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# Neighborhood Interval Observer Based Coordination Control for Multi-agent Systems with Disturbances ⋆

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**Abstract:** This paper focuses on multi-agent systems with uncertain disturbances, in which only the bounding functions on the disturbances and the bounds on the initial state of each agent are known. By designing a neighborhood interval observer for this kind of multi-agent system, the estimation of the sum of the relative state of each agent associated with itself and its neighbors is firstly realized. Then, on the basis of these estimated information, local control algorithm is designed to drive the system to achieve bounded consensus.

*Keywords:* Multi-agent system, disturbance, neighborhood interval observer, bounded consensus.

## 1. INTRODUCTION

The wave of research on multi-agent systems has been inspired by some pioneering publications, such as (Jadbabaie et al. (2003); Olfati-Saber and Murray (2004); Olfati-Saber (2006)), since the beginning of this century. During these two decades, investigations on multi-agent systems emerge from different perspectives (Shi et al. (2004); Lin et al. (2003); Liu et al. (2008); Yu et al. (2011, 2009); Su et al. (2011); Li et al. (2015); Liu et al. (2015); Wang et al. (2017)). Consensus which aims at guiding all agents in the system to reach an agreement is a fundamental coordination behavior of multi-agent systems. In (Olfati-Saber and Murray (2004); Yu et al. (2011); Li et al. (2010); Su et al. (2011); Li et al. (2015); Wang et al. (2017)), the consensus can be achieved on the basis of the state information. Notice that in some practical application, it needs huge economic loss or it is difficult or even impossible to obtain the state information. In this case, some observer based consensus results turn out by using the output information (Li et al. (2010); Zhang et al. (2011); Su et al. (2014); Zhang et al. (2016); Li et al. (2017, 2018)). In (Zhang et al. (2016)), the authors studied consensus tracking of nonlinear multi-agent systems with disturbances bounded by a constant by using the output feedback technique. Li *et al.* investigated the robust consensus of multi-agent systems with additive perturbations upper bounded by an $\mathcal{H}_\infty$ level constant in (Li et al. (2018)). However, the disturbances considered in (Zhang et al. (2016)) and (Li et al. (2018)) are some special kinds of disturbances, there are many disturbances in the real engineering cannot be bounded by a constant or by an $\mathcal{H}_\infty$ level constant.

In this paper, we take the multi-agent systems with the more common disturbances into consideration, where the disturbances are bounded by two functions named the upper bounded function and the lower bounded one, respectively. These two bounding functions may not be bounded by a constant as in (Zhang et al. (2016)) or may not meet the $\mathcal{H}_\infty$ assumption as in (Li et al. (2018)). What is more, due to the existence of this kind of disturbances, the initial state of each agent is also unknown but only the

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bounds can be obtained. But, could we estimate the state of each agent? How can we design a control algorithm on the basis of the bound information mentioned above to drive the system to achieve cooperative control? In (Wang et al. (2019)), the author devoted themselves to solving these problems by using the interval observer technique. As stated in (Wang et al. (2019)), the interval observer technique used in multi-agent systems can be dated from the uncertain single agent system (Hadj-Sadok and Gouzé (1998); Mazenc and Bernard (2011)). An interval observer should be a framer at first. A framer of a multi-agent system contains two dynamic system in the form of the Luenberger observer, which can describe the bounds on the real-time state of each agent. Notice that the framer design involves only the bound information of the unknown disturbances. Then, a distributed control algorithm can be designed on the basis of the information of the framer. If the control algorithm in the framer can drive the uncertain multi-agent system to achieve coordination behavior, then the framer is an interval observer for the multi-agent system.

This paper can be seen as an improvement of (Wang et al. (2019)). In (Wang et al. (2019)), a framer were firstly designed for each agent to estimate the bounds on the absolute state of itself. We name this kind of framer the local framer by following the way of naming in (Zhang et al. (2011)). Then, control algorithm was constructed for each agent based on the sum of relative information associated with the estimated information obtained by the framer of each agent and its neighbors. We call this kind of control algorithm the neighborhood controllers. What is more, we name the interval observer in (Wang et al. (2019)) the local interval observer. In this paper, we design interval observer for each agent from the perspective of the sum of the relative states associated with each agent and all its neighbors. We first design a neighborhood framer for each agent by using the sum of relative information of each agent and its neighbors, then we construct a local control algorithm. The interval observer designed in this paper is called the neighborhood interval observer. In line with the neighborhood interval observer, we solve the bounded consensus of multi-agent systems with uncertain disturbances with the help of cooperativity theory and Lyapunov stability theory. Beyond, the time-varying coordinate transformation is used to get rid of the Metzler dependence on the observer gain matrix. In comparison with the local interval observer based results in (Wang et al. (2019)), the superiorities of the neighborhood interval observer based bounded consensus are mainly in two-fold. First, the acquirement of the relative information is easily to implemented than the absolute information. Second, the neighborhood interval observer based cooperative behavior processes more robust to the local interval observer based one. It will show that consensus can be achieved if each agent is with the same disturbance for the neighborhood interval observer based results, while only the bounded consensus can be reached even though each agent is with the same disturbance according to the local interval observer based results given in (Wang et al. (2019)).

