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Algebraic tests for the asymptotic stability of parametric linear systems ⋆

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Abstract: In this paper, algebraic tests are proposed to establish asymptotic stability of parametric linear systems assuming that the parameters lie in a given variety. In particular, by using tools borrowed from algebraic geometry, necessary and sufficient condition are proposed to test whether a continuous-time or a discrete-time linear parametric system is asymptotically stable for all the specializations belonging to a given variety. Both the cases of zero dimensional and non-zero dimensional varieties are considered.

Keywords: Linear systems, Asymptotic stability, Algebraic tests.

1. INTRODUCTION

In the literature, several tests have been proposed to test the asymptotic stability of linear, time-invariant systems, such as the Routh-Hurwitz criterion (Hurwitz, 1895), the Jury criterion (Ogata, 1995), and linear matrix inequality (LMI) methods (Boyd et al., 1994). On the other hand, if the coefficients of the considered system depend on some parameters, the problem of determining whether the plant is asymptotically stable is much more challenging (Yeung, 1983; Bhattacharyya and Keel, 1995; De Oliveira et al., 1999; Ramos and Peres, 2002). If the system depends just on a single parameter, in Zhang et al. (2003), it has been shown that asymptotic stability is equivalent to the existence of a polynomial Lyapunov function satisfying two matrix inequalities. On the other hand, if upper and lower bounds on the coefficients of the characteristic polynomial of the system are known, then asymptotic stability can be established by evaluating the roots of the four so-called Kharitonov polynomials (Kharitonov, 1978).

A drawback of the Kharitonov-like approaches is that they implicitly assume that the coefficients of the characteristic polynomial of the plant vary independently (Fu et al., 1989). Assuming that these coefficients depend polynomially on the parameters, in Keel and Bhattacharyya (2010), a sufficient stability test, using results on sign-definite decomposition, is proposed. An attempt to reduce the conservativeness of these conditions has been made in Sánchez and Bernal (2017) by means of convex optimization techniques (see also Elizondo-González, 2011 for a survey of methods to study the asymptotic stability of linear time-invariant systems with parametric uncertainty).

The main objective of this paper is to design algebraic tests to establish the asymptotic stability of parametric polynomial systems assuming that the parameters lie in a given variety (see Section 2 for its formal definition). In particular, in Section 3, it is shown how to use algebraic geometry tools to establish asymptotic stability of parametric linear systems in the case that the ideal defining the variety is zero dimensional (such an hypothesis is removed in Section 4). Examples of application of the given methods are reported in Section 5, while conclusions are given in Section 6.

2. ALGEBRAIC GEOMETRY CONCEPTS

Let $x = [\, x_1 \ \cdots \ x_n \,]^\top$ be a vector of variables. A *monomial* in x is a product of the form $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where $\alpha_1, \ldots, \alpha_n$ are non-negative integers. Let $\mathbb{K}$ be a *field*, as, *e.g.*, the sets of real $\mathbb{R}$, rational $\mathbb{Q}$, and complex $\mathbb{C}$ numbers. A *polynomial* p in x with coefficients in $\mathbb{K}$ is a finite $\mathbb{K}$–linear combination of monomials. Let $\mathbb{K}[x]$ denote the *ring* of all the polynomials in $x_1, \ldots, x_n$ with coefficients in $\mathbb{K}$ and let $\mathbb{K}(x)$ denote the *field* of all the rational functions in $x_1, \ldots, x_n$ with coefficients in $\mathbb{K}$. A field $\mathbb{K}$ is *algebraically closed* if every non-constant polynomial $p \in \mathbb{K}[x]$ has a root in $\mathbb{K}$. Let $\overline{\mathbb{K}}$ denote the *algebraic closure of* $\mathbb{K}$, *i.e.*, the smallest extension of $\mathbb{K}$ that is algebraically closed.

Given $p_1, \ldots, p_s \in \mathbb{K}[x]$, the *ideal of* $p_1, \ldots, p_s$ is $\langle p_1, \ldots, p_s \rangle := \{q \in \mathbb{K}[x] : \exists h_i \in \mathbb{K}[x] \text{ s.t. } q = \sum_{i=1}^s h_i p_i\}$, while the *variety in* $\mathbb{K}$ defined by $p_1, \ldots, p_s$ is the set

$$\mathbf{V}_{\mathbb{K}}(p_1, \ldots, p_s) := \{x \in \mathbb{K}^n : p_i(x) = 0, \, i = 1, \ldots, s\}.$$

Given an ideal $\mathcal{I}$ of $\mathbb{K}[x]$, the *variety of* $\mathcal{I}$ is

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$$\mathbf{V}_{\mathbb{K}}(\mathcal{I}) := \{x \in \mathbb{K}^n : p(x) = 0, \, \forall p \in \mathcal{I}\}.$$

On the other hand, given any $\mathcal{S} \subset \mathbb{K}^n$, the *ideal of* $\mathcal{S}$ is

$$\mathbf{I}(\mathcal{S}) := \{p \in \mathbb{K}[x] : p(x) = 0, \, \forall x \in \mathcal{S}\}.$$

The set $\mathcal{S}^{\complement} = \mathbf{V}_{\mathbb{K}}(\mathbf{I}(\mathcal{S}))$ is denoted *Zariski closure of* $\mathcal{S}$ and is the smallest (with respect to set-theoretic inclusions) variety containing $\mathcal{S}$. Given a variety $\mathcal{A}$ let $\mathcal{S} \subset \mathcal{A}$ be the set where a certain property does not hold. The property is said to hold for *almost all* $x \in \mathcal{A}$ if $\mathcal{S}^{\complement} \neq \mathcal{A}$.

A variety $\mathcal{V}$ of $\mathbb{K}^n$ is *irreducible* if whenever $\mathcal{V}$ is written in the form $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are varieties of $\mathbb{K}^n$, then either $\mathcal{V}_1 = \mathcal{V}$ or $\mathcal{V}_2 = \mathcal{V}$. The *dimension* of $\mathcal{V}$ in $\mathbb{K}^n$ is the maximal length r of the chains $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_r$ of distinct irreducible varieties contained in $\mathcal{V}$.

An ideal $\mathcal{I}$ of $\mathbb{K}[x]$ is *radical* if $f^m \in \mathcal{I}$ implies $f \in \mathcal{I}$. The *radical of* $\mathcal{I}$ is an ideal of $\mathbb{K}[x]$ and is defined as $\sqrt{\mathcal{I}} = \{f \in \mathbb{K}[x] : f^m \in \mathcal{I}, \text{ for some integer } m \geq 1\}$.

