

# Input-to-state stability results w.r.t. global attractors of semi-linear reaction-diffusion equations<sup>★</sup>

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**Abstract:** We establish local input-to-state stability and asymptotic gain results for a class of nonlinear infinite-dimensional systems with respect to the global attractor of the respective undisturbed system. We apply our results to a large class of reaction-diffusion equations with nontrivial global attractors.

*Keywords:* input-to-state stability, asymptotic gain, global attractor, infinite-dimensional system, reaction-diffusion equation.

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## 1. INTRODUCTION

The notion of input-to-state stability (ISS) was introduced by Sontag (1989) for finite dimensional systems and is now recognized to be very fruitful in many applications of control and stability theory due to very well elaborated characterizations for ISS and related weaker notions, Lyapunov methods including small-gain results as well as their extensions to different classes of systems. In recent years, many of these extensions were developed for systems given in terms of partial differential equations (PDEs), see for example the works Dashkovskiy et al. (2013), Mironchenko (2017), Mironchenko (2019), Mironchenko et al. (2019), Karafyllis and Krstic (2019), Schmid et al. (2018), Schmid (2019).

ISS-like notions characterize how robust the asymptotic stability of a system is when a disturbing signal enters the system. In case of PDEs a disturbance can enter in domain and/or at the boundary of the domain on which the PDE is defined. The ISS framework for infinite-dimensional systems is not as well-developed as for the finite-dimensional case. In this work, we contribute some new results in this direction.

Here we consider nonlinear parabolic equations having a global attractor in the unperturbed case and study some ISS-like properties with respect to this attractor in case distributed perturbations enter to the system. In particular, we will show that under certain conditions on the nonlinearity in our disturbed reaction-diffusion equations,

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the local input-to-state stability and the asymptotic gain properties are satisfied with respect to this global attractor.

In this paper, we provide only sketches of proofs for our main results – complete proofs and more general results will be published elsewhere.

## 2. NOTATION

$R$  denotes the set of real numbers and  $R^+ := [0, \infty)$ . By  $L^2$  we denote the space of Lebesgue measurable and square integrable functions and  $L^\infty$  denotes the space of Lebesgue measurable essentially bounded functions.

The distance of  $x \in X$  to a set  $\Theta \subset X$  in a Banach space  $X$  is defined by

$$\|x\|_\Theta := \text{dist}(x, \Theta) := \inf_{\theta \in \Theta} \|x - \theta\| \quad (x \in X), \quad (1)$$

$$B_r(\Theta) := \{x \in X : \|x\|_\Theta < r\},$$

$$\overline{B}_r(\Theta) := \{x \in X : \|x\|_\Theta \leq r\},$$

as well as the notation

$$\text{dist}(M, \Theta) := \sup_{x \in M} \|x\|_\Theta$$

for subsets  $M, \Theta \subset X$ .

The following classes of comparison functions are used for the characterization of different stability properties, see, e.g., Sontag et al. (1996). We say that  $\gamma : [0, \infty) \rightarrow [0, \infty)$  is of class  $\mathcal{K}$  if it is continuous, strictly increasing and  $\gamma(0) = 0$ . If additionally the function  $\gamma$  is unbounded, then it said to be of the class  $\mathcal{K}_\infty$ . By  $\mathcal{L}$  we denote the set of continuous and strictly decreasing functions  $\alpha : [0, \infty) \rightarrow [0, \infty)$  such that  $\alpha(t) \rightarrow 0$  as  $t \rightarrow \infty$ . We say that a function  $\beta : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$  is of class

$\mathcal{KL}$  if it is continuous, with  $\beta(\cdot, t) \in \mathcal{K}$  for every  $t \in [0, \infty)$  and  $\beta(r, \cdot) \in \mathcal{L}$  for every  $r \in (0, \infty)$ .

Finally, the upper right-hand Dini derivatives will be denoted by

$$\bar{\partial}_t^+ v(t) := \limsup_{\tau \rightarrow 0^+} \frac{v(t+\tau) - v(t)}{\tau}.$$

