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Conformally Mapped Polynomial Chaos Expansions for Uncertain Dynamical Systems

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Abstract: Polynomial Chaos expansions are among the most popular tools for uncertainty quantification. In this work, we address surrogate modeling for random dynamical systems in the frequency domain, where the randomness accounts for uncertainties in system input parameters. It has been pointed out several times that Polynomial Chaos methods may converge slowly if such systems operate close to resonances or if the input randomness is large. As a remedy, we propose to use conformal mappings to enhance the accuracy of Polynomial Chaos expansions in a certain frequency range. These transformations may enlarge the region of analyticity of the underlying function to be approximated and hence, improve the cost accuracy ratio. We will explain the underlying mechanism and derive transformed Polynomial Chaos expansions which still feature the desired orthogonality properties. The algorithmic development will be complemented by several numerical examples which demonstrate the effectiveness of the proposed approach.

Keywords: Uncertainty, multivariable polynomials, convergence of numerical methods, random variables, frequency domains.

1. INTRODUCTION

Polynomial Chaos (PC) expansions, going back to Wiener (1938), are powerful tools to propagate uncertainties in model input parameters to output quantities in engineering models, see Ghanem and Spanos (2003) as well as Matthies and Keese (2005). Any second order random variable can be accurately represented via the Hermite PC over a family of Gaussian random variables. For different probability distributions, orthogonal PC basis functions can be obtained through the Askey scheme, as described in Xiu and Karniadakis (2002). The convergence properties of these generalized expansions have been analyzed in Ernst et al. (2012). For arbitrary densities, generalized Polynomial Chaos (gPC) can be constructed as outlined in Soize and Ghanem (2004). Nowadays, sparse, adaptive gPC schemes have reached a certain level of maturity, enabling analysts to handle uncertainties in complex systems with a growing number of uncertain inputs.

For dynamical systems with random input data, it has been observed several times that gPC may converge slowly when the system is operated close to a resonance, i.e., when variations in random parameters cause strong variations of the system's response. This is reflected by large gradients in the parametric domain which may demand for high polynomial degrees to achieve a certain level of accuracy. Several remedies have been proposed in the literature to overcome these limitations. In Yaghoubi et al. (2017) a frequency rescaling technique has been presented. Rational surrogate modeling, based on a Padé-Legendre expansion has been put forth in Chantrasmi et al. (2009); Chantrasmi and Iaccarino (2012). Bayesian updating with rational surrogate models has been addressed recently in Schneider et al. (2019). Rational surrogate modeling appears quite natural for dynamical systems in the frequency domain, however, constructing Padé-type approximations in multiple dimensions is non-trivial, as already pointed out in Chantrasmi et al. (2009). This represents a drawback compared to polynomial methods, where sparsity is a rather well-understood concept. Other alternatives are given by wavelet approaches Le Maıtre et al. (2004).

In this work we present a method for accelerating the convergence of gPC expansions for surrogate modeling. The key idea consists of a transformation of the random parameter domain via conformal maps. In particular, we employ a transformation which preserves the PC-orthogonality, which is a crucial property to obtain immediate access to statistical measures of interest, such as variance-based sensitivity indices or statistical moments. We present a fully non-intrusive realization of the conformally-mapped-accelerated gPC expansion.

We proceed with an introduction of the dynamical system with random input data, before describing the accelerated gPC method in the next section. Numerical results for a simple mass-spring benchmark problem will be given at the end.

Let the frequency domain dynamical system be given by

$$\left(-\omega^2\mathbf{M}(\boldsymbol{\xi}) + i\omega\mathbf{D}(\boldsymbol{\xi}) + \mathbf{K}(\boldsymbol{\xi})\right)\mathbf{x}(\omega,\boldsymbol{\xi}) = \mathbf{f}, \tag{1}$$

where i denotes the imaginary unit, ω the angular frequency and $\mathbf{M}, \mathbf{D}, \mathbf{K}$ refer to the mass, damping and stiffness matrix, respectively. Also, $\boldsymbol{\xi}$ refers to the realization of a random vector with $\boldsymbol{\xi} \in \Gamma \subset \mathbb{R}^N$. For simplicity, we assume that the random vector $\boldsymbol{\xi}$ only takes values in a bounded domain Γ. Such a system may originate from a finite element discretization of a mechanical structure or represent a discrete mass-spring-damper system with multiple degrees of freedoms. We assume that the joint probability distribution of the random vector is known and given as $f_{\boldsymbol{\xi}}(\boldsymbol{\xi}) = f_{\xi_1}(\xi_1)f_{\xi_2}(\xi_2)\cdots f_{\xi_N}(\xi_N)$, i.e., the input random variables are independent. Much interest has recently been devoted to construct gPC expansions in the case of dependent input parameters, see for instance Jakeman et al. (2019), yet, we do not consider such a setting here, for simplicity. We also restrict ourselves to a single input single output system, where u represents the scalar input such that $\mathbf{f}(u)$ and $y = g(\mathbf{x})$ denotes the output quantity which depends both on ω and $\boldsymbol{\xi}$. The goal of the numerical method outlined in the following is to approximate the frequency response function

$$\mathcal{F} = \frac{|y(\cdot,\boldsymbol{\xi})|}{|u|}, \tag{2}$$

which implicitly depends on the random parameter $\boldsymbol{\xi}$ through the dynamical system. Hence, in addition to a frequency response approximation technique a method for approximating the $\boldsymbol{\xi}$-dependence is required.

