

Relative Normalized Gain Array-based interaction indicator for non-square multivariable control systems: properties and application

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Abstract: This paper extends the square Relative Normalized Gain Array (RNGA) to non-square multivariable systems with the detailed derivation of non-square RNGA λ^{RN} properties with column-major, i.e. $\lambda^{RN} \in R^{r \times n}$, $r < n$. Non-square RNGA in this paper has a row-column inequality. The developed interaction indicator is applied to a non-square multivariable radiator laboratory test setup for evaluating control-loop interactions. Closed-loop results, as well as sensitivity analysis for the RNGA-based control-loop pairing in comparison to the RGA-based control-loop pairing, are presented in the paper. The results demonstrate the effectiveness of the proposed non-square RNGA over RGA for non-square multivariable systems to have minimum interactions and better control.

Keywords: Relative normalized gain array, Binet-Cauchy relation, generalized inverse, decentralized control, control-loop pairing, non-square multivariable radiator system.

1. INTRODUCTION

The solution to a potential problem of control-loop pairing for non-square systems lies with the decentralized control to achieve minimum interactions among the control loops. The importance of a decentralized control scheme lies with its simplicity to implement for field engineers. The primary step is to decide the control-loop pairing based on the loop-interactions measure (Engell and Konik, 1985). The minimal loop-interactions confirm that the closed-loop performance is not deteriorated. Bristol (1966) introduced a relative gain array to decide the loop pairing for square systems, which constructs gain array symmetric matrix using the Schur product. The transient information of the system is not accounted for achieving the gain matrix. Chang and Yu (1990) extended Bristol's RGA to its non-square version.

Several different forms of RGA are available, i.e. Relative Disturbance Gain Array (RDGA) (Chang and Yu, 1992), Dynamic Relative Gain Array (DRGA) (McAvoy et al., 2003), Relative Effective Gain Array (REGA) (Xiong et al., 2005), Relative Time-Averaged Gain Array (RTAGA) (Tang et al., 2018) concerning different approaches for interaction measurements. He et al. (2009) presented an RNGA based new control-loop pairing criterion for square systems. The RNGA accounts for steady-state gain, dead time and time constant parameters. As compared to DRGA, RNGA provides loop interactions independent of the controller type. Moreover, the computation of square RNGA has proven useful and straightforward to work with, for the field engineers to carry out the loop pairing decision of practical industrial problems (He et al., 2009). The universality of the non-square system with the less output-more input structure can be found in practical problems: mixing-tank process (Reeves and Arkun, 1989), Air-path scheme of a

turbocharged diesel engine (Saidi et al., 2019), Shell control problem (Valchos et al., 2002), etc. The properties of non-square RNGA for less output-more input structures are not explored yet. Hence, it merits investigations.

This paper presents the RNGA for non-square multivariable systems with application to a radiator control system having less outputs-more inputs in its setup. Its Properties with the systematic derivation of their proofs are developed. To test the effectiveness, the proposed technique is applied to a non-square multivariable radiator laboratory test setup. The system transfer function matrix is obtained from the experimental step test readings. The suitable control-loop pairing of the non-square radiator system is then decided. The minimal interactions pairing achieved here pinpoints to a quick and superior control setting i.e. achieving the desired through better controller and sensitivity performance indices. Internal Model Control (IMC) tuned PID controllers are designed for the decentralized control of the non-square system. The closed-loop performance resulting from the pairing suggested by non-square RNGA-based configuration is compared to that of the control configuration of the conventional RGA.

Notations: Consider $G(s)$ is a $r \times n$ non-square process transfer matrix with $r < n$, where $Y(s) = G(s)U(s)$. The term column-major in this paper indicates that the non-square matrix has more columns with less associated rows, i.e., more inputs and less outputs.

2. DEFINITION OF THE NON-SQUARE RNGA

Consider $Y(s)$ is an $r \times 1$ output vector and $U(s)$ is an $n \times 1$ input vector.