### 3. CONTRIBUTIONS

In this paper, we are concerned with disturbed nonlinear reaction-diffusion equations of the form

$$\begin{aligned} \partial_t y(t, \zeta) &= \Delta y(t, \zeta) + g(y(t, \zeta)) + h(\zeta)u(t) & (\zeta \in \Omega) \\ y(t, \zeta) &= 0 & (\zeta \in \partial\Omega) \end{aligned} \quad (2)$$

on a bounded domain  $\Omega \subset R^n$  with smooth boundary  $\partial\Omega$ , where  $g \in C^1(R)$  and  $h \in L^2(\Omega)$  and the disturbance  $u$  belongs to  $\mathcal{U} := L^\infty([0, \infty))$ . It is well-known (Robinson (2001)) that the corresponding undisturbed equation

$$\begin{aligned} \partial_t y(t, \zeta) &= \Delta y(t, \zeta) + g(y(t, \zeta)) & (\zeta \in \Omega) \\ y(t, \zeta) &= 0 & (\zeta \in \partial\Omega) \end{aligned} \quad (3)$$

has a global attractor  $\Theta \subset X := L^2(\Omega)$  under suitable growth and upper-boundedness conditions on the nonlinearity  $g$  and its derivative  $g'$  respectively, see (7) below. As usual, a global attractor for (3) is defined to be a compact subset of  $X$  that is invariant and uniformly attracting w.r.t. the semiflow generated by (3).

What we show in this paper is that the disturbed reaction-diffusion equations (2) are locally input-to-state stable and of asymptotic gain w.r.t. the global attractor  $\Theta$  of the undisturbed equation (3).

The local input-to-state stability means that there exist comparison functions  $\beta \in \mathcal{KL}$ ,  $\gamma \in \mathcal{K}$  and radii  $r_{0x}, r_{0u} > 0$  such that

$$\|y(t, y_0, u)\|_\Theta \leq \beta(\|y_0\|_\Theta, t) + \gamma(\|u\|_\infty) \quad (t \in [0, \infty)) \quad (4)$$

for every  $(y_0, u) \in X \times \mathcal{U}$ ,  $\|y_0\|_\Theta \leq r_{0x}$ ,  $\|u\|_\infty \leq r_{0u}$ , and the asymptotic gain property means that there exists a comparison function  $\gamma \in \mathcal{K}$  such that

$$\limsup_{t \rightarrow \infty} \|y(t, y_0, u)\|_\Theta \leq \gamma(\|u\|_\infty) \quad (5)$$

for every  $(y_0, u) \in X \times \mathcal{U}$ .

In the relations (4) and (5) above,  $y = y(\cdot, y_0, u)$  denotes the global weak solution of (2) with initial value  $y_0$  and disturbance  $u \in \mathcal{U}$ .

As far as we know, our results are essentially the first input-to-state stability results w.r.t. global attractors of infinite-dimensional systems like (2). All input-to-state stability results for concrete PDE systems – like those from Dashkovskiy et al. (2013), Jacob et al. (2018), Jacob et al. (2018), Karafyllis et al. (2016), Karafyllis et al. (2017), Mazenc et al. (2011), Mironchenko et al. (2019), Mironchenko (2019), Schmid et al. (2018), Tanwani et al. (2017), Zheng et al. (2017), Zheng et al. (2017) – establish either theoretical characterization of input-to-state stability in terms of abstract conditions or input-to-state stability only w.r.t. an equilibrium point of the respective undisturbed system. In particular, the results from those papers do not cover the Chaffee–Infante equation, for

example, that is, the reaction-diffusion equation (2) with nonlinearity  $g$  given by

$$g(r) := -r^3 + \lambda r \quad (r \in R),$$

just because the respective undisturbed system only has a non-singleton attractor  $\Theta \neq \{0\}$  (Section 11.5 of Robinson (2001)). With our results, by contrast, we can cover the Chaffee–Infante equation and many more nonlinearities. We refer to Kapustyan et al. (2015), Gorban et al. (2014), Gorban et al. (2015), Dashkovskiy et al. (2017) for other interesting results about non-trivial global attractors of nonlinear, impulsive, or even multi-valued semigroups.

### 4. A LOCAL INPUT-TO-STATE STABILITY RESULT

To establish our main results we have to consider the family of problems (2) with initial data  $(s, y_s) \in R^+ \times X$

$$\begin{aligned} \partial_t y(t, \zeta) &= \Delta y(t, \zeta) + g(y(t, \zeta)) + h(\zeta)u(t), & t > s, \zeta \in \Omega \\ y(t, \cdot)|_{\partial\Omega} &= 0 & \text{and } y(s, \cdot) = y_s. \end{aligned} \quad (6)$$

Suppose that for some  $p \in [2, \infty)$ ,  $q \in (1, 2]$ ,  $1/p + 1/q = 1$  there exist constants  $\alpha_1, \alpha_2, \kappa, \lambda \in (0, \infty)$  such that for every  $r \in R$

$$-\kappa - \alpha_1|r|^p \leq g(r)r \leq \kappa - \alpha_2|r|^p, \quad g'(r) \leq \lambda. \quad (7)$$

A simple class of functions  $g$  satisfying the three inequalities from (7) is given by the polynomials of odd degree with negative leading coefficient:

$$g(r) = \sum_{i=0}^{2m-1} c_i r^i \quad (r \in R)$$

with  $c_{2m-1} < 0$ , where  $m \in N$ . (Choose  $p := 2m$ .) In particular, the nonlinearity of the Chaffee–Infante equation given by  $g(r) := -r^3 + \lambda r$  falls into that class.

