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Control and Filtering Problems for Linear Time-Varying Systems Based on Ellipsoidal Reachable Sets [⋆]

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Abstract: The paper is devoted to reachable sets of linear time-varying continuous systems under uncertain initial states and disturbances with a bounded uncertainty measure. The uncertainty measure is the sum of a quadratic form of the initial state and the integral over the finite-time interval from a quadratic form of the disturbance. It is shown that the reachable set of the system under this assumption is an evolving ellipsoid with a matrix being a solution to the linear matrix differential equation. This result is used to synthesize the optimal observer providing the minimal ellipsoidal set as the estimate of the system state, as well as optimal controllers steering the system state into a final target ellipsoidal set or keeping the entire system trajectory in a prescribed ellipsoidal tube under all admissible initial states and disturbances. The relationship between the optimal ellipsoidal observer and the Kalman filter are established. Numerical modeling with the Mathieu equation for parametric vibrations of a linear oscillator illustrates the results.

Keywords: Ellipsoidal reachable set, generalized H_2 norm, Kalman filter, optimal ellipsoidal control, LMI, LTV system.

1. INTRODUCTION

We consider problems of estimation, filtering and control in dynamic systems under uncertainty, which arise when the initial state of the system is not known exactly and the system dynamics and output measurements are corrupted by disturbances. In such situations, there exist two basic approaches, stochastic and deterministic. The first one is to model the initial state as a random vector and the state and measurement disturbances as additive stochastic processes and minimize the expectation of a cost functional. The second approach considers uncertainties as unknown except for the fact that they belong to prescribed, bounded sets. The optimal solution is the one that achieves the best performance under the worst possible uncertainties. The central concept that emerges in the latter approach is that of the reachable set, which is the set of states to which the system can be steered under all possible initial states and disturbances. Characterization of the reachable set under uncertainties allows to synthesize systems with the reachable sets included at a given moment or during a given time interval in the desired target set or tube with optimal characteristics.

The problem of finding or evaluating reachable sets has been actively studied since the late 60s of the last century and still continues to attract the attention of specialists in the field of control theory and its applications. See Schweppe (1968); Bertsekas and Rhodes (1971); Kurzhanski and Valyi (1997); Chernousko and Ovseevich (2004) for more details. By virtue of the linearity of the system, the state at a given moment is the sum of two vectors, the state of the undisturbed system with an uncertain initial state and the state of the disturbed system with zero initial state. If the set of the initial states is chosen ellipsoidal and the reachable set of the disturbed system is approximated by an ellipsoid, then the problem arises of describing the geometric sum of two ellipsoids, which is a convex set, but not an ellipsoid. Moreover, in problems of recurrent estimation, the need arises to find the smallest ellipsoid that includes the intersection of two ellipsoids. To describe such sets, Schweppe (1968); Boyd et al. (1994); Kurzhanski and Valyi (1997); Chernousko and Ovseevich (2004); Kurzhanskiy and Varaiya (2011); Wang et al. (2019) have derived upper and lower ellipsoidal approximations for reachable sets. All this led to the development of the technique of operating with ellipsoids. Despite significant progress in this direction, associated with the use of linear matrix inequalities and related software, the problem remains open due to the fact that the methods based on ellipsoidal approximations of reachable sets are difficult to apply for the synthesis of optimal control systems with the exception of very simple cases.

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A recent paper Balandin et al. (2019) introduced the concept of a maximum output deviation for a linear time-varying system over a finite-time interval under uncertain initial states and disturbances, which was called the generalized H_2 norm with transients. Essentially, this is the induced norm of the operator generated by the system and mapping the pair, consisting of an initial state vector and a vector-function of the disturbance, to the output. The squared "size" of the pair is measured by the sum of the quadratic form of the initial state and the integral of the quadratic form of the disturbance, while the "size" of the output is measured by the maximum of the peak value of its Euclidean norm over a finite-time interval. Such an operator is also used in Balandin et al. (2019); Amato et al. (2019) in characterizing the finite-time boundedness of linear time-varying systems. For a linear time-invariant system on an infinite time interval under zero initial conditions, a similar characteristic was introduced by Wilson (1989) and called the generalized H_2 norm of the system.

The maximum output deviation was characterized by Balandin and Kogan (2019); Balandin et al. (2019) in terms of solutions to both the linear matrix differential equation and inequalities, and the optimal control minimizing the maximal output deviation was synthesized. These results led the authors to the idea that when the sum of the quadratic form of the initial state and the integral of the quadratic form of the disturbance is bounded above by a specified value, the state of the system belongs to an ellipsoid with a matrix satisfying the above-mentioned linear differential equation. Confirmation of this assumption was found, at least partially, in Kurzhanski and Valyi (1997), where the dynamic programming method showed that, under a similar constraint, the reachable set is the ellipsoid with a matrix being the solution of the Riccati differential equation. Based on this result, an optimal observer was synthesized in Kurzhanski and Valyi (1997), providing an ellipsoidal estimate of the system state. However, these results were obtained under rather burdensome assumptions of non-degeneracy of the quadratic forms of the initial state and disturbances, which means that the initial state must belong to a non-degenerate ellipsoid, and disturbances must be present in the equation for each state component and in the measurement of each output component.

