$. In this study, we leave the non-preservation cases because the improvement control will recover the component size even when the component size goes below the threshold.

Here we discuss a necessary condition of the preservation control. From the definition of tearing, there is no tearing $\mathcal{E}_T$ with keeping the graph $\mathcal{H}$ connected when only one link exists from the node i to the remaining giant component $\mathcal{C}_M$, i.e. $|\mathcal{N}_i \cap \mathcal{C}_M| = 1$. This indicates that the remaining giant component size $\mathcal{C}_M$ with $|\mathcal{N}_i \cap \mathcal{C}_M| = 1$ will be preserved by a global connectivity control. Therefore, we design the giant component preserving control works only if the giant component $\mathcal{C}_M$ has two or more links $|\mathcal{N}_i \cap \mathcal{C}_M| \geq 2$, for avoiding to be conservative.

4. CONTROL STRATEGY

In this section, we provide a control law to improve and preserve the remaining giant component size C_M using the properties shown in the above section. The motion of each the robot $k \in \mathcal{V}$ is given by a single integrator dynamics as

$$\dot{p}_k(t) = u_k(t), \tag{9}$$

where $t \geq 0$ denotes continuous time, $\dot{p}_k$ denotes the time differential of the robot position dp_k/dt, and u_k denotes a control input. The control input u_k is given by

$$u_k = \kappa_1 u_k^{\text{task}} + \kappa_2 u_k^{\text{con}} + \kappa_3 u_k^{\text{rob}}, \tag{10}$$

where u_k^{task} denotes a control low for a team task, u_k^{con} denotes a global connectivity preserving control law introduced in Sabattini et al. (2011), and u_k^{rob} denotes the giant component robustification control law defined in this section. The parameters $\kappa_1, \kappa_2, \kappa_3 \geq 0$ are the control gains respectively.

The improvement control policy is to increase the value $\lambda_2^\varepsilon/\varepsilon$ for some link addition when $C_M^{(i)} < \bar{C}$. Therefore the improvement control law u_{ki}^{imp} for the robot k to improve $C_M^{(i)}$ is given as

$$u_{ki}^{\text{imp}} = \frac{\partial}{\partial p_k}\frac{\lambda_2^{(i)}(\varepsilon)}{\varepsilon} = \sum_{j \in \mathcal{N}_k} \frac{(v_j^{(i)}(\varepsilon) - v_j^{(i)}(\varepsilon))^2}{\varepsilon}\frac{\partial a_{kj}^{(i)}(\varepsilon)}{\partial p_k}. \tag{11}$$

Here we can see the right hand term satisfies

$$\frac{(v_j^{(i)}(\varepsilon) - v_k^{(i)}(\varepsilon))^2}{\varepsilon}\frac{\partial a_{kj}^{(i)}(\varepsilon)}{\partial p_k}$$

$$= \frac{\partial a_{kj}^{(i)}(\varepsilon)}{\partial p_k} \times \begin{cases} (v_j^{(i)}(\varepsilon) - v_k^{(i)}(\varepsilon))^2/\varepsilon, & \text{if } j \neq i \wedge k \neq i, \\ (v_j^{(i)}(\varepsilon) - v_k^{(i)}(\varepsilon))^2, & \text{if } j = i \vee k = i. \end{cases} \tag{12}$$

Therefore the robots k will approach towards the robot i, because it holds that $(v_k^{(i)}(\varepsilon) - v_i^{(i)}(\varepsilon))^2 > 0$ and

$$\frac{(v_k^{(i)}(\varepsilon) - v_j^{(i)}(\varepsilon))^2}{\varepsilon} \simeq 0, \text{if } a_{kj} \gg \varepsilon \wedge k \neq i \wedge j \neq i,$$

from Lemma 1. As the result, all the neighbors $k \in \mathcal{N}_i$ will be sufficiently close to get new links between them, and the remaining giant component size $C_M(i)$ will be improved.