Suppose that (7) is satisfied and let  $s \in R^+$ ,  $y_s \in X$ . A function  $y \in C([s, \infty), X)$  is called a *global weak solution* of (6) iff  $y(s) = y_s$  and for every  $T \in (s, \infty)$  one has  $y|_{[s, T]} \in L^2([s, T], H_0^1(\Omega)) \cap L^p([s, T], L^p(\Omega))$  and there exists a (then unique)  $z \in L^2([s, T], H_0^1(\Omega)^*) + L^q([s, T], L^q(\Omega))$  such that

$$\begin{aligned} \int_s^T \langle z(t), \varphi(t) \rangle dt &= - \int_s^T \int_\Omega \nabla y(t)(\zeta) \cdot \nabla \varphi(t)(\zeta) d\zeta dt + \\ &+ \int_s^T \int_\Omega g(y(t)(\zeta)) \varphi(t)(\zeta) d\zeta dt + \int_s^T \int_\Omega h(\zeta)u(t)\varphi(t)(\zeta) d\zeta dt \end{aligned} \quad (8)$$

for every  $\varphi \in L^2([s, T], H_0^1(\Omega)) \cap L^p([s, T], L^p(\Omega))$ , where  $\langle \cdot, \cdot \rangle$  stands for the dual pairing of  $H_0^1(\Omega)^* + L^q(\Omega)$  and  $H_0^1(\Omega) \cap L^p(\Omega)$ .

It is known (Chepyzhov et al. (2002)) that for every  $s \in R^+$ ,  $y_s \in X$  and  $u \in L_{loc}^2([0, \infty))$  the initial boundary value problem (6) has a unique global weak solution  $y(\cdot) = y(\cdot, s, y_s, u)$ .

A *semiprocess family* on  $X$  is a family of maps  $S_u : R_d^2 \times X \rightarrow X$  for every  $u \in \mathcal{U}$  such that

$$\begin{aligned} S_u(s, s, x) &= x, \quad S_u(t, s, S_u(s, r, x)) = S_u(t, r, x), \\ S_u(t + \tau, s + \tau, x) &= S_u(\cdot, \cdot, x) \end{aligned} \quad (9)$$

for all  $(t, s), (s, r) \in R_d^2$ ,  $\tau \geq 0$ ,  $x \in X$  and  $u \in \mathcal{U}$ , where we used the abbreviation  $R_d^2 := \{(s, t) \in R^+ \times R^+ : t \geq s\}$ .

See Chepyzhov et al. (2002) for more information on semiproduct families.

*Lemma 1.* Suppose that (7) is satisfied. Then  $(S_u)_{u \in \mathcal{U}}$  defined by

$$S_u(t, s, y_s) := y(t, s, y_s, u) \quad (10)$$

is a semiproduct family on  $X$  and, additionally,

$$\|S_0(t, 0, y_{01}) - S_0(t, 0, y_{02})\| \leq e^{\lambda t} \|y_{01} - y_{02}\|, \quad t \in R^+ \quad (11)$$

$$\|S_u(t, 0, y_0) - S_0(t, 0, y_0)\| \leq 2e^{2\lambda} \|h\| \|u\|_\infty t, \quad t \in [0, 1] \quad (12)$$

for all  $y_0, y_{01}, y_{02} \in X$  and all  $u \in \mathcal{U}$ .

We remark for later reference that our semiproduct family  $(S_u)_{u \in \mathcal{U}}$ , like any other semiproduct family, satisfies the following so-called cocycle property:

$$S_u(t + \tau, 0, x) = S_{u(\cdot + \tau)}(t, 0, S_u(\tau, 0, x)) \quad (13)$$

for all  $t, \tau \in R^+$ ,  $x \in X$  and  $u \in \mathcal{U}$ . In particular,  $S_0$  satisfies the semigroup property

$$S_0(t + \tau, 0, x) = S_0(t, 0, S_0(\tau, 0, x)), \quad t, \tau \in R^+, \quad x \in X. \quad (14)$$

A *global attractor* of semigroup  $S_0$  is a compact set  $\Theta \subset X$  such that

- (i)  $\Theta$  is invariant, i.e.,  $S_0(t, 0, \Theta) = \Theta$ ,  $t \in R^+$ ;
- (ii)  $\Theta$  is uniformly attracting, i.e., for every bounded  $B \subset X$  one has

$$\text{dist}(S_0(t, 0, B), \Theta) = \sup_{x \in B} \|S_0(t, 0, x)\|_\Theta \rightarrow 0, \quad t \rightarrow \infty.$$

It directly follows from this definition that a global attractor of  $S_0$  is minimal among all closed uniformly attracting sets of  $S_0$  and maximal among all bounded invariant sets of  $S_0$ . And from this fact, in turn, it immediately follows that if  $S_0$  has any global attractor then it is already unique.