The rest of the this paper is organized as follows. Section 2 states the problem which will be settled, while Section 3 gives the main theoretical results of this paper. Section 4 concludes the whole paper.

**Notation:** Throughout the paper, $\otimes$ represents the Kronecker product. For $x \in \mathbb{R}^{n\times 1}$, $\|x\|_\infty = \max_{i=1,\cdots,N} |x_i|$ denotes the infinite norm of $x$, while $\|x\|$ is the 2-norm of $x$. For any square matrix $Q$, $\det(Q)$ denotes the determinant of $Q$, $Q \prec (\preceq) 0$ means that $Q$ is a negative-definite (semi-negative-definite) matrix, whereas $Q \succ (\succeq) 0$ means that $Q$ is a positive-definite (semi-positive-definite) matrix. $\mathbf{sign}(\cdot)$ represents the signum function. for any two matrices $X$ and $Y$ (or vectors) with the same dimension, $X \geq (\leq \text{ or } =)\ Y \leq (\geq \text{ or } =)$ should be understood componentwise. For matrices $A = (a_{ij})$ and $B = (b_{ij})$ with the same dimension, $C = (c_{ij}) = \max\{A, B\}$ denotes a matrix with $c_{ij} = \max\{a_{ij}, b_{ij}\}$. Moreover, for any matrix or vector $S = (S_{ij}) \in \mathbb{R}^{n\times m}$, $S^+ = (S_{ij}^+) \in \mathbb{R}^{n\times m}$ with $S_{ij}^+ = \max\{0, S_{ij}\}$ and $S^- = S^+ - S$.

## 2. PROBLEM STATEMENT

In this paper, we consider a networked systems including $N$ agents interplays on an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$. In the undirected graph $\mathcal{G}$, $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$ and $\mathcal{E} = \{(v_i, v_j)|$ if there exists an edge between node $v_i$ and node $v_j\}$ denote the node set and edge set, respectively, and $W = (w_{ij}) \in \mathbb{R}^{N\times N}$ with

$$w_{ij} = \begin{cases} 1, \text{if } (v_i, v_j) \in \mathcal{E}; \\ 0, \text{otherwise.} \end{cases}$$

is the adjacency matrix. The degree of the $i-$th agent is $d_i = \sum_{j=1}^{N} w_{ij}$, then on the basis of the adjacency matrix $W$, the Laplacian matrix of $\mathcal{G}$ is $L = D - W$, where $D = \text{diag}\{d_1, d_2, \cdots, d_N\}$. We further denote the eigenvalues of $L$ in the non-decreasing order as $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$. For the connected graph $\mathcal{G}$, we have $\lambda_2 > 0$ (Godsil and Royle (2013)).

On the undirected graph $\mathcal{G}$ defined previously, we consider a general linear networked system, where every agent moves in an $n$-dimensional Euclidean space and regulates itself by following dynamics (1):

$$\begin{cases} \dot{x}_i = Ax_i + Bu_i + \omega_i, i = 1, 2, \cdots, N, \\ y_i = Cx_i, \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}^{n\times 1}$, $u_i \in \mathbb{R}^{m\times 1}$ and $y_i \in \mathbb{R}^{p\times 1}$ are the state, the control input and the output of the $i-$th agent, respectively, $\omega_i \in \mathbb{R}^{n\times 1}$ is the unknown disturbance. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are the system matrices, which satisfy Assumptions 1.

*Assumption 1.* $(A, B)$ is stabilizable and $(C, A)$ is detectable.