In this paper, these results are developed in several directions for linear time-varying continuous systems. The novel contribution of this paper is as follows. First, it is shown that in the case of degenerate quadratic forms in the uncertainty measure and, in particular, in the extreme cases when there are no disturbances or when the initial state is zero, the reachable sets of the system are also ellipsoids. The need to study reachable sets in the case of a degenerate quadratic form of the initial state arises, for example, in the control problems of mechanical systems with shock effects, when some state variables are known while others experience instantaneous uncertain changes. Consideration of this question has required a different approach to substantiate the result, which led to the linear matrix differential equation describing the dynamics of the ellipsoidal reachable sets, instead of the Riccati differential equation. It is also established that the value of the semi-axis of the ellipsoidal reachable set really coincides with the generalized H_2 norm with transients. Secondly, the equations of the optimal ellipsoidal observer and estimator are obtained, which provides the estimate of the state or the vector of unknown parameters in the form of the ellipsoid of minimum size, including the degenerate case, when disturbances in the system and measurements may be absent in some equations. The relationships between the optimal ellipsoidal observer and the Kalman filter is revealed, which creates a bridge between the deterministic and stochastic approaches to linear filtering. Third, it is shown how to synthesize a bounded control under which the state of the system falls into the final target ellipsoidal set or the entire trajectory of the system is retained in a specified ellipsoidal tube. All the results are illustrated by the numerical examples with a linear non-stationary oscillator described by the Mathieu equation.

2. ELLIPSOIDAL REACHABLE SETS

Consider a dynamic system described by the linear non-stationary differential equation

$$\dot{x} = A(t)x + B(t)v, \quad x(t_0) = x_0, \quad t \in [t_0, t_f], \tag{1}$$

where $x \in \mathrm{R}^{n_x}$ is the state, $v \in \mathrm{R}^{n_v}$ is the disturbance input. Denote

$$|a|_Q^2 = a^{\mathrm{T}} Q^{-1} a, \quad \|b\|^2_{M\,[t_0,t]} = \int\limits_{t_0}^{t} b^{\mathrm{T}}(\sigma) M^{-1}(\sigma) b(\sigma)\, d\sigma$$

for non-degenerate matrices Q and $M(\sigma)$. The Euclidean norm (corresponding to $Q = I$) and the L_2-norm are denoted by $|a|$ and $\|b\|_{[t_0,t]}$.

Assume that initial state $x(t_0)$ and disturbance $v = v(\sigma)$, $\sigma \in [t_0, t]$ belong to a set of pairs of admissible initial states and disturbances defined as follows

$$\begin{aligned} \mathcal{S}(t, \tau; R, G) = \{(x, v(\sigma)) : \, x = R^{1/2} w_1, \\ v(\sigma) = G^{1/2}(\sigma) w_2(\sigma), \, |w_1|^2 + \|w_2\|^2_{[\tau,t]} \leq 1\} \end{aligned} \tag{2}$$

for a given $(n_x \times n_x)$-matrix $R^{\mathrm{T}} = R \geq 0$ and a given $(n_v \times n_v)$-matrix function $G^{\mathrm{T}}(\sigma) = G(\sigma) \geq 0$, $\sigma \in [\tau, t]$, $\tau \geq t_0$. If $R > 0$ and $G(\sigma) > 0$, $\sigma \in [\tau, t]$, let us pick w_1 and $w_2(\sigma)$ out first and second equalities in (2) and insert them in the third inequality, which result in the constraint

$$|x(\tau)|^2_R + \|v\|^2_{G\,[\tau,t]} \leq 1. \tag{3}$$

Thus, the definition (2) extends the constraint (3) to the case of the degenerate matrices R and/or $G(\sigma)$.

The sum of the quadratic form of the initial state and the integral of the quadratic form of the disturbance can intuitively be interpreted as the uncertainty measure squared, and the constraint (3) itself can be "explained" as follows. The system state at the moment t depends linearly on the initial state and disturbance and their "growth" results in the corresponding "growth" of the current system state. Therefore, in order to characterize the behavior of the system under uncertain initial states and disturbances, it is reasonable to normalize the current value of the Euclidean norm of the state by the value equal to the specified sum or constrain the specified sum by one, which is the same for linear systems.