It is well-known Robinson (2001) that under conditions (7) the semigroup  $S_0$  generated by (3) possesses the global attractor  $\Theta$ . Moreover, it is connected stable subset of  $X$  and the following structural formula holds

$$\Theta = \{x_0 \in X \mid x_0 \text{ lies on a complete orbit } x(\cdot)\}$$

$$\text{such that } \text{dist}(x(t), \mathfrak{R}) \rightarrow 0 \text{ as } t \rightarrow -\infty\},$$

where  $\mathfrak{R}$  is the set of equilibria, that is, solutions of the stationary problem

$$\begin{aligned} \Delta y(\zeta) + g(y(\zeta)) &= 0 & (\zeta \in \Omega) \\ y(\zeta) &= 0 & (\zeta \in \partial\Omega) \end{aligned}$$

The set  $\Theta$  can be complicated even in the simplest cases. For the particular example Chafee-Infante equation with  $g(r) = -r^3 + \lambda r$ ,  $\Omega = (0, \pi)$  and  $n^2 < \lambda < (n + 1)^2$  the set  $\mathfrak{R}$  consists of  $2n + 1$  equilibria and the dimension of  $\Theta$  equals  $n$ .

It is worth to be mention that in this case there is a natural Lyapunov function

$$V(x) = \int_0^\pi \left( \frac{1}{2} (x'(s))^2 + \frac{1}{4} x^4(s) - \frac{\lambda}{2} x^2(s) \right) ds$$

But it is well-defined only in the phase space  $H_0^1(0, \pi)$ . So it cannot be useful for proving ISS property in the space  $X = L^2(0, \pi)$ . Moreover, even in the space  $H_0^1(0, \pi)$  it can serve as a ISS Lyapunov function only in the case  $\lambda = 0$  when  $\Theta = \{0\}$  (see Dashkovskiy et al. (2013)).

*Theorem 2.* Suppose that (7) is satisfied and let  $\Theta$  be the global attractor of the undisturbed system  $S_0$ . Then the disturbed system  $(S_u)_{u \in \mathcal{U}}$  is locally input-to-state stable w.r.t.  $\Theta$ , that is, there exist comparison functions  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}$  and radii  $r_{0x}, r_{0u} > 0$  such that

$$\|S_u(t, 0, x_0)\|_\Theta \leq \beta(\|x_0\|_\Theta, t) + \gamma(\|u\|_\infty), \quad t \in R^+ \quad (15)$$

for all  $(x_0, u) \in X \times \mathcal{U}$  with  $\|x_0\|_\Theta \leq r_{0x}$  and  $\|u\|_\infty \leq r_{0u}$ .

**Proof.** We give here only the key steps of the proof.

*Step 1.* The global attractor  $\Theta$  is uniformly globally asymptotically stable for  $S_0$ , that is, there exists a comparison function  $\beta_0 \in \mathcal{KL}$  such that

$$\|S_0(t, 0, x)\|_\Theta \leq \beta_0(\|x\|_\Theta, t) \quad t \in R^+, \quad x \in X. \quad (16)$$

Indeed, it immediately follows from the invariance of  $\Theta$  under  $S_0$  and from the estimate (11) that for every  $\epsilon > 0$  and every  $T \in (0, \infty)$  there exists a  $\delta \in (0, 1]$  such that for every  $t \in [0, T]$  and  $x \in B_\delta(\Theta)$

$$\|S_0(t, 0, x)\|_\Theta \leq \inf_{\theta \in \Theta} \|S_0(t, 0, x) - S_0(t, 0, \theta)\| < \epsilon.$$

And from this and the uniform attractivity of  $\Theta$  for  $S_0$ , in turn, it follows that for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that

$$\|S_0(t, 0, x)\|_\Theta < \epsilon \quad (t \in R^+) \quad (17)$$

for every  $x \in B_\delta(\Theta)$ .