We can enhance the improvement according to the idea of Ghedini et al. (2015). Assuming the communication is sufficiently fast such that each robots can send information towards 2-hop neighbors in a control period, the neighbor set $\mathcal{N}_k$ given in (11) can be replaced with $\mathcal{N}_k \cup \mathcal{N}_i$. As a result, the improvement motion will be enhanced since this control makes two non-adjacent (but 2-hop neighbors) robots approach each other.

Once the condition $C_M^{(i)} \geq \bar{C}$ is satisfied by the improvement control, then it will need to be preserved by the preservation control defined here. According to the discussion in Section 3, the preservation control law u_{ki}^{pre} for the robot k to preserve $C_M^{(i)} \geq \bar{C}$ is given as below

$$u_{ki}^{\text{pre}} = -\frac{\partial}{\partial p_k}\coth\left(\frac{\lambda_2^{(i)}(\varepsilon)}{\varepsilon} - \bar{\lambda}^{(i)}\right)$$

$$= -\text{csch}^2\left(\frac{\lambda_2^{(i)}(\varepsilon)}{\varepsilon} - \bar{\lambda}^{(i)}\right)$$

$$\times \sum_{j \in \mathcal{N}_k} \frac{(v_k^{(i)}(\varepsilon) - v_j^{(i)}(\varepsilon))^2}{\varepsilon}\frac{\partial a_{kj}^{(i)}(\varepsilon)}{\partial p_k}, \tag{13}$$

where

$$\bar{\lambda}^{(i)} = \begin{cases} \min\left(\dfrac{D^{(i)} - D_{m^-}^{(i)}}{N - 1 - \bar{C}}, \dfrac{D_{m^-}^{(i)}N}{\bar{C}(N - \bar{C})}\right), \\ \qquad\qquad \text{if } v_i^{(i)}(\varepsilon)w_M^{(i)}(\varepsilon) < 0, \\ \dfrac{D_M^{(i)} - \bar{C}D_{m^-}^{(i)}/C_{m^-}^{(i)}}{C_M^{(i)} - \bar{C}}, \\ \qquad\qquad \text{if } C_M(i) > \bar{C} \wedge v_i^{(i)}(\varepsilon)w_M^{(i)}(\varepsilon) > 0, \end{cases}$$

for $\bar{C} \in (N/2, N - 1)$, and $\coth(\cdot)$, $\text{csch}(\cdot)$ are hyperbolic cotangent and hyperbolic cosecant respectively. The hyperbolic function is introduced not to go below the threshold $\bar{\lambda}$ referring to Sabattini et al. (2011). Note that the robot k preserves only its neighbors' giant components, since our control method requires the component weights and the giant component size.

To summarize the methods discussed above, the robustification control low u_k^{rob} is given by

$$u_k^{\text{rob}} = \sum_{i \in \mathcal{N}_k^{\text{imp}}} u_{ki}^{\text{imp}} + \sum_{i \in \mathcal{N}_k^{\text{pre}}} u_{ki}^{\text{pre}}, \tag{14}$$

where $\mathcal{N}_k^{\text{imp}}$ and $\mathcal{N}_k^{\text{pre}}$ are modified neighbour sets defined by

$$\mathcal{N}_k^{\mathrm{imp}} = \{j \in \mathcal{N}_k \mid C_M^{(j)} < \bar{C}\},$$

$$\mathcal{N}_k^{\mathrm{pre}} = \{j \in \mathcal{N}_k \mid C_M^{(j)} \geq \bar{C} \wedge |\mathcal{N}_k \cap \mathcal{C}_M^{(j)}| \geq 2 \wedge \lambda_2^{(j)} > \varepsilon \bar{\lambda}^{(j)}\}.$$

As the definition of the sets $\mathcal{N}_k^{\mathrm{imp}}$ and $\mathcal{N}_k^{\mathrm{pre}}$ implies, our control method allows the existence of a robot which gets neither the improvement nor the preservation. The reason is that our control method can not guarantee the strict preservation because the method only tries to preserve the value $\dot{\lambda}_2^{(i)}$ in a distributed fashion. To avoid the conservativeness due to the sufficient conditions, we employ this control law and verify it in the next section.