The weighting matrices R and $G(\sigma)$ reflect relative importance of the uncertainty in initial states over the uncertainty in disturbances. From (2) it follows that the set of initial states coincides with the ellipsoid $\mathcal{E}(R) = \{x = R^{1/2}w \quad |w| \leq 1\}$. For $R > 0$ we arrive at the standard form $\mathcal{E}(R) = \{x : |x|^2_R \leq 1\}$. If $R \geq 0$, than $\mathcal{E}(R)$ is a degenerate ellipsoid with affine dimension equal to rank of the matrix R (see, for example, Boyd and Vandenberghe (2004)). We denote by $\varphi(t; \tau, x, v)$ the solution of the equation (1) at the moment t with initial state $\varphi(\tau) = x$ under appropriate function $v = v(\sigma)$, $\sigma \in [\tau, t]$. The problem is to describe the set of all system states at the moment t under all admissible initial states and disturbances from set $\mathcal{S}(t, t_0; R, G)$.

Definition. Reachable set of the system (1) at the moment $t \geq \tau$ under all admissible pairs of initial states $x_\tau \in \mathcal{E}(R)$ at the moment τ and disturbances $v(\sigma)$, $\sigma \in [\tau, t]$ from set $\mathcal{S}(t, \tau; R, G)$ is the set of all states $\varphi(t; \tau, x_\tau, v)$ denoted by

$$\begin{aligned} \mathcal{D}(t, \tau, \mathcal{E}(R)) = \{x : \; & \varphi(t; \tau, x_\tau, v) = x \\ & \forall\, (x_\tau, v) \in \mathcal{S}(t, \tau; R, G)\}. \end{aligned} \tag{4}$$

Theorem 1. The reachable set of the system (1) at the moment $t \geq t_0$ under all pairs of initial states and disturbances from set $\mathcal{S}(t, t_0; R, G)$, $t \in [t_0, t_f]$ with $R \geq 0$ and $G(\sigma) \geq 0$, $\sigma \in [t_0, t]$ is the ellipsoid

$$\mathcal{D}(t, t_0, \mathcal{E}(R)) = \mathcal{E}(P(t)) \tag{5}$$

with matrix $P(t) \geq 0$ being the solution to the linear matrix differential equation

$$\dot{P} = A(t)P + PA^{\mathrm{T}}(t) + B(t)G(t)B^{\mathrm{T}}(t) \tag{6}$$

under initial condition $P(t_0) = R$.

Proof: Equation (6) has the solution

$$\begin{aligned} P(t) = \Phi(t, t_0)P(t_0)\Phi^{\mathrm{T}}(t, t_0) + \\ \int_{t_0}^{t} \Phi(t, \tau)B(\tau)G(\tau)B^{\mathrm{T}}(\tau)\Phi^{\mathrm{T}}(t, \tau)d\tau, \end{aligned} \tag{7}$$

where $\Phi(t, \tau)$ is the solution to the equation

$$\frac{d\Phi(t, \tau)}{dt} = A(t)\Phi(t, \tau), \quad \Phi(\tau, \tau) = I.$$

At first, we consider the non-degenerate case when $R > 0$ and $G(\sigma) > 0$, $\sigma \in [t_0, t_f]$. Since $\Phi(t, \tau)$ is a non-degenerate matrix, we get $P(t) > 0$, $t \in [t_0, t_f]$. Consider a positive definite quadratic form $V(t, x) = x^{\mathrm{T}} P^{-1}(t)x$ with the matrix $P(t)$ satisfying equation (6). Compute its derivative with respect to the system (1) taking into account that $d(P^{-1})/dt = -P^{-1}(\dot{P})P^{-1}$:

$$\dot{V} = v^{\mathrm{T}} G^{-1} v - (v - v_*)^{\mathrm{T}} G^{-1}(v - v_*), \tag{8}$$

where $v_*(t) = G(t)B^{\mathrm{T}}(t)P^{-1}(t)x(t)$, and $x(t)$ is a solution to the equation

$$\dot{x} = [A(t) + B(t)G(t)B^{\mathrm{T}}(t)P^{-1}(t)]x. \tag{9}$$

Integrating (8) over $[t_0, t]$ results in

$$|x(t)|^2_{P(t)} = |x(t_0)|^2_R + \|v\|^2_{G\,[t_0,t]} - \|v - v_*\|^2_{G\,[t_0,t]}. \tag{10}$$

Since $R > 0$ and $G(t) > 0$, the condition $(x_0, v(\sigma)) \in \mathcal{S}(t, t_0; R, G)$ is equivalent to inequality (3) at $\tau = 0$. Therefore, $x^{\mathrm{T}}(t)P^{-1}(t)x(t) \leq 1$, i.e. $x(t) \in \mathcal{E}(P(t))$.

Let us show that, for any state $\bar{x} \in \mathcal{E}(P(t))$, there exists a state $\bar{x}_0 \in \mathcal{E}(R)$ and a disturbance $\bar{v}(\sigma)$, $\sigma \in [t_0, t]$ from $\mathcal{S}(t, t_0; R, G)$ such that $\varphi(t; t_0, \bar{x}_0, \bar{v}) = \bar{x}$. Choose $\bar{v}(\sigma) = v_*(\sigma)$, where $x(\sigma)$ is the solution to the equation (9) with the terminal condition $x(t) = \bar{x}$. Evidently, the initial state of this trajectory $\bar{x}_0 = x(t_0)$ is the state required. In view of (10) we have

$$|\bar{x}_0|^2_R + \|v_*\|^2_{G\,[t_0,t]} = \bar{x}^{\mathrm{T}} P^{-1}(t)\bar{x} \leq 1.$$

Thus, $\bar{x}_0^{\mathrm{T}} R^{-1} \bar{x}_0 \leq 1$, i.e. $\bar{x}_0 \in \mathcal{E}(R_0)$.