Also, it is well-known that

$$\|S_0(t, 0, x)\|^2 \leq e^{-2\omega t} \|x\|^2 + \frac{\lambda|\Omega|}{\omega} \quad (t \in R^+), \quad (18)$$

where  $\omega > 0$  is the smallest eigenvalue of  $-\Delta$  in  $H_0^1(\Omega)$ . Since

$$\|S_0(t, 0, x)\|_\Theta \leq \|S_0(t, 0, x)\| + \|\Theta\|$$

and

$$\|x\| \leq \|x\|_\Theta + \|\Theta\|$$

with  $\|\Theta\| := \sup_{\theta \in \Theta} \|\theta\|$ , it follows from (18) that there exists a comparison function  $\sigma \in \mathcal{K}$  and a constant  $c \in (0, \infty)$  such that

$$\|S_0(t, 0, x)\|_\Theta \leq \sigma(\|x\|_\Theta) + c \quad (t \in R^+) \quad (19)$$

for every  $x \in X$ . In the terminology of Mironchenko (2017), the relations (17) and (19) mean that  $\Theta$  is uniformly locally stable and Lagrange-stable for  $(S_u)_{u \in \mathcal{U}_0}$ , respectively. And therefore,  $\Theta$  is uniformly globally stable for  $(S_u)_{u \in \mathcal{U}_0} = S_0$  by virtue of Remark 2.9 of Mironchenko (2017), as desired.

*Step 2.* The undisturbed system (3) has a local Lyapunov function, that is, for every  $r_0 > 0$  there exists a Lipschitz continuous function

$$V : \bar{B}_{r_0}(\Theta) \rightarrow R^+$$

with Lipschitz constant 1 and comparison functions  $\underline{\psi}, \bar{\psi}, \alpha, \sigma \in \mathcal{K}_\infty$  such that for every  $x \in B_{r_0}(\Theta)$ ,  $u \in \mathcal{U}$

$$\underline{\psi}(\|x\|_\Theta) \leq V(x) \leq \bar{\psi}(\|x\|_\Theta), \quad (20)$$

$$\begin{aligned} \dot{V}_u(x) &:= \limsup_{t \rightarrow 0^+} \frac{1}{t} (V(S_u(t, 0, x)) - V(x)) \\ &\leq -\alpha(\|x\|_\Theta) + \sigma(\|u\|_\infty). \end{aligned} \quad (21)$$

We prove this fact using function  $\beta_0$  from (16) and arguments from Henry (1981) (Theorem 4.2.1). More precisely, we can prove properties (20),(21) for the function

$$V(x) := \sum_{k=1}^{\infty} 2^{-k} V^{1/k}(x) \quad (x \in \bar{B}_{r_0}(\Theta)),$$

where for every given  $\epsilon > 0$

$$V^\epsilon(x) := e^{-(\lambda+c_0)T(\epsilon)} \sup_{t \in [0, \infty)} \left( e^{c_0 t} \eta_\epsilon(\|S_0(t, 0, x)\|_\Theta) \right),$$

$c_0 \in (0, \infty)$  is an arbitrary constant,  $\eta_\epsilon(r) := \max\{0, r - \epsilon\}$  and  $T(\epsilon)$  be a time such that

$$\beta_0(r_0, t) \leq \epsilon \quad (t \in [T(\epsilon), \infty)).$$

Indeed,  $V^\epsilon : \bar{B}_{r_0}(\Theta) \rightarrow R_0^+$  is a well-defined map (with finite values) and

$$V^\epsilon(x) \leq e^{-\lambda T(\epsilon)} \sup_{t \in [0, T(\epsilon)]} \left( \eta_\epsilon(\|S_0(t, 0, x)\|_\Theta) \right) \leq \beta_0(\|x\|_\Theta, 0) \quad (x \in \bar{B}_{r_0}(\Theta)), \quad (22)$$

$$\begin{aligned} |V^\epsilon(x) - V^\epsilon(y)| &\leq e^{-(\lambda+c_0)T(\epsilon)} \\ &\times \sup_{t \in [0, T(\epsilon)]} \left| e^{c_0 t} \eta_\epsilon(\|S_0(t, 0, x)\|_\Theta) \right. \\ &\quad \left. - e^{c_0 t} \eta_\epsilon(\|S_0(t, 0, y)\|_\Theta) \right| \\ &\leq e^{-\lambda T(\epsilon)} \sup_{t \in [0, T(\epsilon)]} \left| \|S_0(t, 0, x)\|_\Theta - \|S_0(t, 0, y)\|_\Theta \right| \\ &\leq e^{-\lambda T(\epsilon)} \sup_{t \in [0, T(\epsilon)]} \|S_0(t, 0, x) - S_0(t, 0, y)\| \leq \|x - y\| \\ &\quad (x, y \in \bar{B}_{r_0}(\Theta)). \end{aligned} \quad (23)$$