A *monomial order*, denoted $\prec$, is a total, well ordering relation over the set of monomials x^{α}. Letting a monomial order $\prec$ be fixed, each $p \in \mathbb{K}[x]$ can be written as

$$p = a_1 \, x^{\alpha_1} + a_2 \, x^{\alpha_2} + \cdots + a_r \, x^{\alpha_r},$$

with $a_1 \neq 0$ and $x^{\alpha_r} \prec x^{\alpha_{r-1}} \prec \cdots \prec x^{\alpha_1}$; the term $a_1 \, x^{\alpha_1}$ is the *leading term of* p, denoted $\mathrm{LT}(p) = a_1 \, x^{\alpha_1}$. Given an ideal $\mathcal{I}$ of $\mathbb{K}[x]$, a finite subset $\mathcal{G} = \{g_1, \ldots, g_l\}$ of $\mathcal{I}$ is a *Gröbner basis* of $\mathcal{I}$ if $\langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_l) \rangle = \langle \mathrm{LT}(\mathcal{I}) \rangle$, where the *leading term of* $\mathcal{I}$ is $\mathrm{LT}(\mathcal{I}) := \{c x^{\alpha} : \exists f \in \mathcal{I}, \text{ with } \mathrm{LT}(f(x)) = c x^{\alpha}\}$. By the Hilbert Basis Theorem, there exists a Gröbner basis of any ideal in $\mathbb{K}[x]$.

Let $\mathcal{I}$ be an ideal of $\mathbb{K}[x]$. The polynomials f and g are *congruent modulo* $\mathcal{I}$ if $f - g \in \mathcal{I}$, denoted $f = g \mod \mathcal{I}$. The *equivalence class of* f *modulo* $\mathcal{I}$ is $[\![f]\!]_{\mathcal{I}} := \{g \in \mathbb{K}[x] : g = f \mod \mathcal{I}\}$. The *quotient of* $\mathbb{K}[x]$ *modulo* $\mathcal{I}$, denoted $\mathbb{K}[x]/\mathcal{I}$, is the set of all the equivalence classes modulo $\mathcal{I}$,

$$\mathbb{K}[x]/\mathcal{I} = \{[\![f]\!]_{\mathcal{I}}, \, f \in \mathbb{K}[x]\}.$$

By construction, the quotient of $\mathbb{K}[x]$ modulo $\mathcal{I}$ has the structure of a vector field over $\mathbb{K}$ and is therefore an *algebra*. By the Finiteness Theorem (Theorem 6 at page 251 of Cox et al., 2015), the algebra $\mathfrak{A} = \mathbb{K}[x]/\mathcal{I}$ is finite dimensional if and only if the set $\mathcal{B} := \{x^{\alpha} : x^{\alpha} \notin \langle \mathrm{LT}(\mathcal{I}) \rangle\}$, referred to as *standard monomial basis*, is finite dimensional. If the field $\mathbb{K}$ is algebraically closed, this is equivalent to $\mathbf{V}_{\mathbb{K}}(\mathcal{I})$ being constituted by a finite number of points. In such a case the ideal $\mathcal{I}$ is said to be *zero dimensional.*

Given a zero dimensional ideal $\mathcal{I}$ of $\mathbb{K}[x]$ and $f \in \mathbb{K}[x]$, consider the linear map $\tau_f^{\mathfrak{A}}([\![g]\!]_{\mathcal{I}}) = [\![f \, g]\!]_{\mathcal{I}}$ between $\mathfrak{A} = \mathbb{K}[x]/\mathcal{I}$ and itself. Letting a monomial order $\prec$ (whence a standard monomial basis $\mathcal{B}$) be fixed, such a map can be represented by its associated matrix $T_f \in \mathbb{K}^{d \times d}$, where d denotes the cardinality of $\mathcal{B}$, *i.e.*, $\tau_f^{\mathfrak{A}}([\![g]\!]_{\mathcal{I}}) = T_f \circ [\![g]\!]_{\mathcal{I}}$ for all $[\![g]\!]_{\mathcal{I}} \in \mathfrak{A}$. If f is one of the x_i's, then T_f is called *companion matrix of* $\mathcal{I}$ *with respect to* f. Finally, the *trace form of* $\mathcal{I}$ is the (symmetric) matrix $B \in \mathbb{K}^{d \times d}$ whose (i,j)-th entry is given by $[B]_{i,j} = \mathrm{tr}(T_{x_i} T_{x_j})$, where $\mathrm{tr}(\cdot)$ denotes the trace operator.

Letting $\beta = [\, \beta_1 \, \cdots \, \beta_m \,]^{\top}$ be a vector of parameters and letting $p \in \mathbb{K}[\beta, x]$, the operation of fixing the parameters $\beta_i = \hat{\beta}_i$, $i = 1, \ldots, m$, and coercing $p(\hat{\beta}, x)$ to $\mathbb{K}[x]$ is denoted *specialization.*

Letting λ be a single variable, a polynomial $p \in \mathbb{R}[\lambda]$ is *Hurwitz* if all its roots have negative real part, whereas it is *Shur* if all its roots have absolute value lower than 1.

A system of the form

$$\Delta x(t) = A \, x(t),$$

where, if $t \in \mathbb{R}$, then Δ denotes the time derivative, $\Delta x(t) = \dot{x}(t)$, or, if $t \in \mathbb{Z}$, then Δ denotes the one-step forward shift operator, $\Delta x(t) = x(t+1)$, it is said to be *asymptotically stable* if for each $\varepsilon > 0$ there exists $\delta_{\varepsilon} > 0$ such that $\|x(0)\| \leqslant \delta_{\varepsilon}$ implies that $\|x(t)\| < \varepsilon$, for all $t \geqslant 0$, and $\lim_{t \to +\infty} x(t) = 0$.

3. ALGEBRAIC TESTS FOR THE ASYMPTOTIC STABILITY OF PARAMETRIC LINEAR SYSTEMS

3.1 Continuous-time systems

Consider the time-invariant parametric system

$$\dot{\xi}(t) = A(\beta) \, \xi(t), \tag{1}$$

where $\xi = [\, \xi_1 \, \cdots \, \xi_n \,]^{\top} \in \mathbb{R}^n$ is the state vector, $\beta = [\, \beta_1 \, \cdots \, \beta_m \,]^{\top}$ is a vector of parameters, and $A \in \mathbb{R}^{n \times n}[\beta]$ is a parametric matrix. The main objective of this section is formalized in the following problem.

Problem 1. Let a zero dimensional ideal $\mathcal{I}$ in $\mathbb{R}[\beta]$ be given with $\mathbf{V}_{\mathbb{R}}(\mathcal{I}) \neq \emptyset$. Determine whether system (1) is asymptotically stable for all specializations $\hat{\beta} \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$.

Addressing Problem 1 via numerical techniques may lead to errors, as shown in the following motivating example.