The objective of this paper is to investigate the coordinated behavior of the uncertain networked multi-agent system (1). However, due to the existence of the disturbance $\omega_i$, neither the precise state information nor the initial state of each agent can be obtained, but the bounds on them. Without the state information, we will solve the coordination of system (1) by using the estimated information. As done in the neighborhood observer based results on multi-agent systems (Li et al. (2010, 2017, 2018)), the sum of the relative outputs with respective to all its neighbors, that is $\sum_{j\in N(i)}(y_j - y_i)$, is used. In order to match this term, the observer will be designed to estimate the state of $\sum_{j\in N(i)}(x_j - x_i)$. For brevity, let

$$\delta_i = \sum_{j\in N(i)} g_{ij}\left(x_i - x_j\right),$$

then the neighborhood framer for the uncertain multi-agent system (1) can be designed as:

$$\begin{cases} \dot{\overline{\delta}}_i = A\overline{\delta}_i - BK \sum_{j\in N(i)} g_{ij}\left[\left(\overline{\delta}_i - \overline{\delta}_j\right) + \left(\underline{\delta}_i - \underline{\delta}_j\right)\right] \\ \qquad + F\left[C\overline{\delta}_i - \sum_{j\in N(i)} g_{ij}\left(y_i - y_j\right)\right] + \overline{\Omega}_i, \\ \dot{\underline{\delta}}_i = A\underline{\delta}_i - BK \sum_{j\in N(i)} g_{ij}\left[\left(\overline{\delta}_i - \overline{\delta}_j\right) + \left(\underline{\delta}_i - \underline{\delta}_j\right)\right] \\ \qquad + F\left[C\underline{\delta}_i - \sum_{j\in N(i)} g_{ij}\left(y_i - y_j\right)\right] + \underline{\Omega}_i, \end{cases} \tag{2}$$

where $K \in \mathbb{R}^{n\times m}$ and $F \in \mathbb{R}^{n\times p}$ are the user-defined matrices named the control gain and the observer gain, respectively, $\overline{\Omega}_i$ and $\underline{\Omega}_i$ are the upper bounding and lower bounding of $\Omega_i$ with

$$\Omega_i = \sum_{j\in N(i)} g_{ij}\left(\omega_i - \omega_j\right). \tag{3}$$

That is,

$$\underline{\Omega}_i \le \Omega_i \le \overline{\Omega}_i.$$

Notice that the form of $\Omega_i$ given in (3) is the due to the neighborhood information interaction through the communication topology described by the graph $\mathcal{G}$. Furthermore, the bounded information on the initial states of system (1) are given as

$$\underline{\delta}_i(0) \le \delta_i(0) \le \overline{\delta}_i(0), \tag{4}$$

which is also in the neighborhood pattern. In analogy to the local controllers in (Li et al. (2010, 2017, 2018)), the control input $u_i$ in (1) can be constructed as

$$u_i = -K\left(\overline{\delta}_i + \underline{\delta}_i\right). \tag{5}$$

Then, the objective problem of this paper is to prove that $\overline{\delta}_i$ and $\underline{\delta}_i$ in (2) with $u_i$ in (5) make up a neighborhood interval observer of (1). That is to provide sufficient conditions to make

(1) $\underline{\delta}_i \le \delta_i \le \overline{\delta}_i$ for $i = 1,\ldots,N$ in case of (4);
(2) the control algorithm (5) can drive (1) to reach consensus if $\underline{\Omega}_i = \overline{\Omega}_i = 0$ $(i = 1,\ldots,N)$.

We say that $\overline{\delta}_i$ and $\underline{\delta}_i$ in (2) form a neighborhood framer of (1) if only (1) is satisfied. For brevity, we use the pair $\left(\overline{\delta}_i, \underline{\delta}_i\right)$ to denote the two dynamical systems given in (2). Additionally, we call $u_i$ in (5) the local controllers as in Zhang et al. (2011).

## 3. MAIN RESULTS

This section is solve the problems mentioned above. Before moving on, some preliminaries are proposed.

*Definition 1.* (Smith (1995)) A Metzler matrix is a real square matrix with nonnegative off-diagonal entries.

*Lemma 1.* (Skelton et al. (1997)) Under Assumption 1, for any $Q \succ 0$, there exists a unique matrix $P \succ 0$ which solves the following algebraic Riccati equation (ARE):

$$A^TP + PA - PBB^TP + Q = 0.$$

*Lemma 2.* (Smith (1995); Luenberger (1979)) Given a non-autonomous system described by $\dot{x}(t) = Ax(t)+B(t)$, where $A$ is a Metzler matrix and $B(t) \ge 0$. Then, $x(t) \ge 0$ for $\forall t > 0$, provided that $x(0) \ge 0$.

By virtue of these lemmas, one has Proposition 1.

*Proposition 1.* Suppose that the observer gain $F$ is chosen to make $A+FC$ Metzler, under the premise $\underline{\delta}_i(0) \le \delta_i(0) \le \overline{\delta}_i(0)$, the pair $\left(\underline{\delta}_i, \overline{\delta}_i\right)$ form a neighborhood framer for system (1) with $u_i$ given in (5), if $\underline{\Omega}_i \le \Omega_i \le \overline{\Omega}_i$ holds for $t \ge 0$.