Now, $G(s) = (G_{ij}) = \left(\frac{k_{ij} e^{-(t_d)_{ij}s}}{1 + (\tau_c)_{ij}s} \right)$, where $k_{ij}, (t_d)_{ij}, (\tau_c)_{ij}$ are

the process gain, dead time and time constant respectively of the i^{th} output with respect to the j^{th} input and the variables, i and j , run over 1 to r and 1 to n . The steady-state gain and the average residence time (Åström and Hägglund, 2006) of the transfer function entries of the non-square transfer matrix are

$$k = (k_{ij}) = (G_{ij}(0)), \quad b_{ij} = (\tau_c)_{ij} + (t_d)_{ij}. \quad (1a)$$

Now, introducing a convenient computational notation, $\tilde{B} = (\tilde{B}_{ij}) = (b_{ij}^{-1})$, the Normalized Gain Array (NGA) can be defined as $A = K \circ \tilde{B}$, where ‘ \circ ’ is the Schur product between two non-square matrices. Thus,

$$A_{ij} = k_{ij} \tilde{B}_{ij} = k_{ij} b_{ij}^{-1} = \frac{k_{ij}}{(\tau_c)_{ij} + (t_d)_{ij}}. \quad (1b)$$

Using generalized inverse result (Graybill et al., 1966), the non-square RNGA becomes

$$\lambda^{RN} = A \circ (A^+)^T = A \circ (A^T (AA^T)^{-1})^T = A \circ (AA^T)^{-1} A. \quad (2)$$

In the component-wise setting (2) can be recast as

$$\lambda^{RN} = (\lambda_{ij}^{RN}) = (A_{ij} ((AA^T)^{-1})_{ij}). \quad (3)$$

After combining (3) with (1b), we get

$$\begin{aligned} \lambda_{ij}^{RN} &= A_{ij} \sum_{\gamma} \left(\sum_{\phi} A_{i\phi} A_{j\phi} \right)^{-1} A_{i\gamma} A_{j\gamma} \\ &= \sum_{\gamma} \left(\sum_{\phi} \frac{k_{i\phi}}{(\tau_c)_{i\phi} + (t_d)_{i\phi}} \frac{k_{j\phi}}{(\tau_c)_{j\phi} + (t_d)_{j\phi}} \right)^{-1} \frac{k_{j\gamma}}{(\tau_c)_{j\gamma} + (t_d)_{j\gamma}} \frac{k_{i\gamma}}{(\tau_c)_{i\gamma} + (t_d)_{i\gamma}}. \end{aligned} \quad (4)$$

3. PROPERTIES OF THE NON-SQUARE RNGA

Consider the RNGA λ^{RN} contained in the real-valued matrix space $R^{r \times n}$, where $r < n$. The row sum $R(i)$, the column sum $C(j)$, scaling, and permutation properties of the non-square RNGA are derived. Here, we list the following useful RNGA relations:

$$\lambda_{ij}^{RN} = A_{ij} ((A^+)^T)_{ij} = A_{ij} A_{ji}^+, \quad (5)$$

$$\begin{aligned} \sum_j \lambda_{ij}^{RN} &= \sum_j A_{ij} A_{ji}^+ = \sum_{\phi} A_{i\phi} A_{\phi i}^+ = \sum_{\phi} A_{i\phi} A_{\phi j}^+ \\ &= (AA^+)_{ii} = R(i), \end{aligned} \quad (6)$$

$$\begin{aligned} \sum_i \lambda_{ij}^{RN} &= \sum_i A_{ij} A_{ji}^+ = \sum_{\phi} A_{\phi j} A_{i\phi}^+ = \sum_{\phi} A_{\phi j} A_{i\phi}^+ \sum_{i=j} A_{i\phi}^+ A_{\phi j} \\ &= (A^+ A)_{jj} = C(j). \end{aligned} \quad (7)$$

Property 1: Consider the RNGA of an $r \times n$ matrix A is λ^{RN} , where $r < n$. The sum of elements in each row of the non-square RNGA λ^{RN} is always equal to ‘1’, $R(i) = 1, 1 \leq i \leq r$.

Proof: Making the use of the generalized inverse result of Graybill et al. (1966, p. 523), Penrose (1955) and (6), we get

$$R(i) = (AA^+)_{ii} = (A A^T (AA^T)^{-1})_{ii} = (I_r)_{ii} = 1.$$

Property 2: The sum of elements in each column of the non-square RNGA λ^{RN} , where $r < n$, is always between zero and unity, i.e., $0 \leq C(j) \leq 1, 1 \leq j \leq n$.

