Output sign-consensus of heterogenous multi-agent systems: an observer-based approach *

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Abstract: This paper studies output sign-consensus of heterogeneous multi-agent systems over directed signed graphs. To remove the restriction that every follower knows the dynamics of the leader system, a distributed observer is maintained by each follower nodes, such that the state and dynamics of the leader node is estimated. Then a distributed state feedback control law is designed and analyzed. A simulation example is presented to illustrate the effectiveness of the proposed control law.

Keywords: antagonistic interaction; heterogeneous multi-agent system; sign-consensus; signed graph

1. INTRODUCTION

Over the past two decades, significant interests have been paid to distributed control of multi-agent systems, which has reached a relatively mature status (Ren and Beard, 2008; Lewis et al., 2014), with a primary focus on the consensus problem (Jadbabaie et al., 2003; Qin et al., 2017), assuming that all interactions between agents are collaborative. In other words, interactions/communication networks are modeled by nonnegative graphs, whose adjacency matrices are nonnegative (Berman and Plemmons, 1994).

However, in many practical scenarios, both cooperative and competitive interactions coexist within a group of agents, such as social network, biological network, and political campaign. Such interaction networks should be represented by signed graphs, where positive and negative edges stand for collaborative and antagonistic interactions, respectively. Obviously, adjacency matrix of a signed graph is a general matrix, instead of a nonnegative matrix. Thus, analysis of signed graph is much more challenging than that of nonnegative graph.

In the past several years, many researches have been conducted on distributed control of multi-agent systems over signed graphs, with a seminal work being Altafini (2013), which found a new collective behavior of multi-agent systems, called bipartite consensus. By bipartite consensus, it means that the whole group of agents will split into two subgroups, and agents within each subgroup reach conventional consensus, while consensus values of these two subgroups will have the same magnitude but different signs. This work was soon extended from single integrator dynamics to homogeneous and heterogeneous linear systems (Valcher and Misra, 2014; Zhang and Chen, 2017; Jiao et al., 2019), nonlinear systems (Yu et al., 2019), and to systems with measurement noise (Ma and Qin, 2016).

For bipartite consensus problems, it is always assumed that signed graphs should be structurally balanced (Altafini, 2013). However, this condition is very strong and can be easily violated by inappropriately adding an edge or changing the sign of a single edge. Therefore, investigating collective behaviors over structurally unbalanced signed graphs is important and interesting.

Very recently, Altafini and Lini (2015) studied opinion forming process over structurally unbalanced communication graphs. They found that when the communication graph is eventually positive, all agents will achieve an unanimous opinion, although some might be more convinced than others. In other words, the signs of all opinions will reach a consensus. Thus, we coined the term sign-consensus in Jiang et al. (2017), which extends Altafini and Lini (2015) to general homogeneous linear multi-agent systems. It is noticed that in practice almost all agent dynamics are inherently heterogeneous, which allows different agents to have different dimensions and/or different dynamics. For heterogeneous linear multi-agent systems, we designed distributed control law, which guarantees the output of each agent to achieve sign-consensus, i.e., output sign-consensus (Jiang and Zhang, 2020).

It is well known that the major advantage of multi-agent systems is its distributed feature, i.e., control law for each agent only use information of itself and its neighbors. However, most works of heterogeneous multi-agent sys-

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tems assume that all agent can access the information of dynamics of the exosystem (Wieland et al., 2011; Jiang and Zhang, 2020), which means there exist direct edges from the exosystem to all other agents. This is known as a kind of global information and not realistic. Therefore, how to design a fully distributed control law is of longstanding research interests in the control community. Motivated by the above-mentioned statements, we aim to remove this assumption and design a fully distributed control law for heterogeneous multi-agent systems, such that output sign-consensus can be achieved. Inspired by Cai et al. (2017), we propose an observer based control approach, where a distributed adaptive observer is maintained by each agent which can estimate the dynamics of the exosystem. With this observer, the whole control law can be designed in a fully distributed manner.

The rest of this paper is organized as follows. Section 2 introduces notations, preliminaries of signed graphs and problem formulation. In Section 3, an adaptive distributed observer is designed for each agent to estimate dynamics and states of the exosystem. In Section 4, a state feedback observer is designed for each agent to estimate dynamics. With this observer, the whole control law can be designed in a fully distributed manner.