Consider now the degenerate case when $R \geq 0$ and $G(t) \geq 0$, $t \in [t_0, t_f]$. Introduce matrices

$$R_\varepsilon = R + \varepsilon I > 0, \quad G_\varepsilon(t) = G(t) + \varepsilon I > 0.$$

Let $P_\varepsilon(t) > 0$ be a solution to the equation (6) with $G(t)$ replaced by $G_\varepsilon(t)$ and initial condition $P(t_0) = R_\varepsilon$. Then the non-degenerate ellipsoid $\mathcal{E}(P_\varepsilon(t))$ is the reachable set under initial states and disturbances from $\mathcal{S}(t, t_0; R_\varepsilon, G_\varepsilon)$. Since solution $P(t)$ of the equation (6) depends continuously on initial conditions and parameters, we get $P_\varepsilon(t) \to P(t)$, $t \in [t_0, t_f]$ as $\varepsilon \to 0$. Consequently, the reachable set under initial states and disturbances from $\mathcal{S}(t, t_0; R, G)$ in the case of $R \geq 0$ and $G(t) \geq 0$ is the ellipsoid $\mathcal{E}(P(t))$ as well.

Corollary 1. The reachable set of the system (1) under initial states and disturbances from $\mathcal{S}(t, t_0; R, G)$ possesses the evolutionary property

$$\mathcal{D}(t, t_0, \mathcal{E}(R)) = \mathcal{D}(t, \tau, \mathcal{D}(\tau, t_0, \mathcal{E}(R)),$$

since $\mathcal{D}(\tau, t_0, \mathcal{E}(R)) = \mathcal{E}(P(\tau))$ and $\mathcal{D}(t, \tau, \mathcal{E}(P(\tau))) = \mathcal{D}(t, t_0, \mathcal{E}(R))$ for any $\tau \in [t_0, t]$.

Remark 1. Equation (6) coincides with the equation for the state covariance matrix of the system (1), when the initial state and disturbance are zero mean independent white noises with covariance matrices $Ex(t_0)x^{\mathrm{T}}(t_0) = R$ and $Ev(t)v^{\mathrm{T}}(t) = G(t)$ (see, for example, Kwakernaak and Sivan (1972)).

In particular case, when disturbances are absent and the initial state belongs to ellipsoid $\mathcal{E}(R)$, $R \geq 0$, i.e. the set of admissible initial states and disturbances is $\mathcal{S}(t, t_0; R, 0)$, the reachable set is the ellipsoid $\mathcal{E}(P_0(t))$ with matrix $P_0(t) \geq 0$ being a solution to the equation (6) for $G(t) \equiv 0$ with initial condition $P_0(t_0) = R$. In another particular case, when the initial state is zero, i.e. the set of admissible initial states and disturbances is $\mathcal{S}(t, t_0; 0, G)$, the reachable set is the ellipsoid $\mathcal{E}(P_v(t))$ with matrix $P_v(t) \geq 0$ being a solution to the equation (6) with zero initial condition $P_v(t_0) = 0$. Since the solution of the nonhomogeneous equation is of the form

$$P(t) = P_0(t) + P_v(t), \tag{11}$$

then $\mathcal{E}(P_0(t)) \subseteq \mathcal{E}(P(t))$ and $\mathcal{E}(P_v(t)) \subseteq \mathcal{E}(P(t))$.

Let $z = C_z(t)x \in \mathrm{R}^{n_z}$ be an output of the system (1). When a state $x(t)$ belongs to the ellipsoid $\mathcal{E}(P(t))$, i.e. $x(t) = P^{1/2}(t)w$ with $|w| \leq 1$, then $z(t) = C_z(t)P^{1/2}(t)w$. According to Theorem 1 a set of all such vectors is the ellipsoid $\mathcal{E}(C_z(t)P(t)C_z^{\mathrm{T}}(t)) = \{z : z = (C_z(t)P(t)C_z^{\mathrm{T}}(t))^{1/2}g, |g| \leq 1\}$. Hence, the maximal value of the output Euclidean norm coincides with the maximal semi-axis of this ellipsoid, i.e.

$$\max_{(x(t_0),v) \in \mathcal{S}(t,t_0;R,G)} |z(t)| = \lambda^{1/2}_{\max}\big(C_z(t)P(t)C_z^{\mathrm{T}}(t)\big). \tag{12}$$

For $R > 0$ and $G(\sigma) > 0$, the maximum of this value over the given interval $[t_0, t_f]$

$$\sup_{t\in[t_0,t_f]} \max_{(x_0,v)\in\mathcal{S}(t,t_0;R,G)} |z(t)| = \max_{x(t_0)\neq 0, v\not\equiv 0} \frac{\sup\limits_{t\in[t_0,t_f]} |z(t)|}{\left(|x(t_0)|_R^2 + \|v\|^2_{G\,[t_0,t]}\right)^{1/2}} = \sup_{t\in[t_0,t_f]} \lambda^{1/2}_{\max}(C_z(t)P(t)C_z^{\mathrm{T}}(t)) \tag{13}$$

coincides with the generalized H_2 norm with transients for the system (1), $z = C_z(t)x$, which was introduced in Balandin and Kogan (2019); Balandin et al. (2019).