2. POLYNOMIAL CHAOS WITH CONFORMAL MAPPINGS

We proceed by recalling the standard gPC expansion, before giving an extension in terms of conformal mappings in the subsequent subsection.

2.1 Polynomial Chaos

We consider the space of all second order random variables on Γ, i.e., $Y \in L^2(\Gamma, \mathcal{B}(\Gamma), f_{\boldsymbol{\xi}}\mathrm{d}\xi)$, with Borel sigma-algebra $\mathcal{B}(\Gamma)$. In the following, we will simply write L^2, for short. Note that $Y \in L^2$ ensures that the variance of Y is finite. Every $Y \in L^2$ can be represented using gPC as

$$Y(\boldsymbol{\xi}) = \sum_{\boldsymbol{i}} y_{\boldsymbol{i}}\Psi_{\boldsymbol{i}}(\boldsymbol{\xi}), \tag{3}$$

where $\boldsymbol{i} = (i_1, i_2, \ldots, i_N)$ denotes the polynomial degree multi-index such that Ψ is a tensor product polynomial of degree i_j in dimension $j = 1, \ldots, N$. Also, $\Psi_{\boldsymbol{i}}$ denotes a multivariate global polynomial on Γ which satisfies

$$\int_\Gamma \Psi_{\boldsymbol{i}}(\boldsymbol{\xi})\Psi_{\boldsymbol{j}}(\boldsymbol{\xi})\ f_{\boldsymbol{\xi}}(\boldsymbol{\xi})\mathrm{d}\xi = \delta_{\boldsymbol{ij}}, \tag{4}$$

i.e., which are normalized and orthogonal in L^2. The infinite sum in (3) is truncated to obtain computable expressions as

$$Y(\boldsymbol{\xi}) \approx Y_{\mathrm{PC}}(\boldsymbol{\xi}) = \sum_{|\boldsymbol{i}|\le p} y_{\boldsymbol{i}}\Psi_{\boldsymbol{i}}(\boldsymbol{\xi}), \tag{5}$$

where $|\boldsymbol{i}| = i_1 + i_2 + \cdots + i_N$ and p refers to the total polynomial degree. The basis coefficients can be determined by Galerkin projection or non-intrusively by stochastic collocation, regression or discrete projection, see Xiu (2010) for an overview. Such methods converge spectrally, i.e., exponentially fast, if the map $\boldsymbol{\xi} \mapsto Y(\boldsymbol{\xi})$ is analytic.

In the context of approximating the frequency response function $\mathcal{F}$ with PC, as already mentioned, the convergence may be reduced due to strong variations of the system's response. This observation has been analyzed quantitatively in Pagnacco et al. (2017) for a one degree of freedom mass-spring-damper system. In particular, the authors of this paper have introduced a parameter $\frac{\sigma}{\eta\mu}$, where μ, σ refer to the mean value and standard deviation of the uncertain input parameter (the stiffness in their setting) and η is proportional to the damping. Multiple modes in the solution's probability density function may occur if this parameter exceeds a specific threshold, which then results in an increased computational effort associated with PC approximations. Hence, for low damping (small η) and high input uncertainty (large σ) PC expansions may be difficult to apply without dedicated efforts for convergence acceleration.

2.2 Conformal Mappings

Conformal mappings have been used in a couple of references to accelerate numerical methods, in particular quadrature. The idea was introduced in Hale and Trefethen (2008) and used in Jantsch and Webster (2018) and recently by Georg et al. (2020) to accelerate sparse grid quadrature and interpolation. A more detailed derivation of the convergence acceleration of PC expansions with conformal mappings can be found in Georg and Römer (2019). Here, we will briefly present the main steps and investigate the approximation properties for random frequency responses.