**Proof.** For system (1) with $u_i$ in (5), we have

$$\dot{\delta}_i = A\delta_i - BK \sum_{j\in N(i)} g_{ij}\left[\left(\overline{\delta}_i - \overline{\delta}_j\right) + \left(\underline{\delta}_i - \underline{\delta}_j\right)\right] + \Omega_i. \tag{6}$$

Let $\overline{e}_i = \overline{\delta}_i - \delta_i$ and $\underline{e}_i = \delta_i - \underline{\delta}_i$. It follows from (2) and (6) that

$$\dot{\overline{e}}_i = (A + FC)\overline{e}_i + \overline{\Omega}_i - \Omega_i,$$
$$\dot{\underline{e}}_i = (A + FC)\underline{e}_i + \Omega_i - \underline{\Omega}_i.$$

Since $\underline{\delta}_i(0) \le \delta_i(0) \le \overline{\delta}_i(0)$, we have $\overline{e}_i(0) \ge 0$ and $\underline{e}_i(0) \ge 0$. On the other hand, the pre-condition $\underline{\Omega}_i \le \Omega_i \le \overline{\Omega}_i$ implies that $\overline{\Omega}_i - \Omega_i \ge 0$ and $\Omega_i - \underline{\Omega}_i \ge 0$. By Lemma 2, we have $\overline{e}_i \ge 0$ and $\underline{e}_i \ge 0$ for all $t \ge 0$ if $A + FC$ is Metzler, which shows that $\left(\underline{\delta}_i, \overline{\delta}_i\right)$ is a neighborhood framer for system (1) with $u_i$ given in (5).

This completes the proof.

Next, we will design the feedback gain matrix $K$ and the observer gain matrix $F$ as

$$K = B^TP_1, \tag{7}$$

and

$$F = -P_2^{-1}C^T, \tag{8}$$

where $P_1 \succ 0$ and $P_2 \succ 0$ are the solutions of the ARE

$$A^TP_1 + P_1A - 2\mu P_1BB^TP_1 + I = 0, \mu \le \lambda_2,$$

and the LMI

$$A^TP_2 + P_2A - 2C^TC \prec 0,$$

respectively.

Motivated by Proposition 1, a naturally question turns out for $F$ given in (8). That is, how to ensure the Metzler property of the Hurwitz matrix $A + FC$?

In the following, we will introduce the time-varying coordinate transformation to eliminate the Metzler requirement on the Hurwitz matrix $A + FC$.

Firstly, there exists an invertible matrix $T$ such that $A + FC = TJT^{-1}$, where $J = \mathrm{diag}\{J_1,\ldots,J_s\}$ is the Jordan canonical form of $A + FC$. These $s$ matrices $J_k$ $(k = 1,\ldots,s)$ can be divided into two categories according to whether the eigenvalues of $A + FC$ are real or imaginary. Without loss generality, assume that the first $r$ $(r \le s)$ matrices are associated with the $r$ real eigenvalues with multiplicity $n_k$ of $A + FC$, then the rest matrices are associated with the complex eigenvalues with multiplicity

$m_k$ of $A+FC$, and then it turns out $n = \sum_{k=1}^{r} n_k + 2\sum_{k=r+1}^{s} m_k$. For $k = r+1, \ldots, s$, denote

$$J_k = \begin{pmatrix} \Lambda_k & I_2 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda_k & I_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \Lambda_k & I_2 \\ 0 & 0 & 0 & \cdots & 0 & \Lambda_k \end{pmatrix} \in \mathbb{R}^{2m_k \times 2m_k}, \quad (9)$$

where

$$\Lambda_k = \begin{pmatrix} -\alpha_k & \beta_k \\ -\beta_k & -\alpha_k \end{pmatrix} \in \mathbb{R}^{2\times 2},$$

with $\alpha_k > 0$ and $\beta_k > 0$.

Then, for system $\dot{e} = (A+FC)e$, according to the statement in (Wang et al. (2019)), there would exist a time-varying coordinate transformation $\zeta = S(t)e$, such that $\dot{\zeta} = M\zeta$, where

$$\begin{aligned} S(t) &= \mathrm{diag}\left\{I_q, \Gamma_{r+1}(t), \ldots, \Gamma_s(t)\right\} T, \\ M &= \mathrm{diag}\Big\{J_1, \ldots, J_r, M_{r+1}, \ldots, M_s\Big\} \in \mathbb{R}^{n\times n}, \end{aligned} \quad (10)$$