Further, the result formulated in Theorem 1 can be generalized to the case when the initial state ellipsoid is not centered at the origin and its center is in a given point x_*. Indeed, let the initial state and disturbance $(x_0, v(\sigma))$ satisfy the constraint

$$x_0 = x_* + R^{1/2}w_1,\; v(\sigma) = G^{1/2}(\sigma)w_2(\sigma),$$
$$|w_1|^2 + \int_{t_0}^{t} |w_2(\sigma)|^2\, d\sigma \le 1,$$

and, in the case of non-degenerated matrices R and $G(\sigma)$, the constraint

$$|x_0 - x_*|_R^2 + \|v\|^2_{G\,[t_0,t]} \le 1.$$

It means that the initial state belongs to the ellipsoid $\mathcal{E}(R, x_*)$ centered at x_* with matrix R. Due to linearity of the system its solution is presented as a sum of two terms

$$\varphi(t; t_0, x_0, v) = \varphi(t; t_0, x_0 - x_*, v) + \varphi_*(t),$$
$$\varphi_*(t) = \varphi(t; t_0, x_*, 0),$$

where the first term is the solution of the system with initial state $x_0 - x_*$ and disturbance $v(\sigma)$, while the second term is the solution of the system without disturbances with initial state x_*. It follows then from Theorem 1 that the reachable set at the moment t is the ellipsoid with matrix $P(t)$ centered at the point $\varphi_*(t)$, i.e.

$$\mathcal{D}(t, t_0, \mathcal{E}(R, x_*)) = \mathcal{E}(P(t), \varphi_*(t)).$$

3. OPTIMAL ELLIPSOIDAL FILTERING

Consider a filtering problem for the linear system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)v(t),\\ y(t) &= C(t)x(t) + D(t)v(t), \end{aligned} \tag{14}$$

where the initial state $x(t_0)$ and disturbance $v(\sigma)$ are unobserved, and $y(\sigma)$, $\sigma \in [t_0, t]$ is the measured output. Assume that the admissible initial states and disturbances are presented in the form

$$\begin{aligned} x(t_0) - x_* = R^{1/2}w_1, &\quad v(\sigma) = G^{1/2}(\sigma)w_2(\sigma),\\ |w_1|^2 + \|w_2\|^2_{[t_0,t]} \le 1, &\quad t \in [t_0, t_f] \end{aligned} \tag{15}$$

with given matrix $R^{\mathrm{T}} = R \ge 0$ and matrix function $G^{\mathrm{T}}(\sigma) = G(\sigma) \ge 0$, $\sigma \in [t_0, t]$. Let an observer be given by the equation

$$\dot{\widehat{x}}(t) = A(t)\widehat{x}(t) + L(t)[y(t) - C(t)\widehat{x}(t)],\; \widehat{x}(t_0) = x_*, \tag{16}$$

where $\widehat{x}(t)$ is the estimate of the state $x(t)$ and $L(t)$ is the observer gain matrix to be determined. Denote estimation error $e(t) = x(t) - \widehat{x}(t)$ that satisfies the equation

$$\dot{e}(t) = A_c(t)e(t) + B_c(t)v(t), \quad e(t_0) = x(t_0) - x_*, \tag{17}$$

where $A_c(t) = A(t) - L(t)C(t)$, $B_c(t) = B(t) - L(t)D(t)$. According to Theorem 1 the reachable set of the system (17) at the moment t is the ellipsoid $\mathcal{E}(P(t))$ with the matrix $P(t)$ being the solution to the equation

$$\dot{P} = A_c(t)P + PA_c^{\mathrm{T}}(t) + B_c(t)G(t)B_c^{\mathrm{T}}(t) \tag{18}$$

with initial condition $P(t_0) = R$. This means that the state $x(t)$ of the system (14) belongs to the ellipsoid $\mathcal{E}(P(t), \widehat{x}(t))$ with the center $\widehat{x}(t)$ determined by the observer equation (16). It is reasonable to call this set an ellipsoidal estimate of the state $x(t)$. The observer with the gain matrix $L_*(t)$ providing, for example, the minimal trace of the matrix $P(t)$ is called optimal ellipsoidal one.