2. PRELIMINARIES

2.1 Notations

$\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}^n$ represent real space, positive real space and n-dimensional real space, respectively. The empty set is denoted as $\emptyset$. $\otimes$ denotes the Kronecker product. $1_n$ denotes an n dimensional vector, whose entries are all ones. A column vector $x \in \mathbb{R}^n$ is defined by $x = [x_1, x_2, \cdots, x_n]^T$ where $x_i$ is the i-th entry. A diagonal matrix $\Lambda \in \mathbb{R}^{n \times n}$ is denoted as $\Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Matrix $A \in \mathbb{R}^{n \times n}$ with entries defined as $a_{ij}$ is denoted as $A = [a_{ij}]$. If the entries of matrix $A$ are all positive, it is said to be positive, which is denoted as $A > 0$. A positive vector $v \in \mathbb{R}^n$, whose entries are all positive, is denoted as $v > 0$. Eigenvalues of $A \in \mathbb{R}^{n \times n}$ are denoted by $\lambda_i(A)$, $i = 1, 2, \cdots, n$. The identity matrix is denoted as $I_n \in \mathbb{R}^{n \times n}$. The spectral radius of $A$, denoted as $\rho(A)$, represents the smallest positive number such that $\rho(A) \geq |\lambda_i(A)|$, $i = 1, 2, \cdots, n$. For $X_i \in \mathbb{R}^{m \times n}$, $\text{col}(X_i) = [X_i^1, X_i^2, \cdots, X_i^n]^T$. For a column vector $x = [x_1, x_2, \cdots, x_q] \in \mathbb{R}^q$, denote $M^q_i(x) = [x_1, x_2, \cdots, x_q]^T \in \mathbb{R}^{q \times q}$. For a matrix $X \in \mathbb{R}^{m \times n}$, its vector valued function is defined as $\text{vec}(X) = [X^1_1, X^2_1, \cdots, X^1_m]^T \in \mathbb{R}^{mn}$ where $X_i \in \mathbb{R}^{m \times n}$ is the i-th column of the matrix $X$. The entrywise sign function of $x = [x_1, x_2, \cdots, x_n]^T$ is defined as

$$\text{sgn}(x) = [\text{sgn}(x_1), \text{sgn}(x_2), \cdots, \text{sgn}(x_n)]^T$$

with

$$\text{sgn}(x_i) = \begin{cases} 1, & x_i > 0 \\ 0, & x_i = 0 \\ -1, & x_i < 0 \end{cases}, \quad i = 1, 2, \cdots, n$$

2.2 Signed graph

The communication or interaction of multi-agent systems can be naturally described as a graph, where each node or vertex represents an agent and each edge represents the communication/interaction between two agents. A directed graph is denoted as $G = (V, E)$, where $V = \{v_1, v_2, \cdots, v_N\}$ is the node set and $E \subseteq V \times V$ is the edge set. The topology of a graph can be completely depicted by its adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N \times N}$, where $a_{ij}$ represents the $(i, j)$th entry and $a_{ij} \neq 0$ implies that node $i$ can obtain information from node $j$, i.e., $(v_j, v_i) \in E$, and $v_i$ is thus called a neighboring agent of $v_j$; otherwise $a_{ij} = 0$. The set of all indices of neighboring agents of $v_i$ can be denoted as $N_i = \{j | (v_j, v_i) \in E, j = 1, 2, \cdots, N\}$, called the neighboring set of node $i$. In this paper, only simple graph is considered, i.e., graph with no self-loops and no multiple edges. Particularly, for graph $G(A)$, if all weights of edges are nonnegative, i.e. $a_{ij} \geq 0$, for all $i, j = 1, 2, \cdots, N$, it is called nonnegative graph; otherwise it is signed graph.

Definition 1. (Altafini and Lini, 2015)

A matrix $A \in \mathbb{R}^{n \times n}$ is said to have strong Perron–Frobenius property if the spectral radius $\rho(A)$ is a simple eigenvalue of $A$, and the associated eigenvector $v_r$ is positive.

Definition 2. (Altafini and Lini, 2015)

A matrix $A \in \mathbb{R}^{n \times n}$ is said to be eventually positive, which can be denoted as $A \geq 0$, if a positive integer $k_0$ exists and such that $A^k \geq 0$ for all $k \geq k_0$. $\rho(A)$ is said to be eventually positive if $A \geq 0$.