Proof: Consider the non-square RNGA

$$\lambda_{ij}^{RN} = \frac{1}{2} \frac{A_{ij}}{\alpha} \frac{d\alpha}{dA_{ij}}, \quad (8)$$

where $\alpha = \det(AA^T)$, the sizes of the matrices AA^T and $A^T A$ are $r \times r$ and $n \times n$, where $r < n$. Here, we recast the Binet-Cauchy relation (Gantmacher, 1977) for the case $r < n$, i.e.

$$\det(AA^T) = \sum_l \det(A^T A)_l = \sum_{1 \leq \kappa \leq \binom{n}{r}} (\det A(\kappa))^2,$$

Note that $1 \leq \kappa \leq \binom{n}{r}$, l is a subset contained in the product space, where $1 \leq l \leq \binom{n}{r}$, κ has r tuples (Vidyasagar, 1985).

By using the matrix calculus (Magnus and Neudecker, 2007: p. 169),

$$\frac{d\alpha}{dA_{ij}} = \frac{d}{dA_{ij}} \left(\sum_{1 \leq \kappa \leq \binom{n}{r}} (\det A(\kappa))^2 \right) = 2 \sum_{1 \leq \kappa \leq \binom{n}{r}} \det A(\kappa) \frac{d \det A(\kappa)}{dA_{ij}}. \quad (9)$$

Note that

$$\begin{aligned} \frac{d \det A(\kappa)}{dA_{ij}} &= \sum_m \frac{d \nabla_{pq}(\kappa') A_{pq}(\kappa)}{dA_{ij}} \\ &= \sum_m \left(\frac{d \nabla_{pq}(\kappa')}{dA_{ij}} A_{pq}(\kappa) + \nabla_{pq}(\kappa') \frac{dA_{pq}(\kappa)}{dA_{ij}} \right) = \nabla_{iq}(\kappa'). \end{aligned} \quad (10)$$

The term $\nabla_{iq}(\kappa')$ is the cofactor of the element $A_{iq}(\kappa)$, where

$1 \leq q \leq r, 1 \leq i \leq r, 1 \leq \kappa' \leq \binom{n-1}{r-1}$ and $1 \leq \kappa \leq \binom{n}{r}$ that must include the j^{th} column of (A_{ij}) . The size of $(\nabla_{iq}(\kappa'))$ is $(r-1) \times (r-1)$. After combining (9) and (10), we get

$$\frac{d\alpha}{dA_{ij}} = 2 \sum_{1 \leq \kappa' \leq \binom{n-1}{r-1}} \det A(\kappa') \nabla_{iq}(\kappa'). \quad (11a)$$

For convenience, replace $\nabla_{iq}(\kappa')$ with $\det(A^{ij}(\kappa'))$ in (11a).

The matrix $A^{ij}(\kappa')$ has the size $(r-1) \times (r-1)$, which is a consequence of $\frac{dA_{pq}(\kappa)}{dA_{ij}}$ and $A(\kappa')$ is an $r \times r$ matrix.

Thus,

$$\frac{d\alpha}{dA_{ij}} = 2 \sum_{1 \leq \kappa' \leq \binom{n-1}{r-1}} \det A(\kappa') \nabla_{iq}(\kappa') = 2 \sum_{1 \leq \kappa' \leq \binom{n-1}{r-1}} \det A(\kappa') \det(A^{ij}(\kappa')). \quad (11b)$$

Equation (7) in conjunction with (8) and (11b) becomes

$$\sum_i \lambda_{ij}^{RN} = \sum_i \frac{1}{2} \frac{A_{ij}}{\alpha} \frac{d\alpha}{dA_{ij}} = \frac{\sum_i A_{ij} \left(\sum_{1 \leq \kappa' \leq \binom{n-1}{r-1}} \det A(\kappa') \det(A^{ij}(\kappa')) \right)}{\sum_{1 \leq \kappa \leq \binom{n}{r}} (\det A(\kappa))^2}. \quad (12)$$

On further simplifications (12) reduces to

$$\begin{aligned} \sum_i \lambda_{ij}^{RN} &= \frac{\sum_{1 \leq \kappa' \leq \binom{n-1}{r-1}} \det A(\kappa') \sum_i A_{ij} \det(A^{ij}(\kappa'))}{\sum_{1 \leq \kappa \leq \binom{n}{r}} (\det A(\kappa))^2} \\ &= \frac{\sum_{1 \leq \kappa' \leq \binom{n-1}{r-1}} (\det A(\kappa'))^2}{\sum_{1 \leq \kappa \leq \binom{n}{r}} (\det A(\kappa))^2}. \end{aligned} \quad (13)$$

Equation (13) is a consequence of the Laplace expansion formula for the determinant. The term κ runs over 1 to $\binom{n}{r}$

and κ' runs over 1 to $\binom{n-1}{r-1}$. Each term within the summation sign is non-negative. Thus, the condition $0 \leq C(j) \leq 1$ holds.

