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Robustness of Constant-Delay Predictor Feedback with Respect to Distinct Uncertain Time-Varying Input Delays ⋆

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Abstract: This paper addresses the robustness of the constant-delay predictor feedback in the case of distinct and uncertain time-varying input delays. Specifically, we consider the case of a predictor feedback that is designed based on the knowledge of the nominal value of the time-varying delay in each control input channel. We derive an LMI-based sufficient condition ensuring the exponential stability of the closed-loop system for small enough variations of the distinct time-varying input delays around their nominal value. Then we apply these results to the feedback stabilization of a class of diagonal infinite-dimensional boundary control systems exhibiting distinct time-varying delays in the boundary control inputs.

Keywords: Time-varying delay control, Predictor feedback, Robust stability, Partial Differential Equations (PDEs), Boundary control.

1. INTRODUCTION

Following the early works of Artstein (1982), linear predictor feedback has emerged as an efficient tool for the feedback stabilization of Linear Time-Invariant (LTI) systems in the presence of constant arbitrarily long input delays. Since then, many extensions of the original linear predictor feedback have been reported in various directions; see e.g. Krstic (2009) and references therein. Because the exact value of the input delay is in general unknown, a number of studies have been concerned with the robustness assessment of predictor feedback control strategies w.r.t delay mismatches. This includes the cases of constant (Krstic, 2008; Li et al., 2014) and time-varying (Bekiaris-Liberis and Krstic, 2013; Karafyllis and Krstic, 2013; Selivanov and Fridman, 2016; Lhachemi et al., 2019a) input delays.

The works cited above deal with a delay input that is common to all the scalar control inputs. However, in practice, one might expect distinct input delays in each scalar control input. To tackle this problem, various extensions of the predictor feedback to distinct input delays were reported in the literature (Artstein, 1982; Bekiaris-Liberis and Krstic, 2016; Tsubakino et al., 2016; Bresch-Pietri and Di Meglio, 2017). Input-to-state stability property w.r.t additive plant disturbances and robustness to constant multiplicative uncertainties in the inputs were studied in (Cai et al., 2019). In order to tackle uncertainties in either the plant model or in the knowledge of the distinct input delays, adaptive control strategies were developed in (Zhu et al., 2018b,a).

In this paper, we are concerned with the robustness of the constant-delay predictor feedback in the case of distinct and uncertain time-varying input delays. The result presented in this paper extends (Lhachemi et al., 2019a), which dealt with a common input delay, and takes the form of an LMI-based sufficient condition ensuring the exponential stability of the closed-loop system for small enough deviations of the distinct time-varying delays around their nominal value.

The obtained stability result is applied to the feedback stabilization of a class of diagonal infinite-dimensional boundary control systems in the presence of distinct time-varying delays in the boundary control inputs. The adopted control strategy, inspired by (Russell, 1978) in the case of a delay-free feedback control, consists of a predictor feedback designed on a finite-dimensional truncated model capturing the unstable modes of the infinite-dimensional system. This approach was first reported in (Prieur and Trélat, 2019) for the exponential stabilization of a reaction-diffusion equation with a constant delay in the boundary control and was then further developed in (Lhachemi and Prieur, 2020; Lhachemi et al., 2019a,b). The objective of the present paper is to extend these results to the case of distinct input delays.

This paper is organized as follows. The robustness of the constant predictor feedback w.r.t distinct, uncertain, and time-varying delays is investigated in Section 2. The

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extension of this result to the feedback stabilization of a class of diagonal infinite-dimensional boundary control systems is presented in Section 3. The results are applied in Section 4, followed by concluding remarks in Section 5.

2. DELAY-ROBUSTNESS OF PREDICTOR FEEDBACK FOR LTI SYSTEMS

The sets of non-negative integers, positive integers, real, non-negative real, positive real, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{N}^*$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_+^*$, and $\mathbb{C}$, respectively. The real and imaginary parts of a complex number z are denoted by $\operatorname{Re} z$ and $\operatorname{Im} z$, respectively. The field $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. The set of n-dimensional vectors over $\mathbb{K}$ is denoted by $\mathbb{K}^n$ and is endowed with the Euclidean norm $\|x\| = \sqrt{x^* x}$. The set of $n \times m$ matrices over $\mathbb{K}$ is denoted by $\mathbb{K}^{n\times m}$ and is endowed with the induced norm denoted by $\|\cdot\|$. For any symmetric matrix $P \in \mathbb{R}^{n\times n}$, $P \succ 0$ (resp. $P \succeq 0$) means that P is positive definite (resp. positive semi-definite). The set of symmetric positive definite matrices of order n is denoted by $\mathbb{S}_n^{+*}$. For any symmetric matrix $P \in \mathbb{R}^{n\times n}$, $\lambda_m(P)$ and $\lambda_M(P)$ denote the smallest and largest eigenvalues of P, respectively. For $M = (m_{i,j}) \in \mathbb{C}^{n\times m}$, we introduce