Example 1. Consider system (1) with $[A(\beta)]_{i,j} = a_{i,j}$,

$$a_{1,1} = \tfrac{1}{3}(2\beta_2\beta_1^2 + 2(\beta_2^2 - 1)\beta_1 - 2\beta_2 + 1),$$
$$a_{2,1} = \tfrac{1}{3}(-2\beta_2\beta_1^2 - 2(\beta_2^2 - 1)\beta_1 + 2\beta_2 + 5),$$
$$a_{3,1} = \tfrac{1}{3}(2\beta_2\beta_1^2 + 2(\beta_2^2 - 1)\beta_1 - 2\beta_2 + 1),$$
$$a_{4,1} = \tfrac{2}{3}(\beta_2\beta_1^2 + (\beta_2^2 - 1)\beta_1 - \beta_2 - 1),$$
$$a_{1,2} = \tfrac{1}{3}(-\beta_1^2 - 4\beta_2\beta_1 - \beta_2^2 - 2),$$
$$a_{2,2} = \tfrac{1}{3}(\beta_1^2 + 4\beta_2\beta_1 + \beta_2^2 - 1),$$
$$a_{3,2} = \tfrac{1}{3}(-\beta_1^2 - 4\beta_2\beta_1 - \beta_2^2 + 1),$$
$$a_{4,2} = \tfrac{1}{3}(-\beta_1^2 - 4\beta_2\beta_1 - \beta_2^2 + 1),$$
$$a_{1,3} = \tfrac{1}{3}(\beta_2 - 2)\beta_2\beta_1^2 - \tfrac{2}{3}(\beta_2^2 + 1)\beta_1 - \tfrac{2\beta_2}{3} - \tfrac{33}{100},$$
$$a_{2,3} = -\tfrac{1}{3}(\beta_2 - 2)\beta_2\beta_1^2 + \tfrac{2}{3}(\beta_2^2 + 1)\beta_1 + \tfrac{2\beta_2}{3} + \tfrac{33}{100},$$
$$a_{3,3} = \tfrac{1}{3}(\beta_2 - 2)\beta_2\beta_1^2 - \tfrac{2}{3}(\beta_2^2 + 1)\beta_1 - \tfrac{2\beta_2}{3} - \tfrac{33}{100},$$
$$a_{4,3} = \tfrac{1}{3}(\beta_2 - 2)\beta_2\beta_1^2 - \tfrac{2}{3}(\beta_2^2 + 1)\beta_1 - \tfrac{2\beta_2}{3} + \tfrac{67}{100},$$
$$a_{1,4} = -\tfrac{1}{3}(\beta_2^2 + 1)\beta_1^2 - \tfrac{2}{3}(2\beta_2 + 1)\beta_1 - \tfrac{\beta_2^2}{3} - \tfrac{2\beta_2}{3} - \tfrac{67}{100},$$
$$a_{2,4} = \tfrac{1}{3}((\beta_2^2 + 1)\beta_1^2 + (4\beta_2 + 2)\beta_1 + \beta_2^2 + 2\beta_2 + \tfrac{201}{100}),$$
$$a_{3,4} = -\tfrac{1}{3}(\beta_2^2 + 1)\beta_1^2 - \tfrac{2}{3}(2\beta_2 + 1)\beta_1 - \tfrac{\beta_2^2}{3} - \tfrac{2\beta_2}{3} + \tfrac{33}{100},$$
$$a_{4,4} = -\tfrac{1}{3}(\beta_2^2 + 1)\beta_1^2 - \tfrac{2}{3}(2\beta_2 + 1)\beta_1 - \tfrac{\beta_2^2}{3} - \tfrac{2\beta_2}{3} + \tfrac{33}{100},$$

and let $\mathcal{I} = \langle \beta_1\beta_2 - 1, (\beta_1 - 10^5)(\beta_1^2 - \beta_1 + 2) + 10^{-3} \rangle$. By computing numerically $\mathbf{V}_{\mathbb{R}}(\mathcal{I})$ using machine precision and the homotpy method (Chow et al., 1978), one obtains $\mathbf{V}_{\mathbb{R}}(\mathcal{I}) = \{\hat{\beta}\}$, with $\hat{\beta} = [\, 100000 \;\, 0.00001 \,]^{\top}$. Thus, by evaluating numerically with machine precision the characteristic polynomial $\hat{p}$ of $A(\hat{\beta})$, $\hat{p}(s) = \det(sI - A(\hat{\beta})) = s^4 + 200000s^3 + 10^{10}s^2 - 6.43768 \cdot 10^7 s + 2.54885 \cdot 10^8$, and noting that there is a negative coefficient in $\hat{p}(s)$, one may conclude that the system is unstable. Nonetheless, as shown in the subsequent Example 2, the specialization $A(\hat{\beta})$ of system (1) is actually asymptotically stable. In fact, in this simple example, exact computations can be carried out by using radicals. In particular, the Cardano formula (Abramowitz et al., 1988) can be used to determine an algebraic expression for the unique real root $\hat{\beta}_1$ of $(\beta_1 - 10^5)(\beta_1^2 - \beta_1 + 2) + 10^{-3}$. Hence, the corresponding value of $\hat{\beta}_2$ can be determined as $\hat{\beta}_2 = \frac{1}{\hat{\beta}_1}$. By substituting these closed-form expressions for $\hat{\beta}_1$ and $\hat{\beta}_2$ in $p(\beta, s) = \det(sI - A(\beta))$, it is possible to determine the complex roots of p by using the general formula to compute the roots of a quartic polynomial (Faucette, 1996). By performing these operations, it can be verified that the polynomial $p(\hat{\beta}, s)$ has four roots with negative real part. Clearly, it would not be generically possible to perform these operations if either $n > 4$ or the cardinality of the standard monomial basis of $\mathcal{I}$ is greater than 4 due to the fact that, by Abel's Impossibility Theorem (Abel, 1826), the roots of a polynomial whose degree is greater than 4 cannot be generically expressed in terms of finite number of additions, subtractions, multiplications, divisions, and root extractions of its coefficients.

In view of Example 1, it is of interest to design an algebraic test that is capable of exactly determine a solution to Problem 1. The following theorem provides an algebraic test that allows one to solve Problem 1 in the case that the variety $\mathbf{V}_{\mathbb{C}}(\mathcal{I})$ just contains real points.

Theorem 1. *Let a zero dimensional ideal $\mathcal{I}$ in $\mathbb{R}[\beta]$ be given such that $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) = \mathbf{V}_{\mathbb{R}}(\mathcal{I})$. Let p be the (parametric) characteristic polynomial of $A(\beta)$,*

$$p(\beta, s) = \det(s\,I - A(\beta)) \in \mathbb{R}[\beta, s].$$

Thus, letting g be a generator of the principal ideal

$$(\mathcal{I} + \langle p \rangle) \cap \mathbb{R}[s],$$

system (1) is asymptotically stable for all specializations $\hat{\beta} \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$ if and only if g is Hurwitz.