5. NUMERICAL SIMULATION

Here we show numerical examples of the proposed control method. The task control input u_k^{task} is defined by the coverage control law in a non-convex environment introduced in Kantaros et al. (2015). Our objective in these simulations is to confirm the validity of our control law, and to confirm robustness of the generated network topologies of the multi-robot system.

We consider the multi-robot system with $N = 20$, and the system covers a target area shown in Fig. 3, and the initial configuration of the system in each cases are also shown there. The black dots represent the robots' positions, the blue lines are the communication links, and the colored area shows the coverage area of the system. Communication range is $r_{\max} = 1.7$ and the sensing radius is $c = 1.02$. The sensing radius is set to be greater than $0.5r_{\max}$ since the coverage area will be almost proportional to the connected component size when the sensing radius c is sufficiently small. The link weight function $w(x)$ is given by

$$w(x) = \begin{cases} 1 - \dfrac{1}{1 + \exp(-80x + 76r_{\max})}, & \text{if } x \leq r_{\max}, \\ 0, & \text{otherwise}, \end{cases}$$

The control period is set by 0.033[sec], and the simulations are executed from $t = 0$ to 165 seconds (5000 steps). The control gains are $\kappa_1 = 0.2, \kappa_2 = 0.5, \kappa_3 = 0.2$.

Figure 4 shows the coverage results on the area A. The color of the coverage area represents the giant component size $C_M^{(i)}$ of the each robot i (a robot with redder area has a smaller giant component). We can confirm that the giant component size of the system and the number of robots with a smaller giant component increase as the threshold $\bar{C}$ decreases, and the covered area also seems to increase as the threshold $\bar{C}$ decreases.

### 5.1 Evaluation of control method

We evaluate the giant component size of the resulting configuration in this subsection. Since strict preservation is difficult as discussed in the last section, and the time series results are unsteady, here we evaluate a moving time average of the giant component size, defined by

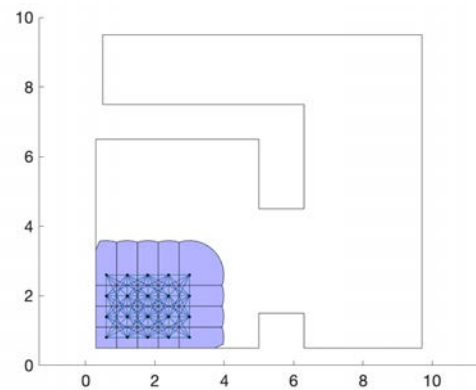$$\hat{C}_M(t) = \frac{1}{2T} \int_{t-T}^{t+T} \min_{i \in \mathcal{V}} C_M^{(i)}(\tau) d\tau,$$

where T denotes the window of the moving average and it is set as $T = 5$[sec]. Figure 5 shows the average giant component size in cases the system on the area A with
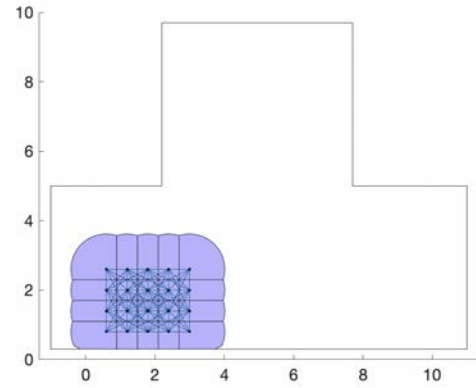
(a) Area A
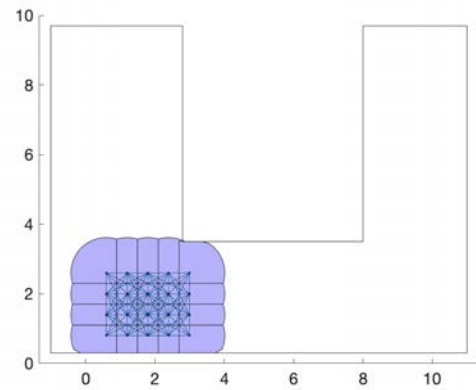
(b) Area B

(c) Area C

Fig. 3. Areas for coverage simulation and initial configurations

the thresholds $\bar{C} \in \{19, 16, 13, 10, 0\}$. We can see that the average $\hat{C}_M(t)$ is almost always greater than or equal to the threshold $\bar{C}$ in each cases. Although the component size temporally falls below the threshold in some cases, the value recovers thanks to the improvement control. Since the preservation control law does not consider the non-neighbors' component sizes, there is a possibility that the component size falls below the threshold.