We first represent the parameter domain through the product of univariate intervals as

$$\Gamma = [-1, 1]^N, \tag{6}$$

assuming that each parametric interval is mapped to $[-1, 1]$. For instance, the (inverse) CDF-transform $\tilde{\xi} = -1 + 2F_\xi(\xi)$ could be used to that end. Then, we consider conformal mappings that preserve this domain, i.e., $g([-1, 1]) = [-1, 1]$. An example is given by

$$g(s) = \frac{1}{53089}(40320s + 6720s^3 + 3024s^5 + 1800s^7 + 1225s^9), \tag{7}$$

which represents a Taylor expansion to the inverse sin-function and, due to its visual appearance, is called a *sausage* mapping in the literature. This is illustrated in Fig. 1, where the mapping is applied to a Bernstein ellipse. Such Bernstein ellipses arise in the convergence analysis of polynomial-based methods, cf. Hale and Trefethen (2008). In particular, for functions with an analytic continuation on a large Bernstein ellipse, fast convergence can be expected. The mapping g is designed to map these Bernstein ellipses to *straighter* regions, which can be observed in Fig. 1. Such a region may be better suited for parametric problems, where the derived regions of analyticity often resemble strips, see Babuška et al. (2007).

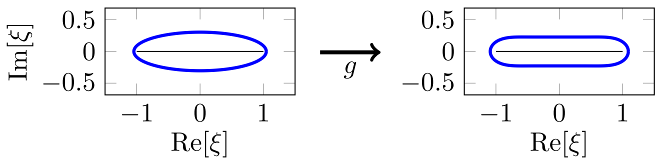

Fig. 1. Conformal map of a Bernstein ellipse, from Georg et al. (2020).

The transformation g gives rise to a new density

$$\tilde{f}_\xi(s) = \frac{f_\xi(g(s))}{|(g^{-1})'(g(s))|}. \quad (8)$$

We then introduce the mapped generalized PC basis by setting up an orthogonal basis with respect to $\tilde{f}_\xi$, denoted $\{\Phi_i\}_i$ which is then transformed as

$$\tilde{\Psi}_i = \Phi_i \circ g^{-1}. \quad (9)$$

The intuition behind this construction is that the polynomial basis $\tilde{\Psi}_i$ exploits a larger region of analyticity if $\mathcal{F}$ has poles close to the imaginary axis. It is important to note that the $\tilde{\Psi}_i$ remain orthogonal to f_ξ, which means that many important statistical quantities (moments, variance-based sensitivities) can still be derived by simple post-processing steps. In higher dimensions, we simply employ conformal mappings in each dimension and construct the multivariate polynomial via tensor product constructions, as usual.

Numerical methods for quadrature can equally be accelerated using conformal maps, by transforming the quadrature nodes and rescaling the weights. The interested reader is refered to Hale and Trefethen (2008) for details. Such quadrature methods are useful to directly compute statistical moments and are used here in the context of pseudo-spectral projection, to compute the PC coefficients.

3. NUMERICAL EXAMPLE

We consider a mass-spring-damper system introduced in Lohmann and Eid (2007). The geometry is depicted in Fig. 2. We employ two beta-distributed random parameters ξ_1, ξ_2 with joint probability density function

$$f_{\boldsymbol{\xi}} = \begin{cases} \frac{1225}{1024}(1-\xi_1^2)^3(1-\xi_2^2)^3, & -1 \le \xi_1, \xi_2 \le 1, \\ 0, & \text{else.} \end{cases} \quad (10)$$

These are used to represent an uncertain stiffness $c_1 = (27 + 4\xi_1)\,\mathrm{N\,m^{-1}}$ and mass $m_1 = (1 \pm 0.1\xi_2)\,\mathrm{kg}$.

Fig. 3 presents the frequency response for $\omega \in [5.5, 7]\,\mathrm{s^{-1}}$. In particular, the nominal value, the mean value as well as a 2σ band arround the mean value are shown. It can observed that for frequencies close to the resonance, the uncertain parameters have a large impact on $\mathcal{F}$. At each of 51 frequency sample points ω_i, we construct repeatedly gPC and mapped gPC approximations of increasing order, until a discrete approximation of the L^1-norm with a cross-validation sample of size 1000 indicates an absolute error below 0.05. In all cases, the coefficients of the (mapped) gPC expansions of total degree p are computed by pseudo-spectral projection using (mapped) quadrature of the same degree. Fig. 4 depicts the respective numbers of model evaluations needed for the prescribed accuracy. It can be observed that for frequencies close to the resonance, the mapped approach outperforms gPC, while otherwise the