Remark 1. The hypothesis $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) \subset \mathbb{R}$ can be easily verified by means of the trace form. Indeed, letting a monomial order $\prec$ be fixed and letting $\mathcal{B}$ be a standard monomial basis of $\mathcal{I}$ with respect to $\prec$, by Corollary 2.9 of Sturmfels (2002), one has $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) \subset \mathbb{R}$ if and only if the signature (*i.e.*, the number of positive minus the number of negative eigenvalues) of the trace form B of $\mathcal{I}$ equals the cardinality of $\mathcal{B}$. Note that, by Lemma 2.10 of Sturmfels (2002), the number of positive eigenvalues of B equals the number of sign changes in the coefficient sequence of its characteristic polynomial.

The following corollary shows that if $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) \not\subseteq \mathbb{R}$, then the conditions of Theorem 1 are still sufficient to establish asymptotic stability of the parametric system (1) considering only real values of β.

Corollary 1. *Let a zero dimensional ideal $\mathcal{I}$ in $\mathbb{R}[\beta]$ be given, possibly with $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) \not\subseteq \mathbb{R}$. If the polynomial g defined as in Theorem 1 is Hurwitz, then system (1) is asymptotically stable for all specializations $\hat{\beta} \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$.*

The following example illustrates Corollary 1.

Example 2. Consider the same system of Example 1. The trace form of the ideal $\mathcal{I}$ is

$$B = \begin{bmatrix} 3 & \frac{100000000001}{1000000} & \frac{2100}{199999999} \\ \frac{100000000001}{1000000} & \frac{9999999996000000000001}{1000000000000} & 3 \\ \frac{2100}{199999999} & 3 & -\frac{19999999897994999999}{1999999980000000000500} \end{bmatrix},$$

whose characteristic polynomial has two sign changes. Therefore, the variety $\mathbf{V}_{\mathbb{C}}(\mathcal{I})$ contains a single real point, whence $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) \not\subseteq \mathbb{R}$. Nonetheless, by computing a generator of the ideal $(\mathcal{I} + \langle p \rangle) \cap \mathbb{R}[s]$, one obtains a polynomial g such that the first column of the corresponding Routh array has not sign changes (the explicit expressions of g and of the corresponding column of the Routh array are omitted for compactness). Thus, letting $\hat{\beta}$ be the single point in $\mathbf{V}_{\mathbb{R}}(\mathcal{I})$, by Corollary 1, the specialization $A(\hat{\beta})$ of system (1) is asymptotically stable.

The next example shows that when $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) \not\subseteq \mathbb{R}$, then the fact that p is Hurwitz is only sufficient to establish that system (1) is asymptotically stable for all $\hat{\beta} \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$.

Example 3. Let $n = 2$ and

$$\mathcal{I} = \langle \beta_2^2 + \beta_1\beta_2 - 5\beta_1 - 8,\\ \beta_2^3 + 2\beta_2^2 - 6\beta_2 - 17\beta_1 - 26 \rangle,$$

$$A(\beta) = \begin{bmatrix} -\beta_2^2 & 0 \\ \beta_1 + \beta_2 & \beta_1 \end{bmatrix}.$$

By computing the reduced Gröbner basis of $(\mathcal{I}+\langle p \rangle)\cap\mathbb{R}[s]$, one obtains the polynomial

$$g = s^6 + 10s^5 + 36s^4 + 81s^3 + 146s^2 + 164s + 72,$$

that is not Hurwitz by the Routh-Hurwitz stability criterion. Nonetheless, by considering that

$$\mathbf{V}_{\mathbb{C}}(\mathcal{I}) = \left\{ \begin{bmatrix} -2 \\ 1 - \imath \end{bmatrix}, \begin{bmatrix} -2 \\ 1 + \imath \end{bmatrix}, \begin{bmatrix} -1 \\ \frac{1}{2}(1 - \sqrt{13}) \end{bmatrix}, \begin{bmatrix} -1 \\ \frac{1}{2}(1 + \sqrt{13}) \end{bmatrix} \right\} \not\subseteq \mathbb{R},$$

and that $p(\hat{\beta}, s)$ is Hurwitz for $\hat{\beta} = [\,-1 \ \frac{1}{2}(1 \pm \sqrt{13})\,]^\top$, system (1) is asymptotically stable for all $\hat{\beta} \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$.

The following proposition provides an algebraic geometry method to solve Problem 1 when $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) \not\subseteq \mathbb{R}$.

Proposition 1. *Let a zero dimensional ideal $\mathcal{I}$ in $\mathbb{R}[\beta]$ be given, possibly with $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) \not\subseteq \mathbb{R}$. Letting p be defined as in Theorem 1, define the ideal*

$$\mathcal{K} := \sqrt{\mathcal{I} + \langle p \rangle} \subset \mathbb{R}[\beta, s]. \tag{2}$$

Fix a monomial order $\prec$ and let d be the cardinality of a standard monomial basis of $\mathcal{K}$. Thus, let $T_{\beta_1}, \ldots, T_{\beta_m}$, and T_s be the companion matrices of the ideal $\mathcal{K}$ with respect to $\beta_1, \ldots, \beta_m$, and s, respectively, and let J be a matrix that simultaneously diagonalizes $T_{\beta_1}, \ldots, T_{\beta_m}$, and T_s, i.e.,

$$J\,T_{\beta_1}\,J^{-1} = \mathrm{diag}(\hat{\beta}_{1,1}, \ldots, \hat{\beta}_{1,\bar{d}}, \hat{\beta}_{1,\bar{d}+1}, \ldots, \hat{\beta}_{1,d}), \tag{3a}$$

$$J\,T_{\beta_2}\,J^{-1} = \mathrm{diag}(\hat{\beta}_{2,1}, \ldots, \hat{\beta}_{2,\bar{d}}, \hat{\beta}_{2,\bar{d}+1}, \ldots, \hat{\beta}_{2,d}), \tag{3b}$$

$$\vdots \tag{3c}$$

$$J\,T_{\beta_m}\,J^{-1} = \mathrm{diag}(\hat{\beta}_{m,1}, \ldots, \hat{\beta}_{m,\bar{d}}, \hat{\beta}_{m,\bar{d}+1}, \ldots, \hat{\beta}_{m,d}), \tag{3d}$$

$$J\,T_s\,J^{-1} = \mathrm{diag}(\hat{s}_1, \ldots, \hat{s}_d), \tag{3e}$$

with $\hat{\beta}_{i,j} \in \mathbb{R}$ for each $i \in \{1, \ldots, m\}$ and each $j \in \{1, \ldots, \bar{d}\}$, and $\hat{\beta}_{i,j} \in \mathbb{C}\setminus\mathbb{R}$ for each $j \in \{\bar{d}+1, \ldots, d\}$ and some $i \in \{1, \ldots, m\}$. Thus, system (1) is asymptotically stable for all $\hat{\beta} \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$ if and only if

$$\hat{s}_i \in \{s \in \mathbb{C} : \mathrm{real}(s) < 0\}, \tag{4}$$

for $i = 1, \dots, \bar{d}$.