Property 3: The non-square RNGA λ^{RN} of an $r \times n$ matrix A is output scaling-invariant, where $r < n$.

Proof: For the output scaling, an $r \times r$ diagonal matrix Q_r is pre-multiplied to the normalized gain matrix A , where $Q_r = (q_i \delta_{ij}), 1 \leq i \leq r, 1 \leq j \leq r$. Since the property $(Q_r A)^+ = A^+ Q_r^{-1}$ holds (Lawson and Hanson, 1974), the output scaled RNGA can be written as

$$\bar{\lambda}^{RN} = (Q_r A) \circ (A^+ Q_r^{-1})^T = (Q_r A) \circ (Q_r^{-1} (A^+)^T).$$

Thus,

$$\begin{aligned} \bar{\lambda}_{ij}^{RN} &= \left(\sum_{\phi} (Q_r)_{ii} \delta_{i\phi} A_{\phi j} \right) \left(\sum_{\phi} (Q_r)_{ii}^{-1} \delta_{i\phi} A_{j\phi}^+ \right) \\ &= (Q_r)_{ii} A_{ij} (Q_r)_{ii}^{-1} A_{ji}^+ = A_{ij} A_{ji}^+ = \lambda_{ij}^{RN}. \end{aligned}$$

Property 4: The non-square RNGA λ^{RN} of an $r \times n$ matrix A input scaling-variant, where $r < n$.

Proof: For the input scaling, an $n \times n$ diagonal matrix Q_n is post-multiplied to the normalized gain matrix A , i.e., $(A Q_n)^+$. Considering $Q_n = (q_i \delta_{ij}), 1 \leq i \leq n, 1 \leq j \leq n$, and

making the use of the right generalized inverse $A^+ = A^T (A A^T)^{-1}$, we arrive at

$$\begin{aligned} (A Q_n)^+ &= Q_n^T A^T (A Q_n^2 A^T)^{-1} = ((A Q_n^2 A^T)^{-1})^T A Q_n^T, \\ ((A Q_n)^+)^T &= ((A Q_n^2 A^T)^{-1})^T A Q_n = (A^+)^T Q_n^{-1} Q_n^{-1} ((A^T)^+)^T A Q_n. \end{aligned}$$

The $r \times r$ square matrix $(A^+)^T Q_n^{-1} Q_n^{-1} ((A^T)^+)^T$ can be rewritten as

$$(A^+)^T Q_n^{-1} Q_n^{-1} ((A^T)^+)^T = ((A^+)^T)_{ij} (q_i^{-2} \delta_{ij}) ((A^T)^+_{ji}),$$

where $((A^+)^T)_{ij}$ is an $r \times n$ matrix, $(q_i^{-2} \delta_{ij})$ is an $n \times n$ diagonal matrix, $((A^T)^+_{ji})$ is an $n \times r$ matrix. Thus, (i, j) components of the $r \times r$ matrix $(A^+)^T Q_n^{-1} Q_n^{-1} ((A^T)^+)^T$ and the $r \times s$ matrix $A Q_n$ are

$$((A^+)^T Q_n^{-1} Q_n^{-1} ((A^T)^+)^T)_{ij} = \sum_{1 \leq \phi \leq s} \frac{A_{\phi i}^+ A_{\phi j}^+}{q_{\phi}^2}, (A Q_n)_{ij} = A_{ij} q_j$$

respectively. The notation A_{ij}^+ is the $(i, j)^{th}$ component of the right generalized inverse of the $r \times n$ matrix A , where $r < n$. Proving an element of the input scaled non-square RNGA $\bar{\lambda}^{RN}$ is different from that of the non-square RNGA λ^{RN} will suffice the adequacy of the non-square RNGA as an input scaling-variant. Thus, for simplified notation, $\bar{\lambda}_{11}^{RN} \neq \lambda_{11}^{RN}$ is demonstrated. The RNGAs are

$$\bar{\lambda}^{RN} = (A Q_n) \circ (A^+)^T Q_n^{-1} Q_n^{-1} ((A^T)^+)^T A Q_n, \lambda^{RN} = A \circ (A^+)^T.$$

$$\bar{\lambda}_{ij}^{RN} = (A Q_n)_{ij} ((A^+)^T Q_n^{-1} Q_n^{-1} ((A^T)^+)^T A Q_n)_{ij}, \lambda_{ij}^{RN} = A_{ij} A_{ji}^+.$$