Additionally, for every  $x \in B_{r_0}(\Theta)$ , we have  $S_0(\tau, 0, x) \in B_{r_0}(\Theta)$  for  $\tau$  small enough and thus, by the semigroup property,

$$\begin{aligned} V^\epsilon(S_0(\tau, 0, x)) &= e^{-(\lambda+c_0)T(\epsilon)} \sup_{t \in [0, \infty)} \left( e^{c_0 t} \eta_\epsilon(\|S_0(t + \tau, 0, x)\|_\Theta) \right) \\ &\leq e^{-c_0 \tau} V^\epsilon(x) \end{aligned} \quad (24)$$

for every  $x \in B_{r_0}(\Theta)$  and all sufficiently small times  $\tau$ . Consequently,

$$\dot{V}_0^\epsilon(x) = \lim_{\tau \rightarrow 0^+} \frac{1}{\tau} (V^\epsilon(S_0(\tau, 0, x)) - V^\epsilon(x)) \leq -c_0 V^\epsilon(x) \quad (25)$$

for  $x \in B_{r_0}(\Theta)$ . So for function  $V$  we conclude from (23), (24), (25) that for  $x, y \in \bar{B}_{r_0}(\Theta)$

$$V(x) \leq \beta_0(\|x\|_\Theta, 0), \quad x \in \bar{B}_{r_0}(\Theta), \quad (26)$$

$$|V(x) - V(y)| \leq \sum_{k=1}^{\infty} 2^{-k} |V^{1/k}(x) - V^{1/k}(y)| \leq \|x - y\| \quad (27)$$

$$\dot{V}_0(x) \leq \sum_{k=1}^{\infty} 2^{-k} \dot{V}_0^{1/k}(x) \leq -c_0 V(x), \quad x \in B_{r_0}(\Theta). \quad (28)$$

Since  $\sup_{t \in [0, \infty)} (e^{c_0 t} \eta_{1/k}(\|S_0(t, 0, x)\|_\Theta)) \geq \eta_{1/k}(\|x\|_\Theta)$  for all  $x \in X$ , we also conclude that

$$V(x) \geq \sum_{k=1}^{\infty} 2^{-k} e^{-(\lambda+c_0)T(1/k)} \eta_{1/k}(\|x\|_\Theta), \quad x \in \bar{B}_{r_0}(\Theta). \quad (29)$$

In view of these estimates, we now define the comparison functions  $\bar{\psi}$ ,  $\underline{\psi}$  and  $\alpha$  in the following way:

$$\bar{\psi}(r) := \beta_0(r, 0) + r, \quad \underline{\psi}(r) := \sum_{k=1}^{\infty} 2^{-k} e^{-(\lambda+c_0)T(1/k)} \eta_{1/k}(r)$$

and  $\alpha(r) := c_0 \psi(r)$  for  $r \in R_0^+$ . It is easy to verify that  $\bar{\psi}$ ,  $\underline{\psi}$  and hence  $\alpha$  belong to  $\mathcal{K}_\infty$ .

And, moreover, by virtue of (26), (27), (28), (29), we obtain estimate (20) and

$$\dot{V}_0(x) \leq -\alpha(\|x\|_\Theta) \quad (x \in B_{r_0}(\Theta)).$$

After that, defining  $\sigma \in \mathcal{K}_\infty$  by  $\sigma(r) := 2e^{2\lambda} \|h\| r$  for all  $r \in R_0^+$ , we can see that for every  $x \in B_{r_0}(\Theta)$  and every  $u \in \mathcal{U}$  we have

$$\begin{aligned} \dot{V}_u(x) &\leq \overline{\lim}_{\tau \rightarrow 0^+} \frac{1}{t} (V(S_0(t, 0, x)) - V(x)) \\ &\quad + \overline{\lim}_{\tau \rightarrow 0^+} \frac{1}{t} (V(S_u(t, 0, x)) - V(S_0(t, 0, x))) \\ &\leq -\alpha(\|x\|_\Theta) + \overline{\lim}_{\tau \rightarrow 0^+} \frac{1}{t} \|S_u(t, 0, x) - S_0(t, 0, x)\| \\ &\leq -\alpha(\|x\|_\Theta) + \sigma(\|u\|_\infty), \end{aligned}$$

so (21) holds.