As noted in Remark 1, equation (18) is the same equation that describes dynamics of the covariance matrix $Ee(t)e^{\mathrm{T}}(t)$ when the initial state and disturbance are modeled as independent white noise processes with covariance matrices $Ex(t_0)x^{\mathrm{T}}(t_0) = R$ and $Ev(t)v^{\mathrm{T}}(t) = G(t)$, respectively. Since the trace of matrix $P(t)$ of the ellipsoidal estimate coincides with the variation of the estimation error in the stochastic case, the equation of the optimal observer coincides with the equation of the Kalman filter for the system (14) under the above-mentioned covariances. Note that additive stochastic state and measurement disturbances $\xi_1(t) = B(t)v(t)$ and $\xi_2(t) = D(t)v(t)$ in the system (14) are correlated with each other and have covariances

$$E\xi(t)\xi^{\mathrm{T}}(t) = \begin{pmatrix} B(t)G(t)B^{\mathrm{T}}(t) & B(t)G(t)D^{\mathrm{T}}(t) \\ * & D(t)G(t)D^{\mathrm{T}}(t) \end{pmatrix}, \tag{19}$$

where $\xi(t) = \mathrm{col}\left(\xi_1(t), \xi_2(t)\right)$. Taking into account the standard requirement within the framework of the Kalman filtering that the covariance matrix of measurement disturbances is to be non-degenerate (see, for example, Kwakernaak and Sivan (1972)), we arrive at the following statement.

Theorem 2. If $\det[D(\sigma)G(\sigma)D^{\mathrm{T}}(\sigma)] \neq 0$, $\sigma \in [t_0, t]$, the optimal ellipsoidal observer is determined by the equation (16) with the gain matrix

$$L^{(c)}(t) = [D(t)G(t)B^{\mathrm{T}}(t) + C(t)P_*(t)]^{\mathrm{T}}[D(t)G(t)D^{\mathrm{T}}(t)]^{-1},$$

where $P_*(t) \ge 0$ is the solution to the equation (18) for $L(t) = L^{(c)}(t)$ and $P_*(t_0) = R$.

Remark 2. Optimal ellipsoidal estimate of an output $z(t) = C_z(t)x(t) \in \mathrm{R}^{n_z}$ is the ellipsoid $\mathcal{E}(P_{z\,*}(t), \widehat{z}(t))$, where $P_{z\,*}(t) = C_z(t)P_*(t)C_z^{\mathrm{T}}(t)$, $\widehat{z}(t) = C_z(t)\widehat{x}(t)$ and $\widehat{x}(t)$ is the state estimate determined by the optimal ellipsoidal observer (16).

Remark 3. The correspondence established between the optimal ellipsoidal observer and the Kalman filter creates a bridge between the deterministic and stochastic approaches to linear filtering. Moreover, this relation reveals the following important property of the Kalman filter. Let $\widehat{x}(t)$ be the state estimate of the system (14) obtained by the Kalman filter under given covariances $K_x(t_0) = R$ and $K_v(t) \equiv G(t)$ and $P_*(t)$ be the covariance of the estimation

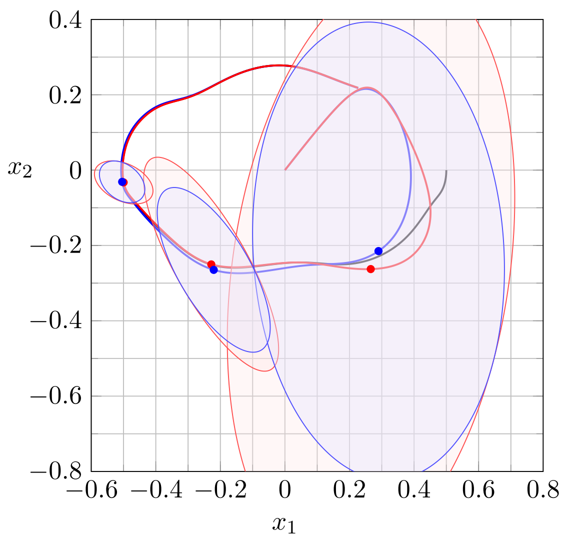

Fig. 1. Dynamics of the optimal ellipsoidal and generalized H_∞-optimal estimates for the Mathieu equation

error. Then one may assert that system state $x(t)$ belongs to the ellipsoid $\mathcal{E}(P_*(t), \widehat{x}(t))$ for any deterministic initial states and disturbances of the form (15).