The following example illustrates the application of Proposition 1 to the parametric system considered in Example 3.

Example 4. Consider the parametric system given in Example 3. Letting the graded reverse lexicographic order with $s \prec \beta_2 \prec \beta_1$ be fixed, the companion matrices of the ideal $\mathcal{K}$ given in (2) with respect to β_1, β_2 and s are

$$T_{\beta_1} = \begin{bmatrix} 0 & -2 & 0 & 8 & 14 & -14 & 0 & 0 \\ 1 & -3 & 0 & 5 & 8 & -8 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 4 & 1 & -8 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 & -3 & 3 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 & 0 & 8 & 0 & -16 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -3 \end{bmatrix},$$

$$T_{\beta_2} = \begin{bmatrix} 0 & 8 & -14 & 0 & 26 & 14 & 0 & 0 \\ 0 & 5 & -8 & 0 & 17 & 8 & 0 & -3 \\ 0 & 0 & 4 & 0 & 0 & 1 & 0 & -11 \\ 1 & 0 & -2 & 0 & 6 & 2 & 0 & -2 \\ 0 & -1 & 3 & 1 & -2 & -3 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 8 & 0 & 0 & 0 & 0 & -18 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \end{bmatrix},$$

$$T_s = \begin{bmatrix} 0 & 0 & 0 & 0 & 14 & 0 & 0 & -10 \\ 0 & 0 & 0 & 0 & 8 & -3 & 0 & -1 \\ 0 & 1 & -8 & 0 & 1 & -11 & 0 & 36 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 & -3 & -3 & 0 & 12 \\ 0 & 0 & -2 & 1 & 0 & -6 & 0 & 12 \\ 1 & 0 & -16 & 0 & 0 & -18 & 0 & 68 \\ 0 & 0 & -3 & 0 & -1 & -1 & 1 & 4 \end{bmatrix}.$$

It can be easily verified that there is $J \in \mathbb{C}^{8\times 8}$ that simultaneously diagonalizes T_{β_1}, T_{β_2}, and T_s, with

$$\begin{aligned} J\, T_{\beta_1} J^{-1} &= \operatorname{diag}(-1, -1, -1, -1, -2, -2, -2, -2), \\ J\, T_{\beta_2} J^{-1} &= \operatorname{diag}(\tfrac{1+\sqrt{13}}{2}, \tfrac{1+\sqrt{13}}{2}, \tfrac{1-\sqrt{13}}{2}, \tfrac{1-\sqrt{13}}{2}, 1+\imath, \\ &\qquad 1+\imath, 1-\imath, 1-\imath), \\ J\, T_s\, J^{-1} &= \operatorname{diag}(\tfrac{-7-\sqrt{13}}{2}, -1, \tfrac{\sqrt{13}-7}{2}, -1, -2\imath, -2, 2\imath, -2). \end{aligned}$$

Hence, by Proposition 1, system (1) is asymptotically stable for all specializations $\hat{\beta} \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$.

3.2 Discrete-time systems

Consider the time-invariant parametric system

$$\xi(k+1) = A(\beta)\, \xi(k), \tag{5}$$

where $\xi = [\, \xi_1 \cdots \xi_n \,]^\top \in \mathbb{R}^n$ is the state vector, $\beta = [\, \beta_1 \cdots \beta_m \,]^\top$ is a vector of parameters, and $A \in \mathbb{R}^{n\times n}[\beta]$ is a parametric matrix. The main objective of this section is formalized in the following problem.

Problem 2. Let a zero dimensional ideal $\mathcal{I}$ in $\mathbb{R}[\beta]$ be given with $\mathbf{V}_{\mathbb{R}}(\mathcal{I}) \neq \emptyset$. Determine whether system (5) is asymptotically stable for all specializations $\hat{\beta} \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$.

The following theorem provides an algebraic test that allows one to solve Problem 2 in the case that the variety $\mathbf{V}_{\mathbb{C}}(\mathcal{I})$ just contains real points.

Theorem 2. *Let a zero dimensional ideal $\mathcal{I}$ in $\mathbb{R}[\beta]$ be given such that $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) = \mathbf{V}_{\mathbb{R}}(\mathcal{I})$. Let q be the (parametric) characteristic polynomial of $A(\beta)$,*

$$q(\beta, z) = \det(z\, I - A(\beta)) \in \mathbb{R}[\beta, z].$$

Thus, letting ℓ be a generator of the ideal

$$(\mathcal{I} + \langle q \rangle) \cap \mathbb{R}[z],$$

system (5) *is asymptotically stable for all specializations $\hat{\beta} \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$ if and only if ℓ is Schur.*

The following corollary shows that if $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) \not\subseteq \mathbb{R}$, then the conditions of Theorem 2 are still sufficient to establish asymptotic stability of the parametric system (5) considering only real values of β.

Corollary 2. *Let a zero dimensional ideal $\mathcal{I}$ in $\mathbb{R}[\beta]$ be given, possibly with $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) \not\subseteq \mathbb{R}$. If the polynomial ℓ defined as in Theorem 2 is Schur, then system* (5) *is asymptotically stable for all specializations $\hat{\beta} \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$.*

As for continuous-time systems, if $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) \not\subseteq \mathbb{R}$, then the fact that ℓ is Schur is only sufficient to establish that system (5) is asymptotically stable for all $\hat{\beta} \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$. Nonetheless, a technique similar to the one given in Proposition 1 can be used to solve Problem 2, as shown in the following proposition.