Alternatively,

$$\bar{\lambda}_{11}^{RN} = (A Q_n)_{11} ((A^+)^T Q_n^{-1} Q_n^{-1} ((A^T)^+)^T A Q_n)_{11}$$

$$= A_{11} \sum_{1 \leq j \leq r} \sum_{1 \leq \phi \leq n} \frac{A_{\phi 1}^+ A_{\phi 1}^+}{q_{\phi}^2} A_{j1} q_1^2, \lambda_{11}^{RN} = A_{11} A_{11}^+.$$

Thus, $\bar{\lambda}_{11}^{RN} \neq \lambda_{11}^{RN}$.

Property 5: Suppose P_r and P_n are two orthogonal matrices of the sizes $r \times r$ and $n \times n$ respectively. Consider the RNGA of an $r \times n$ matrix A is λ^{RN} , and construct the matrix $P_r A P_n$, the RNGA $\tilde{\lambda}^{RN}$ of the matrix $P_r A P_n$ satisfies the condition: $\tilde{\lambda}^{RN} = P_r \lambda^{RN} P_n$, where $\lambda^{RN} = A \circ (A^+)^T$.

Proof: The orthogonal matrices have properties, $P_n^{-1} = P_n^T, P_r^{-1} = P_r^T$.

Now, on applying the definition of the non-square RNGA to the matrix $P_r A P_n$, we have

$$\tilde{\lambda}^{RN} = (P_r A P_n) \circ ((P_r A P_n)^+)^T = (P_r A P_n) \circ (P_n^{-1} A^+ P_r^{-1})^T.$$

After introducing the orthogonal matrix property and extending the property 2 of Grosdidier et al. (1985), we have

$$\begin{aligned} \tilde{\lambda}^{RN} &= (P_r A P_n) \circ (P_n^{-1} A^+ P_r^{-1})^T = (P_r A P_n) \circ (P_r (A^+)^T P_n) \\ &= P_r (A \circ (A^+)^T) P_n = P_r \lambda^{RN} P_n. \end{aligned}$$

4. APPLICATION OF RNGA TO A NON-SQUARE RADIATOR CONTROL PROBLEM

Consider a laboratory radiator setup (Fig. 1). Here, water and air are fed from the hot water tank, and the air blower housing is equipped with 3- ϕ , 5 HP electric motor.

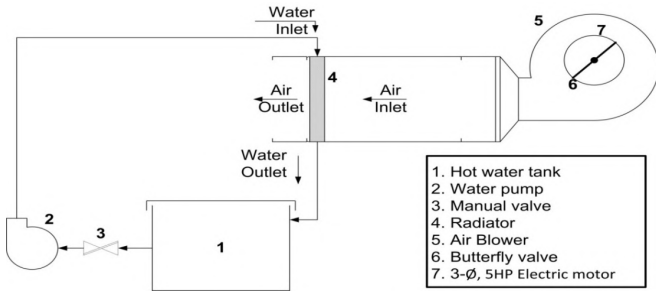


Fig. 1. Schematic diagram of a laboratory radiator experimental setup

At the radiator setup, four input variables are considered: air inlet temperature T_{AIN} , air inlet flow rate F_{AIN} , water inlet temperature T_{WIN} , and water inlet flow rate F_{WIN} . The output variables are air outlet temperature T_{AOUT} and water outlet temperature T_{WOUT} . Fig. 2 depicts the actual laboratory radiator setup available for open-loop experiments. The design parameters of the radiator are given in Table 1. The operating conditions for the system identification are:

$$F_{AIN} = 8.08 \text{ m/s}, F_{WIN} = 8 \text{ LPM}, T_{AIN} = 38.3^\circ \text{C},$$

$$T_{WIN} = 72.5^\circ \text{C}, T_{AOUT} = 44.5^\circ \text{C}, T_{WOUT} = 65^\circ \text{C}.$$