with $q = \sum_{k=1}^{r} n_k$, and for all $k = r+1, \ldots, s$,

$$\begin{aligned} \Gamma_k(t) &= \mathrm{diag}\Bigg\{\underbrace{\varphi_k, \ldots, \varphi_k}_{m_k}\Bigg\} \in \mathbb{R}^{2m_k\times 2m_k}, \\ \varphi_k &= \begin{pmatrix} \cos(\beta_k t) & -\sin(\beta_k t) \\ \sin(\beta_k t) & \cos(\beta_k t) \end{pmatrix}, \\ M_k &= \begin{pmatrix} -\alpha_k I_2 & I_2 & 0 & \cdots & 0 & 0 \\ 0 & -\alpha_k I_2 & I_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha_k I_2 & I_2 \\ 0 & 0 & 0 & \cdots & 0 & -\alpha_k I_2 \end{pmatrix} \\ &\in \mathbb{R}^{2m_k\times 2m_k}. \end{aligned} \quad (11)$$

The form of $M_k (k = r+1, \ldots, s)$ shows that $M$ is a Metzler matrix. Hereafter, $t$ in the time-varying matrix $S(t)$ will be omitted for simplicity. What is more, let $\Upsilon = S^{-1}$. As demonstrated in (Wang et al. (2019)), it turns that

$$\dot{S} = MS - S(A+FC),$$

and

$$\dot{\Upsilon} = -\Upsilon M + (A+FC)S^{-1}.$$

What is more, there hold $\|S\| \le \|T\|$ and $\|\Upsilon\| \le \|T\|$.

With the help of this time-varying coordinate transformation, for $K$ and $F$ given in (7) and (8), respectively, the interval observer can be designed according to Algorithm 1.

**Algorithm 1** The neighborhood interval observer construction for system (1).

**step 1)**: Define two dynamical networked systems:

$$\begin{cases} \dot{\overline{\xi}}_i = M\overline{\xi}_i - SBK\Upsilon \displaystyle\sum_{j\in N(i)} g_{ij}\left[\left(\overline{\xi}_i - \overline{\xi}_j\right) + \left(\underline{\xi}_i - \underline{\xi}_j\right)\right] \\ \quad - SFC\Upsilon\xi_i + S^+\overline{\Omega}_i - S^-\underline{\Omega}_i, \\ \dot{\underline{\xi}}_i = M\underline{\xi}_i - SBK\Upsilon \displaystyle\sum_{j\in N(i)} g_{ij}\left[\left(\overline{\xi}_i - \overline{\xi}_j\right) + \left(\underline{\xi}_i - \underline{\xi}_j\right)\right] \\ \quad - SFC\Upsilon\xi_i + S^+\underline{\Omega}_i - S^-\overline{\Omega}_i, \end{cases} \quad (12)$$

where

$$\begin{cases} \overline{\xi}_i(0) = S^+\overline{\delta}_i(0) - S^-\underline{\delta}_i(0), \\ \underline{\xi}_i(0) = S^+\underline{\delta}_i(0) - S^-\overline{\delta}_i(0). \end{cases} \quad (13)$$

**step 2)**: On the basis of $\left(\overline{\xi}_i, \underline{\xi}_i\right)$ in **step 1)**, define two dynamical systems:

$$\begin{cases} \overline{\delta}_i = \Upsilon^+\overline{\xi}_i - \Upsilon^-\underline{\xi}_i, \\ \underline{\delta}_i = \Upsilon^+\underline{\xi}_i - \Upsilon^-\overline{\xi}_i. \end{cases} \quad (14)$$

**step 3)**: For $\left(\overline{\delta}_i, \underline{\delta}_i\right)$ obtained in **step 2)**, $u_i$ is designed as:

$$u_i = -K\left(\overline{\delta}_i + \underline{\delta}_i\right). \quad (15)$$

*Theorem 3.* Consider a networked system consisting of $N$ interplays on a connected graph $\mathcal{G}$, where each agent is steered by (1). Under the premise $\underline{\delta}_i(0) \le \delta_i(0) \le \overline{\delta}_i(0)$, for $K$ and $F$ given in (7) and (8), respectively, $\left(\overline{\delta}_i, \underline{\delta}_i\right)$ in (14) form a neighborhood interval observer for system (1) with control algorithm (15), if $\underline{\Omega}_i \le \Omega_i \le \overline{\Omega}_i$ holds for $t \ge 0$.

**Proof.** The proof includes two parts. The first part devotes to the explanation that $\left(\overline{\delta}_i, \underline{\delta}_i\right)$ is a neighborhood framer for system (1) with control algorithm (14), that is establishment of the relation $\underline{\delta}_i \le \delta_i \le \overline{\delta}_i$, while the second one switches to the cooperative behavior of the system.