5.2 Network robustness and coverage

Here we discuss the result configurations at the viewpoint of the tradeoff between the network robustness and the
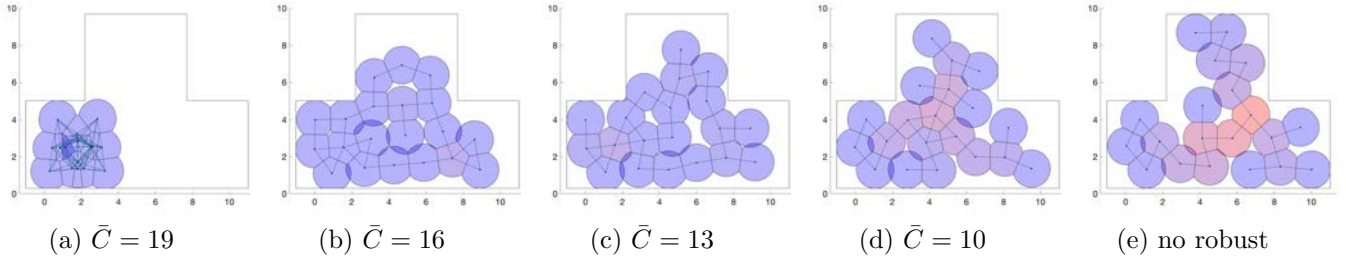
(a) $\bar{C} = 19$ (b) $\bar{C} = 16$ (c) $\bar{C} = 13$ (d) $\bar{C} = 10$ (e) no robust

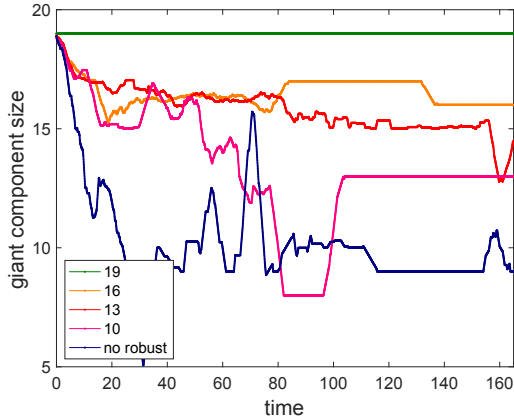Fig. 4. Configuration results on area A (at $t = 165$[sec]).

Fig. 5. Moving average of minimum giant component size $\min_i C_M^{(i)}$ for each thresholds $\bar{C}$ on area A.

coverage area. To evaluate the network robustness against robot failures, we define a robustness indicator by

$$R = \frac{2}{N} \sum_{n=1}^{N} E(\mathcal{G}, n),$$

where $E(\mathcal{G}, n)$ denotes an expectation node number of the giant component when n nodes are randomly removed from the graph $\mathcal{G}$. Clearly $E(\mathcal{G}, 1) = \sum_i C_M^{(i)}/N$ and we aim $E(\mathcal{G}, 1) \geq \bar{C}$ by the proposed control. The network robustness R includes an evaluation of the graph in cases of one or more robots failure. For example, a circle graph topology is robust against any one robot failure since $E(\mathcal{G}, 1) = N - 1$ is the maximum. However in cases when two robots fail, a circle graph is not so robust since $E(\mathcal{G}, 2) \simeq (3N - 5)/4$, this value is not so large comparing with the maximum $N - 2$. The evaluation $R(\mathcal{G})$ reflects such high number failures. The coverage area is defined by

$$H = \int_{\mathcal{A}} \chi(p) dp,$$

where $\mathcal{A}$ denotes a target area to cover by the robot's sensor range, and χ is an indicator function defined by

$$\chi = \begin{cases} 1, & \text{if } \exists i \in \mathcal{V}, \text{ s.t. } \|p - p_i\| \leq c, \\ 0, & \text{otherwise,} \end{cases}$$

where $c > 0$ denotes the sensor range. Simply speaking, H is the colored area in Figs. 3 and 4.