Table 1. Radiator setup parameters

	Parameters	Dimensions
Radiator core	Height (mm)	300
	Length (mm)	24
	Width (mm)	340
Radiator fins (Aluminum)	Length (mm)	24
	Width (mm)	10.89
	Thickness (mm)	0.1
	Depth (mm)	1.8
Radiator tube (Aluminum)	No. of fins	185
	Outer diameter (mm)	8
	Inner diameter (mm)	6
	Thickness (mm)	1

The step test method for the radiator non-square transfer matrix identification is adopted (Ahmed et al., 2008). First, the air inlet flow rate F_{AIN} is varied from its steady-state value, i.e., from 8.08 m/s to 10 m/s, by keeping all other inputs at their steady-state. The effect of change in the air inlet flow rate to the two output temperatures T_{AOUT} and T_{WOUT} respectively is measured and noted to get the process reaction curve. Similarly, the water inlet flow rate F_{WIN} is varied to get a relationship with the two output temperatures.



Fig. 2. Laboratory radiator experimental setup

Subsequently, experimental data from the open-loop step test are taken for the rest of the two inputs. Using the procedure of the step test method, the open-loop transfer matrix is obtained as

$$(Y_1 \ Y_2)^T = G(s)(U_1 \ U_2 \ U_3 \ U_4)^T, \quad (14)$$

where the non-square radiator transfer matrix $G(s)$ is

$$\begin{pmatrix} -0.9826e^{-1374s} & 0.25702e^{-1068s} & 1.09306e^{-1867s} & 0.2154e^{-9.12s} \\ 1+42435s & 1+32922s & 1+7324s & 1+787255s \\ -0.1556e^{-7971s} & 0.8045e^{-1656s} & 0.3023 e^{-1986s} & 1.052e^{-1827s} \\ 1+25162s & 1+30264s & 1+120274s & 1+59261s \end{pmatrix}$$

Note that $(Y_1 \ Y_2)^T = (T_{AOUT} \ T_{WOUT})^T$ and $(U_1 \ U_2 \ U_3 \ U_4)^T = (F_{AIN} \ F_{WIN} \ T_{AIN} \ T_{WIN})^T$.

The closeness of the numerically simulated trajectories with the experimentally generated trajectories is depicted in Fig. 3, and Fig. 4.

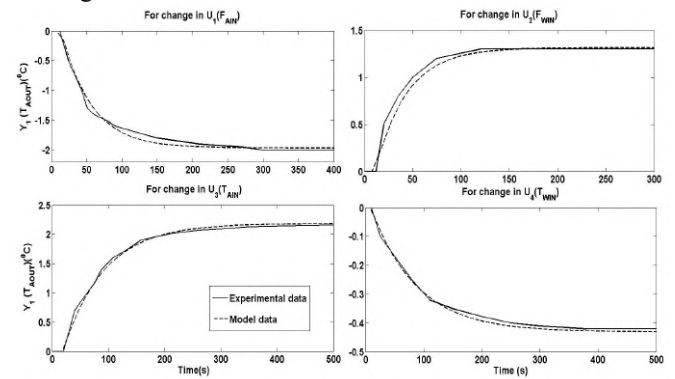


Fig. 3. A Comparison of the open-loop step test for output 1, experimental and model

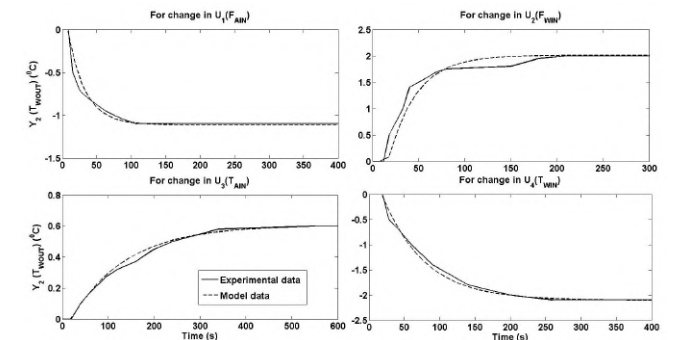


Fig. 4. A Comparison of the open-loop step test for output 2, experimental and model

Now, we choose the best suitable control-loop pairing utilizing the non-square RGA theory in this paper and considering the input-output pairing whose associated RGA matrix element is ≥ 0.5 and closer to unity (Seborg et al., 2004). Considering (1)-(4) in combination with the radiator non-square transfer matrix of (14), the resulting non-square radiator RGA matrix is

$$\lambda^{RN} = \begin{pmatrix} 0.7166 & -0.0370 & 0.3470 & -0.0267 \\ -0.0486 & 0.6350 & -0.0210 & 0.4345 \end{pmatrix}. \quad (15)$$