Proposition 2. *Let a zero dimensional ideal $\mathcal{I}$ in $\mathbb{R}[\beta]$ be given, possibly with $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) \not\subseteq \mathbb{R}$. Letting q be defined as in Theorem 2, define the ideal*

$$\mathcal{K} := \sqrt{\mathcal{I} + \langle q \rangle} \subset \mathbb{R}[\beta, z].$$

Fix a monomial order $\prec$ and let d be the cardinality of a standard monomial basis of $\mathcal{K}$. Thus, let $T_{\beta_1}, \dots, T_{\beta_m}$, and T_z be the companion matrices of the ideal $\mathcal{K}$ with respect to $\beta_1, \dots, \beta_m$, and z, respectively, and let J be a matrix that simultaneously diagonalizes $T_{\beta_1}, \dots, T_{\beta_m}$, and T_z, i.e.,

$$J\, T_{\beta_1} J^{-1} = \operatorname{diag}(\hat{\beta}_{1,1}, \dots, \hat{\beta}_{1,\bar{d}}, \hat{\beta}_{1,\bar{d}+1}, \dots, \hat{\beta}_{1,d}), \tag{6a}$$

$$J\, T_{\beta_2} J^{-1} = \operatorname{diag}(\hat{\beta}_{2,1}, \dots, \hat{\beta}_{2,\bar{d}}, \hat{\beta}_{2,\bar{d}+1}, \dots, \hat{\beta}_{2,d}), \tag{6b}$$

$$\vdots \tag{6c}$$

$$J\, T_{\beta_m} J^{-1} = \operatorname{diag}(\hat{\beta}_{m,1}, \dots, \hat{\beta}_{m,\bar{d}}, \hat{\beta}_{m,\bar{d}+1}, \dots, \hat{\beta}_{m,d}), \tag{6d}$$

$$J\, T_z\, J^{-1} = \operatorname{diag}(\hat{z}_1, \dots, \hat{z}_d), \tag{6e}$$

with $\hat{\beta}_{i,j} \in \mathbb{R}$ for each $i \in \{1, \dots, m\}$ and each $j \in \{1, \dots, \bar{d}\}$, and $\hat{\beta}_{i,j} \in \mathbb{C} \setminus \mathbb{R}$ for each $j \in \{\bar{d}+1, \dots, d\}$ and some $i \in \{1, \dots, m\}$. Thus, system (5) *is asymptotically stable for all $\hat{\beta} \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$ if and only if*

$$\hat{z}_i \in \{z \in \mathbb{C} : |z| < 1\}, \tag{7}$$

for $i = 1, \dots, \bar{d}$.

Remark 2. In order to use the techniques outlined in Propositions 3 and 4, one has to compute a matrix T that simultaneously diagonalizes the companion matrices $T_{\beta_1}, \dots, T_{\beta_m}$, and T_z of the ideal $\mathcal{K}$ with respect to $\beta_1, \dots, \beta_m$, and z. The columns of T^{-1} can be then determined by solving the joint eigenvalues problem

$$\begin{gathered} T_{\beta_1} v = \hat{\beta}_1 v, \\ \vdots \\ T_{\beta_m} v = \hat{\beta}_m v, \\ T_z\, v = \hat{z}\, v, \end{gathered}$$

In particular, the columns of T^{-1} and the corresponding values $\hat{\beta}_1, \dots, \hat{\beta}_m, \hat{z}$ satisfying either (3) or (6) can be computed by solving the following optimization problem

$$\left| \begin{array}{l} \displaystyle\min_{\hat{\beta}_1, \dots, \hat{\beta}_m, \hat{z}, v} \|T_z\, v - \hat{z}\, v\|^2 + \sum_{i=1}^{m} \|T_{\beta_i} v - \hat{\beta}_i v\|^2 \\ \qquad \text{s.t. } \|v\|^2 = 1, \end{array} \right.$$

whose minimal value is 0.

4. EXTENSION TO THE NON-ZERO DIMENSIONAL CASE

The main objective of this section is to extend the results obtained in Section 3 to the case that $\mathcal{I}$ is not zero dimensional. As in Section 3, the continuous-time and the discrete-time cases are considered separately.

4.1 The continuous-time case

The goal of this section is formalized in the next problem.

Problem 3. Let an ideal $\mathcal{I}$ in $\mathbb{R}[\beta]$ be given with $\mathbf{V}_{\mathbb{R}}(\mathcal{I}) \neq \emptyset$. Determine whether system (1) is asymptotically stable for all specializations $\hat{\beta} \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$.

In order to solve Problem 3, note that, by Corollary 4 at page 509 of Cox et al. (2015), the dimension of $\mathbf{V}_{\mathbb{C}}(\mathcal{I})$ equals the largest integer r for which there exist r variables $\beta_{i_1}, \ldots, \beta_{i_r}$ such that $\mathcal{I} \cap \mathbb{R}[\beta_{i_1}, \ldots, \beta_{i_r}] = \{0\}$. Thus, given the ideal $\mathcal{I}$ and letting r be the dimension of $\mathbf{V}_{\mathbb{C}}(\mathcal{I})$, define the field $\mathbb{W} := \mathbb{R}(\beta_{i_1}, \ldots, \beta_{i_r})$ and the vector $\bar{\beta} = [\,\beta_{j_1} \, \cdots \, \beta_{j_{n-r}}\,]^\top$ of the variables β_i not in $\{\beta_{i_1}, \ldots, \beta_{i_r}\}$. Thus, consider the following theorem, which provides a solution to Problem 3.

Theorem 3. *Letting p be as in Theorem 1, define the ideal $\mathcal{J} = \mathcal{I} + \langle p \rangle$, coerce it to the ring $\mathbb{W}[\bar{\beta}]$, and define the ideal*

$$\mathcal{K} := \sqrt{\mathcal{J}}, \tag{8}$$

Fix a monomial order $\prec$ and let d be the cardinality of a standard monomial basis of $\mathcal{K}$. Thus, let $T_{\beta_{j_1}}, \ldots, T_{\beta_{j_{n-r}}}$ and T_s be the companion matrices in $\mathbb{W}^{d\times d}$ of the ideal $\mathcal{K}$ with respect to $\beta_{j_1}, \ldots, \beta_{j_{n-r}}$, and s, respectively. Hence, let J be a matrix that simultaneously diagonalizes $T_{\beta_{j_1}}, \ldots, T_{\beta_{j_{n-r}}}$ and T_s,

$$\begin{aligned} J\, T_{\beta_{j_1}}\, J^{-1} &= \operatorname{diag}(\hat{\beta}_{j_1,1}, \ldots, \hat{\beta}_{j_1,d}), \\ &\vdots \\ J\, T_{\beta_{j_{n-r}}}\, J^{-1} &= \operatorname{diag}(\hat{\beta}_{j_{n-r},1}, \ldots, \hat{\beta}_{j_{n-r},d}), \\ J\, T_s\, J^{-1} &= \operatorname{diag}(\hat{s}_1, \ldots, \hat{s}_d), \end{aligned}$$

with $\hat{\beta}_{j,k} \in \overline{\mathbb{W}}$ and $\hat{s}_k \in \overline{\mathbb{W}}$, $j = j_1, \ldots, j_{n-r}$, $k = 1, \ldots, d$. Thus, system (1) is asymptotically stable for almost all $\hat{\beta} \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$ if and only if for all $(\hat{\beta}_{i_1}, \ldots, \hat{\beta}_{i_r}) \in \mathbb{R}^r$ such that $\hat{\beta}_{j,k}(\hat{\beta}_{i_1}, \ldots, \hat{\beta}_{i_r}) \in \mathbb{R}$ for all $j \in \{j_1, \ldots, j_{n-r}\}$ and some $k \in \{1, \ldots, d\}$, the corresponding $\hat{s}_k(\hat{\beta}_{i_1}, \ldots, \hat{\beta}_{i_r})$ has negative real part.