Figure 6 represents the simulation results on the (H, R) plane for the component size thresholds $\bar{C} \in \{16, 13, 10, 0\}$. We can see that the system with small robustness almost achieves large coverage as we expected. Because the component size preservation is not guaranteed as mentioned in

the last paragraph of Sec. 4, the result with $\bar{C} = 10$ in the area B (Fig. 6(b)) has smaller coverage and robustness than the result with $\bar{C} = 13$.

As the results of the Fig. 6 indicate, we may say that the convergence state $\lim_{t \to +\infty}(H(p), R(p))$ is in a local weak Pareto frontier in terms of the robot positions $p = (p_1, \ldots, p_N)$, since the coverage control law u^{task} achieves the local optimal coverage H. From multi-objective optimization literatures summarised in Marler and Arora (2004), we can state that the converged position $\lim_{t \to +\infty} p(t)$ approximately satisfies a local Pareto optimal of the vector valued objective function

$$f(p) = (H(p), -\coth(\lambda_2(p) - \epsilon), V(p)),$$

where

$$V(p) = \sum_{i \in \mathcal{V}^{\text{imp}}} \lambda_2^{(i)} - \sum_{i \in \mathcal{V}^{\text{pre}}} \coth(\dot{\lambda}_2^{(i)} - \bar{\lambda}^{(i)}),$$

and $\mathcal{V}^{\text{imp}} = \bigcup_{i \in \mathcal{V}} \mathcal{N}_i^{\text{pre}}$, $\mathcal{V}^{\text{pre}} = \bigcup_{i \in \mathcal{V}} \mathcal{N}_i^{\text{rob}}$, if the state $p_i(t)$ converges. This is because the control law $u = (u_1, \ldots, u_N)$ follows a gradient of the weighted sum $\kappa_1 H(p) - \kappa_2 \coth(\lambda_2(p) - \epsilon) + \kappa_3 V(p)$. Further analysis and discussion about the multi-objective optimality are left as future works.

6. CONCLUSION

In this paper, we propose a control method for robustness of multi-robot network by using the concept of the remaining giant component size. As the numerical results indicate, our concept and control method have a potential to perform one of the local optimal states considering a tradeoff between the network robustness and the coverage performance. In practical applications, the fragility threshold will be designed based on the possibility of each robots' failure in order to maximize the expected coverage performance.

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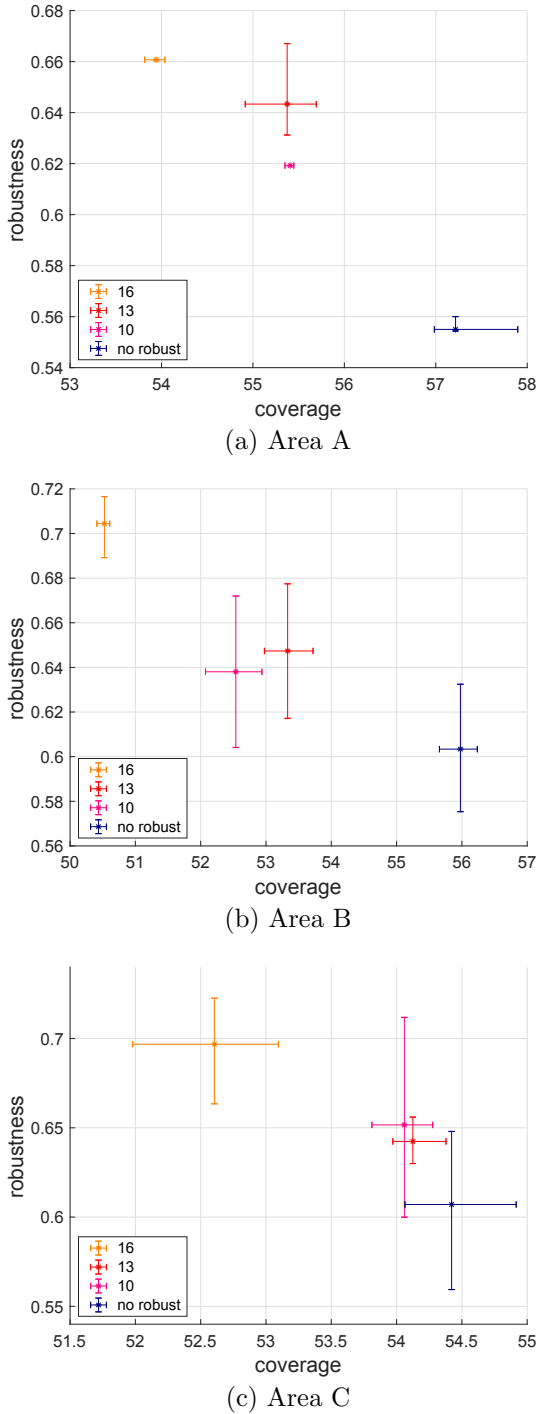
(a) Area A