*Step 3.* For  $\chi(r) := \alpha^{-1}(2\sigma(r))$  we have that for all  $(x_0, u) \in X \times \mathcal{U}$  with  $r_0 \geq \|x_0\|_\Theta \geq \chi(\|u\|_\infty)$

$$\dot{V}_u(x_0) \leq -\alpha(\|x_0\|_\Theta). \quad (30)$$

According to the comparison lemma from Mironchenko et al. (2016) (Corollary 1), we can then choose a comparison function  $\bar{\beta}$  in such a way that for every  $T \in (0, \infty]$  and every function  $v \in C([0, T], R^+)$  with

$$\bar{\partial}_t^+ v(t) \leq -(\alpha \circ \bar{\psi}^{-1})(v(t)) \quad (t \in [0, T])$$

one has  $v(t) \leq \bar{\beta}(v(0), t)$  for all  $t \in [0, T]$ . We now define

$$\beta(r, t) := \underline{\psi}^{-1}(\bar{\beta}(\bar{\psi}(r), t)), \quad \gamma(r) := \underline{\psi}^{-1}(\bar{\psi}(\chi(r))) \quad (31)$$

for  $r, t \in R^+$  and choose  $r_{0x}, r_{0u} \in (0, \infty)$  so small that

$$r_{0x} < r_0, \quad \beta(r_{0x}, 0) < r_0, \quad \gamma(r_{0u}) < r_0. \quad (32)$$

Also, we will write

$$M_u := \{x \in \bar{B}_{r_0}(\Theta) : V(x) \leq \bar{\psi}(\chi(\|u\|_\infty))\} \quad (33)$$

for  $u \in \mathcal{U}$ . After that, treating the case  $x_0 \in M_u$  and the case  $x_0 \notin M_u$  separately, we can prove that with the functions  $\beta$  and  $\gamma$  from (31), the desired estimate (15) is satisfied.

## 5. AN ASYMPTOTIC GAIN RESULT

To prove our asymptotic gain result we need some additional constructions from non-autonomous systems theory.

Suppose  $(S_v)_{v \in \mathcal{V}}$  is a semiprocess family on  $X$ , where  $\mathcal{V}$  is a translation-invariant subset of  $L_{loc}^2(R^+)$ , that is, for every  $u \in \mathcal{V}$ ,  $h \in R^+$  one has  $u(\cdot + h) \in \mathcal{V}$ . We will denote by  $S_{\mathcal{V}}$  the corresponding *set-valued semiprocess* which is defined by

$$S_{\mathcal{V}}(t, s, x) := \{S_v(t, s, x) : v \in \mathcal{V}\}. \quad (34)$$

A subset  $\Theta_{\mathcal{V}} \subset X$  is then called a *global attractor for the set-valued semiprocess  $S_{\mathcal{V}}$*  iff  $\Theta_{\mathcal{V}}$  is compact and the following conditions are satisfied:

- (i)  $\Theta_{\mathcal{V}}$  is *uniformly attracting for  $S_{\mathcal{V}}$* , that is, for every bounded subset  $B \subset X$  one has
 
$$\text{dist}(S_{\mathcal{V}}(t, 0, B), \Theta_{\mathcal{V}}) \rightarrow 0 \quad (t \rightarrow \infty) \quad (35)$$

(ii)  $\Theta_{\mathcal{V}}$  is negatively invariant under  $S_{\mathcal{V}}$ , that is,

$$\Theta_{\mathcal{V}} \subset S_{\mathcal{V}}(t, 0, \Theta_{\mathcal{V}}) \quad (t \in \mathbb{R}^+). \quad (36)$$

It should be noted that if  $\mathcal{V} = \{0\}$ , then the global attractor of the set-valued semiprocess  $S_{\{0\}}$  coincides with the global attractor of the semigroup  $S_0$ .

We choose, for given  $u \in \mathcal{U}$ ,

$$\mathcal{V}(u) := \overline{\{u(\cdot + h) : h \in \mathbb{R}^+\}} \subset L_{\text{loc}}^2(\mathbb{R}^+), \quad (37)$$

where the closure is w.r.t. the weak topology of the locally convex space  $L_{\text{loc}}^2(\mathbb{R}^+)$ . It is well-known (Chepyzhov et al. (2002)) that  $\mathcal{V}(u)$  is metrizable, compact, translation-invariant,  $u \in \mathcal{V}(u)$ ,  $\mathcal{V}(0) = \{0\}$  and

$$\int_s^t \|v(s)\|^2 ds \leq (t-s)\|u\|_{\infty}^2 \quad (v \in \mathcal{V}(u)). \quad (38)$$

*Theorem 3.* Suppose that (7) is satisfied and let  $\Theta$  be the global attractor of the undisturbed system  $S_0$ . Then the disturbed system  $(S_u)_{u \in \mathcal{U}}$  has the asymptotic gain property w.r.t.  $\Theta$ , that is, there exists a comparison function  $\gamma \in \mathcal{K}$  such that

$$\limsup_{t \rightarrow \infty} \|S_u(t, 0, x_0)\|_{\Theta} \leq \gamma(\|u\|_{\infty}) \quad (39)$$

for all  $(x_0, u) \in X \times \mathcal{U}$ .