4.2 The discrete-time case

The goal of this section is formalized in the next problem.

Problem 4. Let an ideal $\mathcal{I}$ in $\mathbb{R}[\beta]$ be given with $\mathbf{V}_{\mathbb{R}}(\mathcal{I}) \neq \emptyset$. Determine whether system (5) is asymptotically stable for all specializations $\hat{\beta} \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$.

Let $\mathbb{W}$, $\beta_{i_1}, \ldots, \beta_{i_r}$, and $\bar{\beta} = [\,\beta_{j_1} \, \cdots \, \beta_{j_{n-r}}\,]^\top$ be defined as in Section 4.1. The following theorem provides a solution to Problem 4.

Theorem 4. *Letting q be as in Theorem 1, define the ideal $\mathcal{J} = \mathcal{I} + \langle q \rangle$, coerce it to the ring $\mathbb{W}[\bar{\beta}]$, and define the ideal*

$$\mathcal{K} := \sqrt{\mathcal{J}}, \tag{9}$$

Fix a monomial order $\prec$ and let d be the cardinality of a standard monomial basis of $\mathcal{K}$. Thus, let $T_{\beta_{j_1}}, \ldots, T_{\beta_{j_{n-r}}}$ and T_s be the companion matrices in $\mathbb{W}^{d\times d}$ of the ideal $\mathcal{K}$ with respect to $\beta_{j_1}, \ldots, \beta_{j_{n-r}}$, and s, respectively. Hence, let J be a matrix that simultaneously diagonalizes $T_{\beta_{j_1}}, \ldots, T_{\beta_{j_{n-r}}}$ and T_s,

$$\begin{aligned} J\, T_{\beta_{j_1}}\, J^{-1} &= \operatorname{diag}(\hat{\beta}_{j_1,1}, \ldots, \hat{\beta}_{j_1,d}), \\ &\vdots \\ J\, T_{\beta_{j_{n-r}}}\, J^{-1} &= \operatorname{diag}(\hat{\beta}_{j_{n-r},1}, \ldots, \hat{\beta}_{j_{n-r},d}), \\ J\, T_s\, J^{-1} &= \operatorname{diag}(\hat{z}_1, \ldots, \hat{z}_d), \end{aligned}$$

with $\hat{\beta}_{j,k} \in \overline{\mathbb{W}}$ and $\hat{s}_k \in \overline{\mathbb{W}}$, $j = j_1, \ldots, j_{n-r}$, $k = 1, \ldots, d$. Thus, system (5) is asymptotically stable for almost all $\hat{\beta} \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$ if and only if for all $(\hat{\beta}_{i_1}, \ldots, \hat{\beta}_{i_r}) \in \mathbb{R}^r$ such that $\hat{\beta}_{j,k}(\hat{\beta}_{i_1}, \ldots, \hat{\beta}_{i_r}) \in \mathbb{R}$ for all $j \in \{j_1, \ldots, j_{n-r}\}$ and some $k \in \{1, \ldots, d\}$, the corresponding $\hat{z}_k(\hat{\beta}_{i_1}, \ldots, \hat{\beta}_{i_r})$ has modulus strictly smaller than 1.

5. EXAMPLES OF APPLICATION

In the following example it is shown how the technique given in Section 3 can be used to provide an algebraic certificate for local exponential stability of the equilibria of a nonlinear polynomial system.

Example 5. Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 = f_1(x) &:= 2x_1^3 + 10x_2x_1^2 - 40x_1^2 + 13x_2^2x_1 - 148x_2x_1 \\ &\quad + 246x_1 - 130x_2^2 + 467x_2 - 486, \quad \text{(10a)} \\ \dot{x}_2 = f_2(x) &:= 13x_2^3 + 10x_1x_2^2 - 22x_2^2 + 2x_1^2x_2 - 40x_2 \\ &\quad + 4x_1^2 - 38x_1 + 92. \quad \text{(10b)} \end{aligned}$$

The main objective of this example is to establish whether the equilibria of system (10) are locally exponentially stable. Thus, letting $f = [\,f_1 \; f_2\,]^\top$, consider the matrix

$$A(x) = \frac{\partial f(x)}{\partial x} = \begin{bmatrix} 3x_1^2 + x_2^2 - 1 & 2x_1x_2 - 1 \\ 2x_1x_2 + 1 & x_1^2 + 3x_2^2 - 1 \end{bmatrix}.$$

By Chapter 4 of Khalil (2002), the equilibria of system (10) are locally exponentially stable if and only if the matrix $A(x)$ is Hurwitz for all $x \in \mathbf{V}_{\mathbb{R}}(f_1, f_2)$. By fixing the graded reverse lexicographic order with $x_2 \prec x_1$, one has that $\mathcal{B} = \{1\}$ is a standard monomial basis of $\mathcal{I} = \langle f_1, f_2 \rangle$ and the trace form of $\mathcal{I}$ is $B = 1$. Therefore, the variety $\mathbf{V}_{\mathbb{C}}(\mathcal{I}) \subset \mathbb{R}$ and is a singleton. Thus, by letting $p(x, s) = \det(sI - A(x))$ and computing a Gröbner basis of $(\mathcal{I} + \langle p \rangle) \cap \mathbb{R}[s]$, one obtains the polynomial $g = 2 + 2s + s^2$, that is Hurwitz by the Descartes' rule of sign. Thus, by Theorem 1, the unique equilibrium of system (10) is locally exponentially stable. Figure 1 depicts the stream plot of the solutions to system (10) and its unique equilibrium point.

As shown by such a figure, although the unique equilibrium point of system (10) is locally exponentially stable, such a property cannot be easily deduced by the stream plot of the trajectories of the system.

The following example shows how the technique given in Section 4 can be used to evaluate the asymptotic stability of a parametric discrete-time system.