The row sum property of the non-square radiator RGA (14) holds, i.e.

$$R(i) = (1.00 \quad 1.00)^T, \quad 1 \leq i \leq 2. \quad (16a)$$

Furthermore, the column sum property of the non-square radiator RGA (15) also holds, i.e. $0 \leq C(j) \leq 1$,

$$(C(j))^T = (0.668 \quad 0.598 \quad 0.326 \quad 0.4078)^T, \quad 1 \leq j \leq 4, \quad 0 \leq C(j) \leq 1. \quad (16b)$$

The input variant, output invariant properties as well as row-column permutations properties, can be tested for the non-square radiator RGA. Since their approach is straightforward, discussions are omitted. The non-square radiator RGA λ^R is

$$\lambda^R = \begin{pmatrix} 0.4884 & -0.0194 & 0.5664 & -0.0354 \\ -0.0250 & 0.3759 & -0.0279 & 0.6770 \end{pmatrix}. \quad (17)$$

The column sum values of the non-square RGA matrix (17) and the value of $\lambda_{ij}^R \geq 0.5$ recommends $(Y_1 - U_3 / Y_2 - U_4)$ pairing. The first two inputs are eliminated from the control loop configuration, which is attributed to smaller entries of the column sum vector, see (17). On the other hand, the non-square radiator RGA (15) recommends a different decentralized control scheme, i.e. $(Y_1 - U_1 / Y_2 - U_2)$ pairing.

Two decentralized IMC tuned PID controllers using the RGA pairing (1-3/2-4) and RGA pairing (1-1/2-2) are designed through the standard IMC tuning procedure (Skogestad, 2003). Controller parameters are listed in Table 2. The controller structure is $G_c(s) = k_c(1 + (1/\tau_i)s + \tau_d s)$.

Table 2. Decentralized PID controllers for both control configurations of radiator example

Loop	1-1/2-2 pairing (RNGA)			Loop	1-3/2-4 pairing (RGA)		
	k_c	τ_i	τ_d		k_c	τ_i	τ_d
1-1	-2.43	49.305	5.912	1-3	2.697	82.576	8.279
2-2	1.928	38.544	6.501	2-4	2.372	68.396	7.914

Fig. 5(a) shows the closed-loop response of air outlet temperature $Y_1(T_{AOUT})$ for a step-change in the reference Y_{r1} . On the other hand, Fig. 5(b) shows the behavioural pattern of water outlet temperature $Y_2(T_{WOUT})$ under interactions from the change in the reference Y_{r1} . Note that the second output is set-point change-free. Figs. 5(c)-5(d) shows the closed-loop responses considering a step set-point change to the reference

Y_{r2} , keeping the set-point of first output unchanged. Figs. 5(a)-5(d) show the corresponding IAE values.

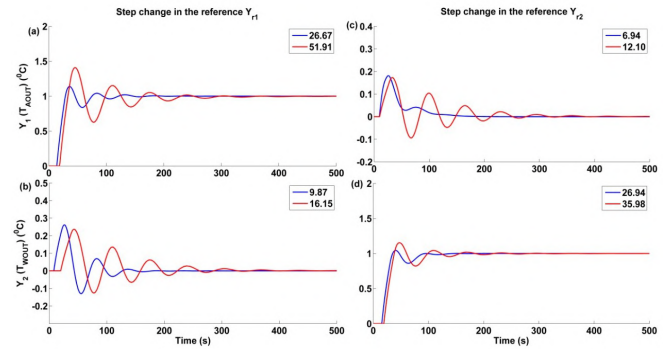


Fig. 5. Closed-loop results for radiator example (Blue lines: RNGA recommended pairing, Red lines: RGA recommended pairing)

Fig. 6 displays the Bode plots of both the pairings (RNGA and RGA) associated with outputs Y_1 and Y_2 respectively. The Bode plots (Fig. 6) reveals that in the RNGA recommended pairing the phase crossover frequency ω_{pc} has increased. As a result, the Gain Margin (GM) has increased by 26.01% and 12.73%, see Table 3. Figs. 7(a) and 7(b) show the absolute sensitivity plot for both the pairings associated with the outputs Y_1 and Y_2 respectively. The peak s_m in the sensitivity plot describes the amplification due to input disturbances and uncertainties.