In order to illustrate the statement of Theorem 2, we consider the famous Mathieu equation describing parametric vibrations of the linear oscillator. Rewrite this equation in the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\omega_0^2(1 + \varepsilon \sin \omega t)x_1 + v, \end{aligned} \tag{20}$$

where ω_0, ω and ε are given parameters, $v = v(t)$ is the external disturbance. It is known (see, for example, Neimark (2003)) that there exists a phenomenon of the so called parametric resonance under certain values of the parameters. Let the measured output be

$$y = x_1 + x_2 + v.$$

The system has the following parameters: $\omega_0 = \pi/6$, $\omega = 2\pi$, $\varepsilon = 0.1$, $R = 10.5I$, and $G(\sigma) \equiv 1$. Fig. 1 shows the system trajectory $x(t)$, denoted by the black line, with initial state $x_1(0) = 0.5$, $x_2(0) = 0$ under disturbance $v(t) = 0.05 \sin \pi t$ as well as the trajectory $\widehat{x}(t)$, denoted by the blue color, corresponding to the optimal ellipsoidal estimate and the appropriate ellipsoids $\mathcal{E}(P_*(t), \widehat{x}(t))$ at moments of time $t_1 = 2$, $t_2 = 4$, $t_3 = 6$. For comparison, the trajectory corresponding to the generalized H_∞-optimal estimate given in Nagpal and Khargonekar (1991) and the appropriate ellipsoids, denoted by the red color, are also presented in Fig. 1. This experiment demonstrates that "sizes" of the ellipsoids decrease significantly in time and that the red ellipsoids are larger than the blue ones.

4. OPTIMAL ELLIPSOIDAL CONTROL

In the previous sections, it was established that the state of a linear system under an uncertain initial state and a disturbance with the uncertainty measure being less or equal to unit belongs at each moment of time to an evolving ellipsoid. Now, we will show how to synthesize a bounded control in the form of non-stationary state-feedback $u = \Theta(t)x$ ensuring that, despite the presence of uncertainties: (i) the final output of the closed-loop system will lie in a prescribed target ellipsoidal set; or (ii) the entire output trajectory will lie inside a target ellipsoidal tube. Such a control law will be called *ellipsoidal control*.

Consider the closed-loop system of the form

$$\begin{aligned} \dot{x}(t) &= [A(t) + B_u(t)\Theta(t)]x(t) + B(t)v(t), \\ z(t) &= [C_z(t) + D(t)\Theta(t)]x(t), \quad x(t_0) = x_0, \end{aligned} \tag{21}$$

where $z(t)$ is the controlled output, $t \in [t_0, t_f]$. Assume that the admissible initial state and disturbance are in the set $\mathcal{S}(t, t_0; R, G)$, control input should be inside the ellipsoid $\mathcal{E}(Q_u(t))$ with $Q_u(t) > 0$, and the target set is the ellipsoid $\mathcal{E}_z(Q(t)) = \{z : \, z^{\mathrm{T}} Q^{-1}(t) z \leq 1\}$ with $Q(t) > 0$.

The following lemma gives conditions of boundedness for the control input. The proof of this lemma is omitted.

Lemma 1. Given $\Theta(t)$, $u(t) = \Theta(t)x(t) \in \mathcal{E}(Q_u(t))$ with $Q_u(t) > 0$ for all $x(t) \in \mathcal{E}(P(t))$ with $P(t) \geq 0$ if and only if the linear matrix inequality

$$\begin{pmatrix} P(t) & * \\ \Theta(t)P(t) & Q_u(t) \end{pmatrix} \geq 0. \tag{22}$$

is feasible with respect to $P(t)$.

According to Theorem 1 the state of the closed-loop continuous-time system (21) is inside the ellipsoid $\mathcal{E}(P(t))$ with the matrix $P(t)$ satisfying the equation

$$\begin{aligned} \dot{P} = A(t)P + PA^{\mathrm{T}}(t) + B_u(t)Z(t) + Z^{\mathrm{T}}(t)B_u^{\mathrm{T}}(t) + \\ B(t)G(t)B^{\mathrm{T}}(t), \quad P(t_0) = R. \end{aligned} \tag{23}$$

where $Z(t) = \Theta(t)P(t)$. Then the controlled output of the closed-loop system belongs to the ellipsoid $\mathcal{E}_z(Q_z(t))$, where

$$Q_z(t) = [C(t) + D(t)\Theta(t)]P(t)[C(t) + D(t)\Theta(t)]^{\mathrm{T}},$$

and, therefore, belongs to the target set provided that $\mathcal{E}_z(Q_z(t)) \subseteq \mathcal{E}(Q(t))$, i.e. $Q_z(t) \leq Q(t)$. Substituting $Q_z(t)$ into the latter inequality, using Schur lemma and Lemma 1, we arrive at the following statement.

Theorem 3. The control law $u = \Theta(t)x$ with $\Theta(t) = Z(t)P^{-1}(t)$ satisfies constraint $u(t) \in \mathcal{E}(Q_u(t))$, $Q_u(t) > 0$, $\forall t \in [t_0, t_f]$ and ensures $z(t_f) \in \mathcal{E}_z(Q(t_f))$ in case (i) $\big(z(t) \in \mathcal{E}_z(Q(t))$, $\forall t \in [t_0, t_f]$ in case (ii)$\big)$ for all admissible initial states and disturbances from the set $\mathcal{S}(t, t_0; R, G)$, if there exist matrix functions $P(t) > 0$ and $Z(t)$ satisfying differential equation (23), inequalities (22), $Q_z(t_f) \leq Q(t_f)$ in case (i) $\big(Q_z(t) \leq Q(t)$, $\forall t \in [t_0, t_f]$ in case (ii)$\big)$.