$$\mathcal{R}(M) \triangleq \begin{bmatrix} \operatorname{Re} M & -\operatorname{Im} M \\ \operatorname{Im} M & \operatorname{Re} M \end{bmatrix} \in \mathbb{R}^{2n\times 2m}$$

where $\operatorname{Re} M \triangleq (\operatorname{Re} m_{i,j}) \in \mathbb{R}^{n\times m}$ and $\operatorname{Im} M \triangleq (\operatorname{Im} m_{i,j}) \in \mathbb{R}^{n\times m}$. For any $t_0 > 0$, we say that $\varphi \in \mathcal{C}^0(\mathbb{R};\mathbb{R})$ is a *transition signal over* $[0, t_0]$ if $0 \leq \varphi \leq 1$, $\varphi|_{(-\infty,0]} = 0$, and $\varphi|_{[t_0,+\infty)} = 1$.

2.1 Problem setting

We study the feedback stabilization of the following LTI system with distinct input delays:

$$\dot{x}(t) = Ax(t) + \sum_{k=1}^{m} B_k u_k(t - D_k(t)), \quad t \geq 0, \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$ and $B_k \in \mathbb{R}^n$. Vectors $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ denote the state and the control input, respectively, while $u_k(t) \in \mathbb{R}$ denotes the k-th component of $u(t)$. The command inputs are subject to distinct and uncertain time-varying delays $D_k \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{R}_+)$. We assume that there exist $D_{0,k} > 0$ and $0 < \delta_k < D_{0,k}$ such that $|D_k(t) - D_{0,k}| \leq \delta_k$ for all $t \geq 0$. In this context, we consider the following constant-delay linear predictor feedback (Artstein, 1982, Example 5.2), which is based on the knowledge of the constants nominal values $D_{0,k}$:

$$u(t) = K \left\{ x(t) + \sum_{i=1}^{m} \int_{t-D_{0,i}}^{t} e^{(t-D_{0,i}-s)A} B_i u_i(s)\,\mathrm{d}s \right\} \tag{2}$$

for $t \geq 0$, where the feedback gains $K_k \in \mathbb{R}^{1\times n}$ are selected such that $A_{\mathrm{cl}} \triangleq A + \tilde{B}K = A + \sum_{k=1}^{m} e^{-D_{0,k}A} B_k K_k$ is Hurwitz with $B = [B_1\ B_2\ \ldots\ B_m]$, $K = \begin{bmatrix} K_1^\top & K_2^\top & \ldots & K_m^\top \end{bmatrix}^\top$, and $\tilde{B} = \begin{bmatrix} e^{-D_{0,1}A} B_1 & e^{-D_{0,2}A} B_2 & \ldots & e^{-D_{0,m}A} B_m \end{bmatrix}$. The existence of such a feedback gain K is ensured under the assumption that the pair (A, B) is stabilizable. This claim follows from the Hautus test because $x^* A = \lambda x^*$ and $x^* \tilde{B} = 0$ for some $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n \backslash \{0\}$ implies that $0 = x^* e^{-D_{0,k}A} B_k = e^{-D_{0,k}\lambda} x^* B_k$ for all $1 \leq k \leq m$ and thus $x^* B = 0$.

In the nominal configuration $D_i = D_{0,i}$, it is well known that (2) ensures the exponential stabilization of (1), see (Artstein, 1982, Example 5.2). In this section, we study the robust exponential stability of the closed-loop system (1-2) w.r.t delay mismatches, i.e. when $D_i \neq D_{0,i}$.

2.2 Preliminary results

For $h > 0$, we denote by W the space of absolutely continuous functions $\psi : [-h, 0] \to \mathbb{R}^n$ with square-integrable derivative endowed with the norm $\|\psi\|_W \triangleq \sqrt{\|\psi(0)\|^2 + \int_{-h}^{0} \|\dot{\psi}(\theta)\|^2\,\mathrm{d}\theta}$. The following preliminary Lemma is a variation of (Fridman, 2006, Thm 1).