Let $\xi_i = S\delta_i$, then it follows from (6) that

$$\begin{aligned} \dot{\xi}_i =& S\dot{\delta}_i + \dot{S}\delta_i \\ =& M\xi_i - SBK\Upsilon \sum_{j\in N(i)} g_{ij}\left[\left(\overline{\xi}_i - \overline{\xi}_j\right) + \left(\underline{\xi}_i - \underline{\xi}_j\right)\right] \\ & - SFC\Upsilon\xi_i + S\Omega_i. \end{aligned} \quad (16)$$

Denote $\overline{E}_i = \overline{\xi}_i - \xi_i$ and $\underline{E}_i = \xi_i - \underline{\xi}_i$, by (12) and (16), one has

$$\begin{aligned} \dot{\overline{E}}_i &= M\overline{E}_i + S^+\overline{\Omega}_i - S^-\underline{\Omega}_i - S\Omega_i, \\ \dot{\underline{E}}_i &= M\underline{E}_i + S\Omega_i - S^+\underline{\Omega}_i + S^-\overline{\Omega}_i, \end{aligned} \quad (17)$$

with

$$\begin{aligned} \overline{E}_i(0) &= \overline{\xi}_i(0) - S\delta_i(0) \\ &= S^+\overline{\delta}_i(0) - S^-\underline{\delta}_i(0) - \left(S^+ - S^-\right)\delta_i(0) \\ &= S^+\left(\overline{\delta}_i(0) - \delta_i(0)\right) + S^-\left(\delta_i(0) - \underline{\delta}_i(0)\right), \\ \underline{E}_i(0) &= S\delta_i(0) - \underline{\xi}_i(0) \\ &= \left(S^+ - S^-\right)\delta_i(0) - \left(S^+\underline{\delta}_i(0) - S^-\overline{\delta}_i(0)\right) \\ &= S^+\left(\delta_i(0) - \underline{\delta}_i(0)\right) + S^-\left(\overline{\delta}_i(0) - \delta_i(0)\right). \end{aligned}$$

Since $S^+ \ge 0$, $S^- \ge 0$ together with $\overline{\delta}_i(0) - \delta_i(0) \ge 0$ and $\delta_i(0) - \underline{\delta}_i(0) \ge 0$, $\overline{E}_i(0) \ge 0$ and $\underline{E}_i(0) \ge 0$. On the other hand, $M$ is Metzler. By Lemma 2, we have $\overline{E}_i \ge 0$ for $t \ge 0$ and $\underline{E}_i \ge 0$ for $t \ge 0$, which further implies that

$$\underline{\xi}_i \leq \xi_i \leq \overline{\xi}_i, t \geq 0. \tag{18}$$

For (18), we have $\Upsilon^+\underline{\xi}_i \leq \Upsilon^+\xi_i \leq \Upsilon^+\overline{\xi}_i$ for all $t \geq 0$ and $\Upsilon^-\underline{\xi}_i \leq \Upsilon^-\xi_i \leq \Upsilon^-\overline{\xi}_i$ for all $t \geq 0$. So, it turns that $\Upsilon^+\underline{\xi}_i - \Upsilon^-\overline{\xi}_i \leq \Upsilon^+\xi_i - \Upsilon^-\xi_i \leq \Upsilon^+\overline{\xi}_i - \Upsilon^-\underline{\xi}_i$ for all $t \geq 0$. That is, $\underline{\delta}_i \leq \delta_i \leq \overline{\delta}_i$ for all $t \geq 0$. Therefore, $\left(\overline{\delta}_i, \underline{\delta}_i\right)$ in (14) is a neighborhood interval observer for system (1) with control algorithm (15).

Next, we devote ourselves to the cooperative behavior of system (1) with control algorithm (15).

For system (1) with control algorithm (15), we have

$$\begin{aligned}\dot{\delta} =& (I \otimes A)\, \delta - \left(L \otimes BB^T P_1\right) \left(\overline{\delta} + \underline{\delta}\right) + \Omega \\ =& (I \otimes A)\, \delta - \left(L \otimes BB^T P_1 \Upsilon\right) \left(\overline{\xi} + \underline{\xi}\right) + \Omega \\ =& \left(I \otimes A - 2L \otimes BB^T P_1\right) \delta - \left(L \otimes BB^T P_1\right) (I \otimes \Upsilon)\, \overline{E} \\ &+ \left(L \otimes BB^T P_1\right) (I \otimes \Upsilon)\, \underline{E} + \Omega.\end{aligned} \tag{19}$$

Construct a Lyapunov function candidate as

$$\begin{aligned}V &= V_1 + V_2 + V_3, \\ V_1 &= \delta^T (I \otimes P_1)\, \delta, \\ V_2 &= \left(\lambda_{\max}(P_1 BB^T P_1) N \|T\|^2 + 1\right) \overline{E}^T (I \otimes Q)\, \overline{E}, \\ V_3 &= \left(\lambda_{\max}(P_1 BB^T P_1) N \|T\|^2 + 1\right) \underline{E}^T (I \otimes Q)\, \underline{E},\end{aligned} \tag{20}$$

where $Q \succ 0$ is the unique solution of the Lyapunov equation $M^T Q + QM + I = 0$. We denote $\Theta = P_1 BB^T P_1$ for simplicity.