The above equation and inequalities can be solved by discretizing the time interval $[t_0, t_f]$ into equally spaced time instances t_k, $k = 0, \ldots, N$. The discretized problem thus can be expressed as the following semi-definite program for $k = 0, \ldots, N$

$$
\begin{aligned}
&Y(k+1) - Y(k) - h\left(A(k)Y(k) + Y(k)A^{\mathrm{T}}(k)\right) - \\
&h\left(B_u(k)Z(k) + Z^{\mathrm{T}}(k)B_u^T(k)\right) - \\
&hB(k)G(k)B^{\mathrm{T}}(k) = 0, \quad Y(0) = R, \quad k \neq N, \qquad (24)\\
&\begin{pmatrix} Y(k) & * \\ Z(k) & Q_u(k) \end{pmatrix} > 0, \quad Y(k) > 0,
\end{aligned}
$$

$$
\text{case (i)} : \begin{pmatrix} Y(N) & * \\ C(N)Y(N) + D(N)Z(N) & Q(N) \end{pmatrix} \geq 0,
$$

$$
\text{case (ii)} : \begin{pmatrix} Y(k) & * \\ C(k)Y(k) + D(k)Z(k) & Q(k) \end{pmatrix} \geq 0
$$

with respect to $Y(k)$, $Z(k)$, where the argument k implies that the matrix concerned is evaluated at $t = t_k$. The gain matrix is computed as follows $\Theta(k) = Z(k)Y^{-1}(k)$.

This numerical procedure allows to compute the ellipsoidal controls for specified matrices $Q_u(t)$ and $Q(t)$. The ellipsoidal control law will be optimal in case (i) if matrix $Q(t_f)$ of the target ellipsoidal set has the minimal trace. In this case, matrix $Q(N)$ becomes an additional variable and the problem $\min \operatorname{tr} Q(N)$ subject to constraints (24), (i) is solved. In case (ii), when the target ellipsoidal tube is described by inequality $z^{\mathrm{T}}Q^{-1}(t)z \leq r^2$ with a prescribed $Q(t) > 0$, the ellipsoidal control law will be optimal under minimal possible value of r. In this case, $Q(k)$, $k = 0, \ldots N$ are replaced by $r^2Q(k)$ and the problem $\min r^2$ subject to constraints (24), (ii) is solved.

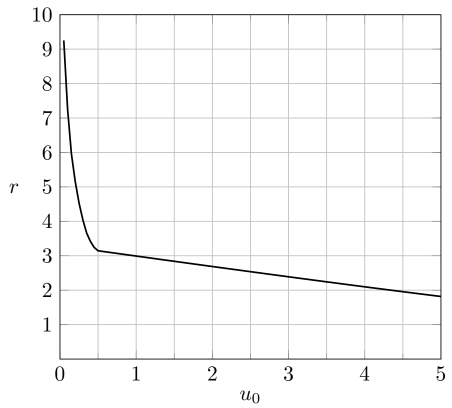

Fig. 2. Dependence of the minimal radius of the tube on the maximal value of the control

As an illustrative example we consider the equations

$$
\dot{x}_1 = x_2, \quad \dot{x}_2 = -\omega_0^2(1 + \varepsilon \sin \omega t)x_1 + u + v \qquad (25)
$$

with $\omega_0 = \pi$, $\omega = 2\pi$, $\varepsilon = 0.1$, describing parametric vibrations of the controlled linear oscillator. Our objective here is to observe the dependence of minimal radius r of the target tube $x_1^2(t) + x_2^2(t) \leq r^2$, $t \in [0, 60]$ on the control bound u_0 under the optimal ellipsoidal control of the form $u = \theta_1(t)x_1 + \theta_2(t)x_2$ subject to constraint $|u| \leq u_0$. Fig. 2 demonstrates this dependence for $R = \mathrm{diag}(1, 1)$ and $G(\sigma) \equiv 1$.

5. CONCLUSION

This paper demonstrates that reachable sets of a linear time-varying continuous system under uncertain initial states and disturbances with a bounded uncertainty measure are evolving ellipsoids. The uncertainty measure is defined as the sum of a quadratic form of the initial state and integral of a quadratic form of the disturbance over finite-time interval. The cases of degenerated quadratic forms are also considered. The matrices of the ellipsoidal reachable sets satisfy the linear matrix differential equation. Application of this result allows to synthesize both the optimal observer providing the ellipsoidal estimate of the system state as well as linear non-stationary controllers to steer the system state into a prescribed final ellipsoidal set or to keep the entire system trajectory in a given ellipsoidal tube. It is shown that the Kalman filter in the state estimation problem is simultaneously the optimal ellipsoidal observer, under deterministic initial states and disturbances with the appropriate bounded uncertainty measure. Illustrative examples for the Mathieu equation demonstrate the effectiveness of the approach proposed.

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