(b) Area B

(c) Area C

Fig. 6. Tradeoff between Robustness R and coverage H. Star marker shows average in $t \in [140, 165]$ and error bar shows min-max range.

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Appendix A. PROOF OF THEOREM 8 AND 9

First we prove Theorem 8. Assume an articulation node i and consider a remaining component $m^- \in \{1, \ldots, M\}$ such that $v_i w_{m^-} < 0$. From the Fiedler vector constraint $v^T \mathbf{1} = 0$, we get

$$-C_{m^-} w_{m^-} = v_i + \sum_{m \neq m^-} C_m w_m \geq (N - C_{m^-})v_i,$$

using Lemma 4. Substituting it for (2), we find

$$\dot{\lambda}_2 = \frac{D_{m^-}}{C_{m^-}} \left(1 - \frac{v_i}{w_{m^-}} \right) \leq \frac{D_{m^-}}{C_{m^-}} \frac{N}{N - C_{m^-}}.$$

Under the condition (7), we get

$$\frac{D_{m^-}}{\bar{C}} \frac{N}{N - \bar{C}} \leq \dot{\lambda}_2 \leq \frac{D_{m^-}}{C_{m^-}} \frac{N}{N - C_{m^-}}.$$

Solving the inequality, we obtain $C_{m^-} \geq \bar{C}$ or $C_{m^-} \leq N - \bar{C}$. Since the value $\dot{\lambda}_2$ is continuous, it holds $4D_{m^-}/N < \dot{\lambda}_2$ which implies $C_{m^-} > N/2$. Therefore $C_{m^-} \leq N - \bar{C}$ is denied and $C_{m^-} \geq \bar{C}$ remains. □

Next we prove Theorem 9. From (5), we get $\dot{\lambda}_2 \leq D_{m^+}/C_{m^+}$ for an arbitrary component $\mathcal{C}_{m^+}$ with $v_i w_{m^+} > 0$. Thus, after a tearing $\mathcal{E}_T$, the value $\dot{\lambda}_2$ satisfies

$$\dot{\lambda}_2 \leq \frac{D_M(\mathcal{G}) - D_m}{C_M(\mathcal{G}) - C_m},$$

where m denotes a newly generated component by the tearing $\mathcal{E}_T$. We also find $D_{m^-}/C_{m^-} \leq \dot{\lambda}_2 \leq D_m/C_m$ from (5), the inequality

$$\frac{D_M(\mathcal{G}) - d_*}{C_M(\mathcal{G}) - \bar{C}} \leq \dot{\lambda}_2(\mathcal{H}) \leq \frac{D_M(\mathcal{G}) - D_m}{C_M(\mathcal{G}) - C_m} \leq \frac{D_M(\mathcal{G}) - d_*}{C_M(\mathcal{G}) - C_m}$$

holds and therefore $C_m(\mathcal{H}) \geq \bar{C}$. □