Taking the derivative of $V_1$ according to (19) yields

$$\begin{aligned}\dot{V}_1 =& \delta^T \left[I \otimes \left(A^T P_1 + P_1 A\right) - 4L \otimes \Theta\right] \delta \\ &- \left[(I \otimes \Upsilon)\, \overline{E}\right]^T (L \otimes \Theta)\, \delta - \delta^T (L \otimes \Theta) \left[(I \otimes \Upsilon)\, \overline{E}\right] \\ &+ \left[(I \otimes \Upsilon)\, \underline{E}\right]^T (L \otimes \Theta)\, \delta + \delta^T (L \otimes \Theta) \left[(I \otimes \Upsilon)\, \underline{E}\right] \\ &+ \Omega^T (I \otimes P_1)\, \delta + \delta^T (I \otimes P_1)\, \Omega \\ \leq& -\frac{1}{2}\delta^T \delta + 2\Omega^T \left(I \otimes P_1^2\right) \Omega \\ &+ \overline{E}^T \left(L \otimes \Upsilon^T \Theta \Upsilon\right) \overline{E} + \underline{E}^T \left(L \otimes \Upsilon^T \Theta \Upsilon\right) \underline{E} \\ \leq& -\frac{1}{2}\delta^T \delta + 2\lambda_{\max}^2(P_1)\Omega^T \Omega \\ &+ \lambda_{\max}(\Theta) N \|T\|^2 \left(\overline{E}^T \overline{E} + \underline{E}^T \underline{E}\right),\end{aligned} \tag{21}$$

while $\dot{V}_2$ and $\dot{V}_3$ according to (17) are as follows:

$$\begin{aligned}\dot{V}_2 =& \left(\lambda_{\max}(\Theta) N \|T\|^2 + 1\right) \Big[-\overline{E}^T \overline{E} \\ &+ \overline{\Omega}^T \left(I \otimes (S^+)^T Q\right) \overline{E} + \overline{E}^T \left(I \otimes QS^+\right) \overline{\Omega} \\ &- \underline{\Omega}^T \left(I \otimes (S^-)^T Q\right) \overline{E} - \overline{E}^T \left(I \otimes QS^-\right) \underline{\Omega} \\ &- \Omega^T \left(I \otimes S^T Q\right) \overline{E} - \overline{E}^T (I \otimes QS)\, \Omega\Big],\end{aligned} \tag{22}$$

and

$$\begin{aligned}\dot{V}_3 =& \left(\lambda_{\max}(\Theta) N \|T\|^2 + 1\right) \Big[-\underline{E}^T \underline{E} \\ &+ \overline{\Omega}^T \left(I \otimes (S^-)^T Q\right) \underline{E} + \underline{E}^T \left(I \otimes QS^-\right) \overline{\Omega} \\ &- \underline{\Omega}^T \left(I \otimes (S^+)^T Q\right) \underline{E} - \underline{E}^T \left(I \otimes QS^+\right) \underline{\Omega} \\ &+ \Omega^T \left(I \otimes S^T Q\right) \underline{E} + \underline{E}^T (I \otimes QS)\, \Omega\Big].\end{aligned} \tag{23}$$

Once $\overline{\Omega} = \underline{\Omega} = 0$, $\Omega = 0$, then by (21), (22) and (23), we have

$$\dot{V} \leq -\frac{1}{2}\delta^T \delta - \overline{E}^T \overline{E} - \underline{E}^T \underline{E},$$

which further implies that $\lim_{t\to\infty} \delta_i = 0$, that is, $\lim_{t\to\infty} (x_i - x_j) = 0$ for all $i, j = 1, \ldots, N$. Thus, $\left(\overline{\delta}_i, \underline{\delta}_i\right)$ in (14) is a neighborhood interval observer for system (1) with control algorithm (15).

This completes the proof.