Proposition 1. (Altafini and Lini, 2015)

For a matrix $A \in \mathbb{R}^{n \times n}$, the following statements are equivalent:

(i). $A$ and $A^T$ have strong Perron–Frobenius property;

(ii). $A \geq 0$;

(iii). $A^T \geq 0$.

2.3 Problem formulation

Consider the heterogeneous linear multi-agent system with $N$ agents:

$$\dot{x}_i = A_i x_i + B_i u_i$$

$$y_i = C_i x_i, \quad i = 1, 2, \cdots, N$$

where $x_i \in \mathbb{R}^{m_i}$, $u_i \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^p$ are the state, input and output, respectively; and $(A_i, B_i)$ is assumed to be stabilizable.

The exosystem, i.e., the leader node, has the following dynamics:

$$\dot{\eta} = S_0 \eta$$

$$w = R \eta$$

where $\eta \in \mathbb{R}^q$ and $w \in \mathbb{R}^p$ are the state and output of the leader node. Matrix $S_0$ is assumed to have no eigenvalues on the left half plane (Huang, 2004).

The objective of this paper is to design a fully distributed controller by integrating an adaptive observer which can
estimation the leader’s signals and dynamics, such that output sign-consensus of all follower nodes is achieved.

**Definition 3.** (Jiang et al., 2018) Let \( y_i = [y_i^1, y_i^2, \cdots , y_i^p]^T \) and \( y^* = [y^*^1, y^*^2, \cdots , y^*_p]^T \), \( i = 1, 2, \cdots , N \). System (1) is said to achieve output sign-consensus if

\[
\lim_{t \to \infty} (sgn(y_i(t)) - sgn(y^*_i(t))) = 0, \quad \forall t \in L_1
\]

Using the same technique as in Example 9.6 of Khalil (2002), we can show that the origin of system (7) is exponentially as \( t \to \infty \).

**Proof.** Define \( \hat{S}_i = S_i - v_r S_0 \) and \( \varepsilon_i = \zeta_i - v_r \eta \).

Part (i). Since \( A v_r = \rho(A) v_r \) and \( \sigma_i = \rho(A) \), it is trivial to show that \( \dot{y}_i \neq 0 \). Then from (3) we have

\[
\dot{S}_i = \gamma_1 \left( -\sigma_i S_i + \sum_{j=1}^{N} a_{ij} S_j + g a_o(v_r S_0 - S_i) \right)
\]

Using the same technique as in Example 9.6 of Khalil (2002), we can show that the origin of system (7) is Hurwitz for any \( \gamma_1 > 0 \). Therefore all \( \hat{S}_i \) will vanish exponentially as \( t \to \infty \).

Part (ii). Similar to proof in Part (i), we have

\[
\dot{\varepsilon}_i = \frac{1}{v_r} S_i \xi_i - S_0 \eta_i + S_0 \xi_i - v_r S_0 \eta
\]

Denote \( G = \text{diag}(g_{10}, g_{20}, \cdots , g_{N0}) \) and \( \hat{S} = \text{col}(S_1, S_2, \cdots , S_N) \). We have

\[
\hat{\dot{S}} = \gamma_1 \left( (C + G) \otimes I_q \right) \hat{S}
\]

Using the same technique as in Example 9.6 of Khalil (2002), we can show that the origin of system (7) is Hurwitz for any \( \gamma_1 > 0 \). Therefore all \( \hat{S}_i \) will vanish exponentially as \( t \to \infty \).
globally exponentially stable. Then system (6) is input-to-state stable with \( \tilde{S}_d(N \otimes \eta) \) as the input. Since \( \tilde{S}_d \) and hence \( \tilde{S}_d(N \otimes \eta) \) vanishes exponentially, it follows that \( \lim_{t \to \infty} \varepsilon_i = 0 \). Therefore, \( \lim_{t \to \infty} \varepsilon_i = 0 \) for all \( i \).

4. CONTROL LAW DESIGN AND ANALYSIS

In this section we will design a state feedback controller to achieve output sign-consensus. Two lemmas are first presented.