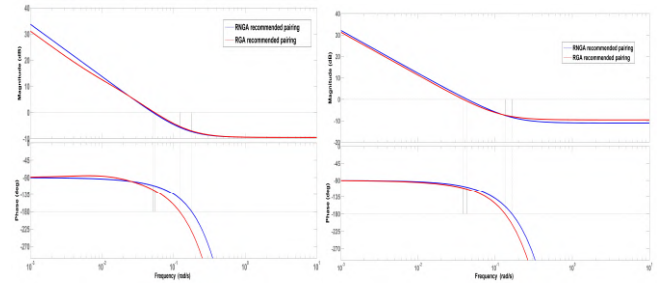


Fig. 6. Bode plot comparison of both the pairings associated with the output Y_1 and Y_2 .

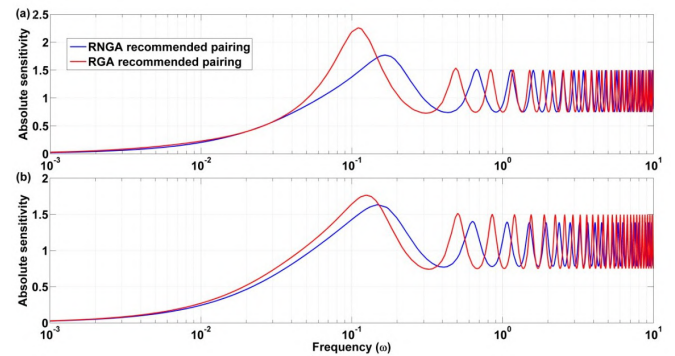


Fig. 7. Graphical representation of absolute sensitivity.

It is observed that the RGA pairing suffers from more amplification in contrast to the RNGA pairing. Table 3 summarizes the characteristics of the Bode plots and the

sensitivity performance indices. Moreover, sensitivity analysis is a good measure of robustness performance as well (Åström and Murray, 2008). The inverse of maximum absolute sensitivity s_m is known as the Stability Margin (SM). The s_m values in Table 3 result in 27.62% and 8.16% rise in SM on utilizing the RNGA recommended pairing for control of output Y_1 and Y_2 respectively. From Figs. 5-7 and Tables 2-3, the proposed non-square RNGA in this paper gives a better suggestion of the control-loop pairing for minimum interactions amongst the loop with enhanced sensitivity and robustness performance.

Table 3. Characteristics of Bode plot and sensitivity performance indices

Loop	RNGA recommended (1-1/2-2) pairing					
	GM	PM	ω_{gc}	ω_{pc}	S_m	ω_{ms}
1-1	2.32	68.9	0.179	0.051	1.77	0.166
2-2	2.63	71.3	0.165	0.043	1.63	0.15
Loop	RGA recommended (1-3/2-4) pairing					
	GM	PM	ω_{gc}	ω_{pc}	S_m	ω_{ms}
1-3	1.84	55.45	0.121	0.054	2.26	0.11
2-4	2.34	68.95	0.134	0.039	1.76	0.12

5. CONCLUSION

In this paper, non-square RNGA for the less output-more input multivariable systems with the systematic derivation of proofs of the non-square RNGA $\lambda^{RN} \in R^{r \times n}, r < n$ properties is presented. The theory of RNGA developed is applied to a non-square radiator laboratory setup with four inputs and two outputs. The results of the experiment carried out on the radiator setup are utilized to obtain the non-square transfer matrix. Then by applying the RNGA to this non-square transfer matrix, the control configuration for the decentralized control is achieved. This proves the usefulness of the proposed method to the real field practical problem. Closed-loop results indicate the usefulness of the proposed non-square RNGA-based pairing over RGA-based pairing for minimum interactions and better control of non-square multivariable systems. The method proposed in this paper is suggestive for field engineers dealing with the control problem of non-square multivariable systems.

ACKNOWLEDGMENT

The Authors express their gratefulness to Dr. Manish Rathod of the mechanical engineering department of the Institute, for useful discussions and support for the radiator experimentation in the laboratory. The authors are also grateful to the IFAC foundation for considering this work eligible for the Young Author Support (YAS) award. Funding provided to the student author in the form of the YAS award is gratefully acknowledged.

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