Back to the proof of Theorem 4, if $\overline{\Omega} \neq 0$ and $\underline{\Omega} \neq 0$, then for $V$ in (20), we have

$$\begin{aligned}\dot{V} \leq& -\frac{1}{2}\delta^T \delta - \frac{1}{2}\overline{E}^T \overline{E} - \frac{1}{2}\underline{E}^T \underline{E} + 2\lambda_{\max}^2(P_1)\Omega^T \Omega \\ &+ \frac{12\|T\|^2 \left(\lambda_{\max}(\Theta) N \|T\|^2 + 1\right)^2 \lambda_{\max}^2(Q)}{\varepsilon} \\ &\times \left(\overline{\Omega}^T \overline{\Omega} + \underline{\Omega}^T \underline{\Omega} + \Omega^T \Omega\right).\end{aligned} \tag{24}$$

Let $\omega^* = \max\left\{\|\underline{\Omega}\|, \|\overline{\Omega}\|\right\}$. Since $\underline{\Omega} \leq \Omega \leq \overline{\Omega}$, then it follows from (24) that

$$\begin{aligned}\dot{V} \leq& -\frac{1}{2}\delta^T \delta - \frac{1}{2}\overline{E}^T \overline{E} - \frac{1}{2}\underline{E}^T \underline{E} + \Omega^* \\ \leq& -\frac{1}{2\lambda_{\max}(P_1)} V_1 - \frac{1}{2\left(\lambda_{\max}(\Theta) N \|T\|^2 + 1\right) \lambda_{\max}(Q)} V_2 \\ &- \frac{1}{2\left(\lambda_{\max}(\Theta) N \|T\|^2 + 1\right) \lambda_{\max}(Q)} V_3 \\ \leq& -\frac{\varsigma}{2} V + \Omega^*,\end{aligned} \tag{25}$$

with

$$\begin{aligned}\varsigma =& \min\left\{\frac{1}{\lambda_{\max}(P_1)}, \frac{1}{\lambda_{\max}(\Theta) N \|T\|^2 + 1 \lambda_{\max}(Q)}\right\}, \\ \Omega^* =& 36\|T\|^2 \left(\lambda_{\max}(\Theta) N \|T\|^2 + 1\right)^2 \lambda_{\max}^2(Q) \left(\omega^*\right)^2 \\ &+ 2\lambda_{\max}^2(P_1) \left(\omega^*\right)^2,\end{aligned}$$

which implies the bounded consensus of system (1).

*Theorem 4.* Consider a networked system consisting of $N$ interplays on a connected graph $\mathcal{G}$, where each agent is steered by (1) with control algorithm (15). Suppose that $\underline{\delta}_i(0) \leq \delta_i(0) \leq \overline{\delta}_i(0)$, then system (1) can achieve bounded consensus if $\underline{\Omega}_i \leq \Omega_i \leq \overline{\Omega}_i$ holds for $t \geq 0$.

**Proof.** By (25), one has $V$ will not decrease until $V \geq \frac{2\Omega^*}{\varsigma}$, which further implies that $\|\delta\| \leq \sqrt{\frac{2\Omega^*}{\varsigma\lambda_{\min}(P_1)}}$. Thus, the bounded consensus can be achieved.

This completes the proof.

*Remark 5.* We name the interval observer for the networked systems with uncertain disturbances in (Wang et al. (2019)) as the local interval observer, where each observer is used to estimate the bound on the absolute output information of each agent. Differently, in this paper, each observer estimates the bound on the sum of relative information associated with each agent and its all neighbors, which is called the neighborhood interval observer according to the way of naming in (Zhang et al. (2011)). What is more, in comparison with the local interval observer in (Wang et al. (2019)), one of the superiorities of the neighborhood interval observer is that the acquirement of the relative information is easily than that of the absolute information. On the other hand, as shown the definition of $\Omega_i$ in (3), once $\omega_i = \omega_j$ for all $i, j = 1, \ldots, N$, that is, system (1) turns to

$$\begin{cases} \dot{x}_i = Ax_i + Bu_i + \omega, i = 1, 2, \cdots, N, \\ y_i = Cx_i, \end{cases} \tag{26}$$

then $\Omega_i = 0$. In this case, if $\overline{\Omega}_i = \underline{\Omega}_i = 0$, then system (26) can achieve consensus in line with the above analysis. This finding further implies that the neighborhood interval observer processes more robust than the local one.

## 4. CONCLUSIONS

In this paper, the concepts of the local interval observer and the neighborhood interval observer for the multi-agent systems are introduced. The main efforts are devoted to the construction of the neighborhood interval observer for multi-agent systems with disturbances. It firstly shows that the Metzler premise on the matrix $A + FC$ is a fundamental requirement for the interval observer construction. It also shows that there exists a time-varying coordinate transformation to eliminate the Metzler requirement on $A + FC$ in the neighborhood interval observer construction. So, we can construct a neighborhood interval observer for any stabilizable and detectable multi-agent systems according to the algorithm provided in this paper.

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