\textbf{Lemma 2.} (Cai et al., 2017) Let \( Q \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \) and \( \text{rank}(Q) = \text{rank}(Q,b) = k \) where \( m, n, k \in \mathbb{R}^+ \). Then \( Q(t) \in \mathbb{R}^{m \times n} \) such that \( \lim_{t \to \infty}(Q(t)-Q) = 0 \). Then for any initial condition and sufficiently large \( \mu > 0 \), the state of system

\[ \dot{x} = -\mu(Q(t))^T(Q(t)x - b) \]

will tend to \( x^* \) exponentially, where \( Qx^* = b \).

\textbf{Lemma 3.} Let \( b_i = vec \left( \begin{bmatrix} 0_{n_i \times q} \\ -R \end{bmatrix} \right) \) and

\[ P_i(t) = S_i^T \otimes \begin{bmatrix} I_{n_i} & 0 \\ 0 & 0 \end{bmatrix} - I_q \otimes \begin{bmatrix} v_r A_i & B_i \\ C_i & 0 \end{bmatrix} \]

Each function \( \dot{\psi}_i = -\gamma_3 P_i(t) \dot{\psi}_i = -b_i, \forall i \in \{1, 2, \cdots, N\} \) with \( \gamma_3 \) positive and sufficiently large, has a unique bounded solution, if the following regulator equations are solvable

\[ v_r \Pi_i S_0 = v_r A_i \Pi_i + B_i \Gamma_i \]

\[ C_i \Pi_i = R \]

And further let

\[ \Psi_i(t) = M^n_{\gamma}(\psi_i(t)) = \left[ \begin{array}{c} \Psi_{1i}(t) \\ \Psi_{2i}(t) \end{array} \right] \]

Then for some solution \((\Pi_i, \Gamma_i)\) of equation (9), we have

\[ \lim_{t \to \infty} \left( \Psi_i(t) - \begin{bmatrix} \Pi_i \\ \Gamma_i \end{bmatrix} \right) = 0. \]

\textbf{Proof.} Rewrite (9) as

\[ v_r \begin{bmatrix} I_{n_i} & 0 \\ 0 & 0 \end{bmatrix} \Pi_i \Gamma_i S_0 - v_r A_i B_i \begin{bmatrix} \Pi_i \\ \Gamma_i \end{bmatrix} = 0_{n_i \times q} \]

which can be further put into the form

\[ P_i \chi_i = b_i \]

where \( P_i = v_r S_0^T \otimes \begin{bmatrix} I_{n_i} & 0 \\ 0 & 0 \end{bmatrix} - I_q \otimes \begin{bmatrix} v_r A_i & B_i \\ C_i & 0 \end{bmatrix} \) and \( \chi_i = vec \left( \begin{bmatrix} \Pi_i \\ \Gamma_i \end{bmatrix} \right) \). From Lemma 1, \( \lim_{t \to \infty}(P_i(t) - P_i) = 0 \). Thus from Lemma 2, \( \lim_{t \to \infty}(\psi_i(t) - \chi_i) = 0 \) by noting that

\[ \Psi_i(t) - \begin{bmatrix} \Pi_i \\ \Gamma_i \end{bmatrix} = M^n_{\gamma}(\psi_i(t) - \chi_i) \]

this completes the proof.

\textbf{Theorem 1.} Consider system (1) and (2) with its control law (3), (4)-(8),(10) and (11). Suppose assumptions 1 and 2 hold, and regulator equations (9) are solvable. Then if \( \gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}^+ \) and \( \gamma_2, \gamma_3 \) are sufficiently large, output sign-consensus can be achieved.

\textbf{Proof.} Let \( \tilde{x}_i = x_i - v_r \Pi_i \eta, \tilde{u}_i = u_i - \Gamma_i \eta, \varepsilon_i = y_i - v_r \xi, K_i = \frac{1}{v_r} (\Gamma_i - v_r \Pi_i) \) and \( \hat{K}_c(t) = \hat{K}_c(t) - K_c \).