**Proof.** We give here only a rough sketch of the proof.

*Step 1.* Using (38) we prove, that there exist continuous monotonically increasing functions  $\sigma, \gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that

$$\|S_v(t, 0, x_0)\| \leq e^{-\omega t} \sigma(\|x_0\|) + \gamma(\|u\|_{\infty}), \quad t \in [0, \infty) \quad (40)$$

for all  $(x_0, v) \in X \times \mathcal{V}(u)$  and all  $u \in \mathcal{U}$ .

*Step 2.* Using the compactness of the semigroup generated by our parabolic problem (see Lemma 15 of Valero et al. (2006)), we prove that whenever  $x_n \rightarrow x$  weakly in  $X$  and  $v_n \rightarrow v$  weakly in  $L_{\text{loc}}^2(\mathbb{R}^+)$  for some  $x_n, x \in X$  and  $v_n \in \mathcal{V}(u_n), v \in \mathcal{V}(u)$  and  $u_n, u \in \mathcal{U}$ , one has the strong convergence

$$S_{v_n}(t_0, 0, x_n) \rightarrow S_v(t_0, 0, x) \quad (n \rightarrow \infty) \quad (41)$$

for every  $t_0 \in (0, \infty)$ .

*Step 3.* With the results of the previous steps at hand, we prove that the global weak solutions of the problems (6) for every  $u \in \mathcal{U}$  generate a semiprocess family  $(S_v)_{v \in \mathcal{V}(u)}$ , the corresponding set-valued semiprocess  $S_{\mathcal{V}(u)}$  has a global attractor  $\Theta_{\mathcal{V}(u)}$  and

$$\text{dist}(\Theta_{\mathcal{V}(u)}, \Theta) \rightarrow 0 \quad (u \rightarrow 0). \quad (42)$$

*Step 4.* We observe that

$$\begin{aligned} \|S_u(t, 0, x_0)\|_{\Theta} &= \inf\{\|S_u(t, 0, x_0) - \theta\| : \theta \in \Theta\} \\ &\leq \|S_u(t, 0, x_0) - \theta_u\| + \text{dist}(\Theta_{\mathcal{V}(u)}, \Theta) \end{aligned}$$

for every  $\theta_u \in \Theta_{\mathcal{V}(u)}$ . So, taking the infimum over  $\theta_u \in \Theta_{\mathcal{V}(u)}$  and using  $S_u(t, 0, x_0) \in S_{\mathcal{V}(u)}(t, 0, x_0)$ , we see that

$$\|S_u(t, 0, x_0)\|_{\Theta} \leq \text{dist}(S_{\mathcal{V}(u)}(t, 0, x_0), \Theta_{\mathcal{V}(u)}) + \text{dist}(\Theta_{\mathcal{V}(u)}, \Theta) \quad (43)$$

for every  $(x_0, u) \in X \times \mathcal{U}$  and  $t \in \mathbb{R}^+$ . Since  $\Theta_{\mathcal{V}(u)}$  is a global attractor for  $S_{\mathcal{V}(u)}$ , we conclude that

$$\limsup_{t \rightarrow \infty} \text{dist}(S_{\mathcal{V}(u)}(t, 0, x_0), \Theta_{\mathcal{V}(u)}) = 0 \quad (44)$$

for every  $(x_0, u) \in X \times \mathcal{U}$ . Since, moreover,  $u \mapsto \Theta_{\mathcal{V}(u)}$  is upper semicontinuous at 0, we further conclude that there exists a  $\gamma \in \mathcal{K}$  such that

$$\text{dist}(\Theta_{\mathcal{V}(u)}, \Theta) \leq \gamma(\|u\|_{\infty}) \quad (u \in \mathcal{U}). \quad (45)$$

Combining now (43), (44), (45), we obtain the claimed asymptotic gain property.

## 6. CONCLUSION

For a class of nonlinear parabolic equations having a global attractor in the unperturbed case we have provided estimates about deviations of the solutions for the case of disturbances entering to the system. These estimates can be further used for the applications of small-gain theorems in case interconnections of such systems need to be analyzed in view of existence and robust stability of the corresponding global attractor.

The results can be useful in applications involving modeling of nonlinear diffusion or heat propagation processes including the cases where instead of a global asymptotic stability property the system possess a nontrivial global attractor.

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