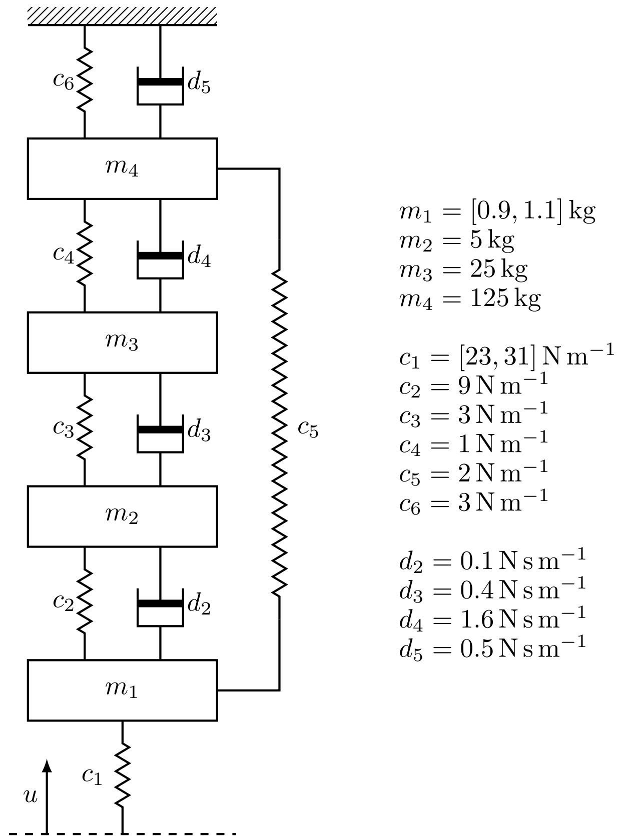

Fig. 2. Mass-spring-damper system with random parameters as proposed by Lohmann and Eid (2007).

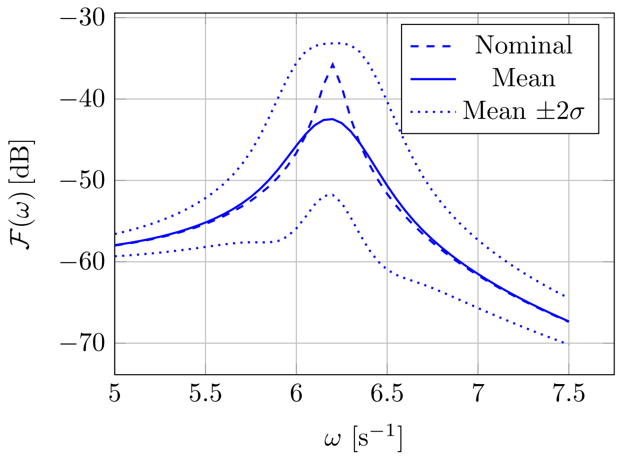

Fig. 3. Frequency response: Nominal value, mean value and 2σ-band.

standard gPC achieves a better efficiency. Hence, conformal mappings may serve as a suitable convergence acceleration technique for dynamical systems, in particular, as the response close to resonances is often of primary interest in applications.

The improved convergence order is further illustrated in Fig. 5 for $\omega = 6.2\,\mathrm{s^{-1}}$, where we depict the error in the expected value. The reference solution for the mean value is obtained up to machine accuracy by Gaussian quadrature of order 100. The gPC approximation of $\mathbb{E}[\mathcal{F}]$

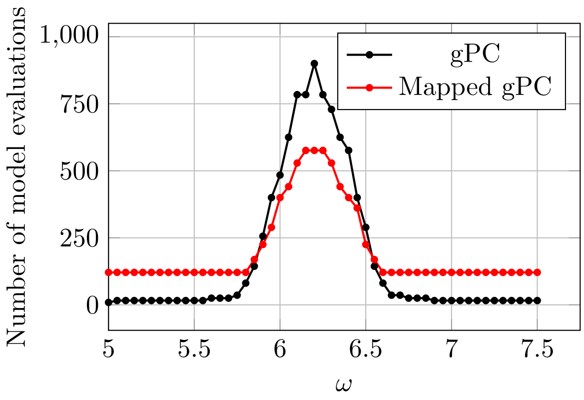

Fig. 4. Computational cost to compute a surrogate model with a prescribed accuracy, w.r.t. the frequency ω.

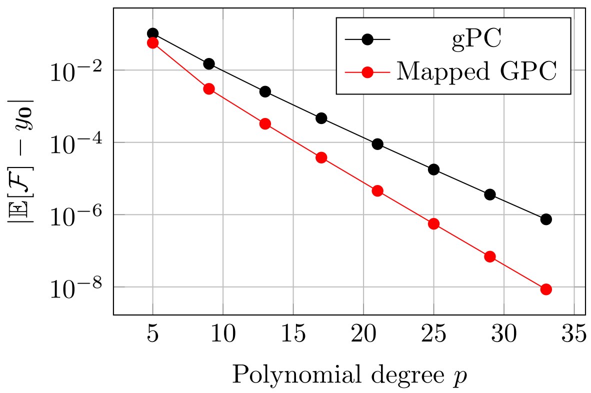

Fig. 5. Convergence of mean value for $\omega = 6.2\,\mathrm{s}^{-1}$.

is simply given by y_0, i.e., the first polynomial coefficient, due to the orthonormality of the basis.

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