For each \( i \in \{1, 2, \cdots, N\} \), by utilizing (9) we obtain that

\[ \dot{x}_i = \dot{\hat{x}}_i + v_r \Pi_i \eta \]

\[ = A_i \dot{x}_i + B_i \dot{u}_i - v_r \Pi_i S_0 \eta \]

\[ = A_i (\dot{\hat{x}}_i + v_r \Pi_i \eta) + B_i (\tilde{u}_i + \Gamma_i \eta) - v_r \Pi_i S_0 \eta \]

\[ = A_i \dot{\hat{x}}_i + B_i \tilde{u}_i \]

\[ e_i = y_i - v_r \xi \]

\[ = C_i \dot{\hat{x}}_i + v_r \xi R \eta \]

\[ = C_i \dot{\hat{x}}_i \]

\[ \hat{u}_i = K_i \dot{x}_i + K_c(t) (\xi_i - \Gamma_i \eta) \]

\[ = K_i (\dot{x}_i + v_r \Pi_i \eta) + K_c(t) (\xi_i - \Gamma_i \eta) \]

\[ = K_i \dot{x}_i + \dot{K}_c(t) \xi_i + (\Gamma_i - v_r \Pi_i) \xi_i + v_r \dot{K}_c(t) \xi_i - K_c(t) \eta \]

\[ = K_i \dot{x}_i + \dot{K}_c(t) \xi_i + v_r \dot{K}_c(t) \eta \]

\[ = K_i \dot{x}_i + \dot{K}_c(t) (\xi_i + v_r \dot{K}_c(t) \eta) \]

Substituting (14) into (12) leads to

\[ \dot{x}_i = (A_i + B_i K_c(t) \dot{x}_i + B_i \tilde{K}_c(t) \eta) \]

When \( \gamma_1 > 0 \) and \( \gamma_2 \) sufficiently large, by Lemma 1, we know \( \hat{S}_i \) vanishes exponentially and \( \varepsilon_i \to 0 \) as \( t \to \infty \). Since

\[ v_r \tilde{K}_c(t) = (\Psi_2i(t) - \Gamma_i - v_r K_i (\Psi_1(t) - \Pi_i) \]

then \( \lim_{t \to \infty} \tilde{K}_c(t) = 0 \) if \( \gamma_3 \) is large enough (see Lemma 3). Thus \( v_r B_i \tilde{K}_c(t) \eta \) tends to zero exponentially. Similar to the proof of Lemma 1, one can show that system (15) is input-to-state stable with the vanishing input term \( B_i \tilde{K}_c(t) \eta \). Therefore for any initial condition \( \ddot{x}_i(0) \), \( \lim_{t \to \infty} \ddot{x}_i = 0 \). Then \( \lim_{t \to \infty} \varepsilon_i = 0 \), which implies \( \lim_{t \to \infty} (y_i - v_r \xi R \eta) = 0 \). This means that output sign-consensus is achieved.

\textbf{Remark.} Although the selection of \( \gamma_2 \) and \( \gamma_3 \) depends on agents’ dynamics and graph topology \( G \), we need not compute them in practice, but to guarantee that they are positive and sufficiently large.

5. SIMULATION EXAMPLE

Consider a group of 6 followers nodes (1) with dynamics

\[ A_i = \begin{bmatrix} -0.5 & -1 \\ 2 & 0 \end{bmatrix}, \forall i \in \{1, 2, 3\} \]

\[ A_i = \begin{bmatrix} 0.5 & -1 \\ 2 & 0 \end{bmatrix}, \forall i \in \{4, 5, 6\} \]
\[ B_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \forall i \in \{1, \ldots, 6\} \]

Obviously \((A_i, B_i)\) is stabilizable.

Consider an exosystem (2) with the following dynamics

\[ S_0 = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

The communication graph \(G\) of follower nodes is adopted from Jiang et al. (2018) and is shown in Fig. 1. Suppose only node 1 has a direct communication with the leader, i.e., \(g_{10} = 1\) and \(g_{i0} = 0\) for \(i = 2, 3, \ldots, 6\). We can easily show that assumptions 1 and 2 are satisfied.

Consider system (1) and (2) with control law (3), (4), (8), (10) and (11). Let \(\gamma_1 = 5\), \(\gamma_2 = 20\) and \(\gamma_3 = 10\). The output trajectories of all agents are shown in Figs. 2 and 3. Output sign-consensus is achieved.

6. CONCLUSION

This paper considered output sign-consensus of heterogeneous multi-agent systems. The objective is to design a distributed control law such that the sign of outputs of all follower nodes can reach consensus, i.e., output sign-consensus. Compared with our recent work (Jiang and Zhang, 2020), this paper removes the restriction that each follower nodes should know the dynamics of the leader node by designing an adaptive distributed observer. A state feedback control law was designed. This paper only considered a fixed graph topology. Output sign-consensus problem over time varying graph topology is worthy of further investigation.

REFERENCES