Example 6. Consider the matrix

$$A(\beta) = \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix},$$

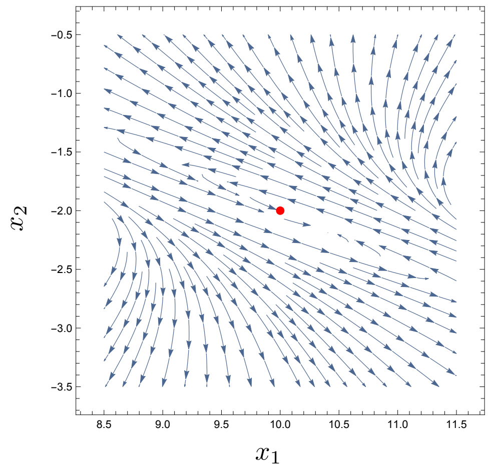

Fig. 1. Stream plot of the solutions to system (10).

and the ideal

$$\mathcal{I} = \langle \beta_1 + \beta_2 - 1, \beta_1 + \beta_3 - 1, \beta_2 + \beta_4 - 1, \beta_3 + \beta_4 - 1 \rangle.$$

Solving Problem 4 with respect to the matrix $A(\beta)$ and the ideal $\mathcal{I}$ given above corresponds to characterize the asymptotic stability of two-dimensional systems whose dynamical matrices have rows and columns summing up to one. Note that the ideal $\mathcal{I}$ is not zero-dimensional, but $\mathcal{I} \cap \mathbb{R}[\beta_4] = \{0\}$. Hence, following Theorem 4, let $\mathcal{J} = \mathcal{I} + \langle \det(zI - A) \rangle$, let $\mathbb{W} = \mathbb{R}(\beta_4)$, coerce $\mathcal{J}$ to $\mathbb{W}[\beta_1, \beta_2, \beta_3]$, and let $\mathcal{K} = \sqrt{\mathcal{J}}$. Fixing the graded reverse lexicographic order with $z \prec \beta_3 \prec \beta_2 \prec \beta_1$, the companion matrices of $\mathcal{K}$ with respect to $z, \beta_3, \beta_2, \beta_1$ are

$$T_{\beta_1} = \begin{bmatrix} \beta_4 & 0 \\ 0 & \beta_4 \end{bmatrix}, \qquad T_{\beta_2} = \begin{bmatrix} 1-\beta_4 & 0 \\ 0 & 1-\beta_4 \end{bmatrix},$$

$$T_{\beta_3} = \begin{bmatrix} 1-\beta_4 & 0 \\ 0 & 1-\beta_4 \end{bmatrix}, \qquad T_z = \begin{bmatrix} 0 & -2\beta_4+1 \\ 1 & 2\beta_4 \end{bmatrix},$$

respectively. It can be easily verified that there is $J \in \overline{\mathbb{W}}$ such that $J\,T_z\,J^{-1} = \operatorname{diag}(1, 2\beta_4 - 1)$. Therefore, since $\hat{s}_1(\hat{\beta}_4) = 1$ for all $\hat{\beta}_4 \in \mathbb{R}$, the system is not asymptotically stable for all specializations $\hat{\beta} \in \mathbf{V}_{\mathbb{R}}(\mathcal{I})$.

6. CONCLUSIONS

In this paper, some algebraic tests have been proposed to establish whether a discrete-time or a continuous-time linear system is asymptotically stable for all the specializations belonging to a given variety. The interest in such tests have been motivated via Example 1, which showed that using numerical techniques to determine if a given system is asymptotically stable for all the specializations in a variety may lead to evaluation errors. Both the cases of zero dimensional and non-zero dimensional varieties have been considered. Examples of application of the proposed techniques have been given all throughout the paper to illustrate and corroborate the theoretical results.

Future work will deal with the extension of the proposed technique to the identification of the specializations that make the system unstable.

REFERENCES

Abel, N.H. (1826). Beweis der unmöglichkeit, algebraische gleichungen von höheren graden als dem vierten allgemein aufzulösen. *Journal für die reine und angewandte Mathematik*, 1, 65–84.

Abramowitz, M., Stegun, I.A., and Romer, R.H. (1988). *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* AAPT.

Bhattacharyya, S.P. and Keel, L.H. (1995). Robust control: the parametric approach. In *Advances in Control Education*, 49–52. Elsevier.

Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V. (1994). *Linear matrix inequalities in system and control theory.* Siam.

Chow, S.N., Mallet-Paret, J., and Yorke, J.A. (1978). Finding zeroes of maps: homotopy methods that are constructive with probability one. *Mathematics of Computation*, 32(143), 887–899.

Cox, D., Little, J., and O'Shea, D. (2015). *Ideals, Varieties, and Algorithms.* Undergraduate Texts in Mathematics. Springer Verlag.

De Oliveira, M.C., Bernussou, J., and Geromel, J.C. (1999). A new discrete-time robust stability condition. *Systems & control letters*, 37(4), 261–265.

Elizondo-González, C. (2011). Parametric robust stability. In A. Mller (ed.), *Recent Advances in Robust Control: Theory and Applications in Robotics and Electromechanics.* IntechOpen.

Faucette, W.M. (1996). A geometric interpretation of the solution of the general quartic polynomial. *The American Mathematical Monthly*, 103(1), 51–57.

Fu, M., Olbrot, A.W., and Polis, M.P. (1989). Introduction to the parametric approach to robust stability. *IEEE Control systems magazine*, 9(5), 7–11.

Hurwitz, A. (1895). Ueber die bedingungen, unter welchen eine gleichung nur wurzeln mit negativen reellen theilen besitzt. *Mathematische Annalen*, 46(2), 273–284.

Keel, L.H. and Bhattacharyya, S.P. (2010). Robust stability via sign-definite decomposition. *IEEE Transactions on Automatic Control*, 56(1), 140–145.

Khalil, H.K. (2002). *Nonlinear systems.* Prentice Hall.

Kharitonov, V. (1978). Asymptotic stability of an equilibrium position of a family of systems of differential equations. *Differential equations*, 14, 1483.

Ogata, K. (1995). *Discrete-time control systems*, volume 2. Prentice Hall.

Ramos, D.C. and Peres, P.L. (2002). An LMI condition for the robust stability of uncertain continuous-time linear systems. *IEEE Transactions on Automatic Control*, 47(4), 675–678.

Sánchez, M. and Bernal, M. (2017). A convex approach for reducing conservativeness of kharitonovs-based robustness analysis. *IFAC-PapersOnLine*, 50(1), 832–837.

Sturmfels, B. (2002). *Solving systems of polynomial equations.* American Mathematical Society.

Yeung, K.S. (1983). Linear system stability under parameter uncertainties. *International Journal of Control*, 38(2), 459–464.

Zhang, X., Tsiotras, P., and Iwasaki, T. (2003). Parameter-dependent Lyapunov function for exact stability analysis of single-parameter dependent LTI systems. In *42nd IEEE International Conference on Decision and Control*, volume 5, 5168–5173. IEEE.