Lemma 1. *Let $M, N_k \in \mathbb{R}^{n\times n}$, $D_{0,k} > 0$, and $\delta_k \in (0, D_{0,k})$ be given. Assume that there exist $\kappa > 0$, $P_1, Q_k \in \mathbb{S}_n^{+*}$, and $P_2, P_3 \in \mathbb{R}^{n\times n}$ such that $\Theta(\Delta, \kappa) \preceq 0$ with $\Delta = (\delta_1, \ldots, \delta_m)$ and $\Theta(\Delta, \kappa)$ defined by (3). Then, there exists $C_0 > 0$ such that, for any $D_k \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{R}_+)$ with $|D_k - D_{0,k}| \leq \delta_k$, the trajectory x of:*

$$\dot{x}(t) = Mx(t) + \sum_{k=1}^{m} N_k \left\{ x(t - D_k(t)) - x(t - D_{0,k}) \right\};$$
$$x(\tau) = x_0(\tau),\ \tau \in [-h, 0]$$

with initial condition $x_0 \in W$, where $h = \max_{1\leq k\leq m} (D_{0,k} + \delta_k)$, satisfies $\|x(t)\| \leq C_0 e^{-\kappa t} \|x_0\|_W$ for all $t \geq 0$.

Proof. First, as $x_0 \in W$, we note that

$$\dot{x}(t) = Mx(t) + \sum_{k=1}^{m} N_k \int_{t-D_{0,k}}^{t-D_k(t)} \dot{x}(\tau)\,\mathrm{d}\tau \tag{4}$$

for all $t \geq 0$. Inspired by (Fridman, 2014, Sec. 3.2), we define $V(t) = V_1(t) + V_2(t)$ with $V_1(t) = x(t)^\top P_1 x(t)$ and

$$V_2(t) = \sum_{k=1}^{m} \int_{-D_{0,k}-\delta_k}^{-D_{0,k}+\delta_k} \int_{t+\theta}^{t} e^{2\kappa(s-t)} \dot{x}(s)^\top Q_k \dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta$$

where $P_1, Q_k \in \mathbb{S}_n^{+*}$. Taking the time derivative we have

$$\begin{aligned} \dot{V}(t) = {} & 2x(t)^\top P_1 \dot{x}(t) + 2\dot{x}(t)^\top \left(\sum_{k=1}^{m} \delta_k Q_k \right) \dot{x}(t) - 2\kappa V_2(t) \\ & - \sum_{k=1}^{m} \int_{-D_{0,k}-\delta_k}^{-D_{0,k}+\delta_k} e^{2\kappa\theta} \dot{x}(t+\theta)^\top Q_k \dot{x}(t+\theta)\,\mathrm{d}\theta \end{aligned} \tag{5}$$

for all $t \geq 0$. Following Fridman (2006), we define $P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}$ for some $P_2, P_3 \in \mathbb{R}^{n\times n}$. Then, using (4) we have

$$\begin{aligned} x(t)^\top P_1 \dot{x}(t) = {} & \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^\top P^\top \begin{bmatrix} 0 & I \\ M & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\ & + \sum_{k=1}^{m} \int_{t-D_{0,k}}^{t-D_k(t)} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^\top P^\top \begin{bmatrix} 0 \\ N_k \end{bmatrix} \dot{x}(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{6}$$

Using $2a^\top b \leq \|a\|^2 + \|b\|^2$, $\forall a, b \in \mathbb{R}^n$, we obtain that

$$\begin{aligned} & 2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^\top P^\top \begin{bmatrix} 0 \\ N_k \end{bmatrix} \dot{x}(\tau) \\ & \leq e^{-2\kappa(\tau-t)} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^\top P^\top \begin{bmatrix} 0 \\ N_k \end{bmatrix} Q_k^{-1} \begin{bmatrix} 0 \\ N_k \end{bmatrix}^\top P \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \end{aligned} \tag{7}$$

$$\Theta(\Delta,\kappa) = \begin{bmatrix} 2\kappa P_1 + M^\top P_2 + P_2^\top M & P_1 - P_2^\top + M^\top P_3 & \delta_1 P_2^\top N_1 & \delta_2 P_2^\top N_2 & \dots & \delta_m P_2^\top N_m \\ P_1 - P_2 + P_3^\top M & -P_3 - P_3^\top + 2\sum_{k=1}^m \delta_k Q_k & \delta_1 P_3^\top N_1 & \delta_2 P_3^\top N_2 & \dots & \delta_m P_3^\top N_m \\ \delta_1 N_1^\top P_2 & \delta_1 N_1^\top P_3 & -\delta_1 e^{-2\kappa D_{0,1}} Q_1 & 0 & \dots & 0 \\ \delta_2 N_2^\top P_2 & \delta_2 N_2^\top P_3 & 0 & -\delta_2 e^{-2\kappa D_{0,2}} Q_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta_m N_m^\top P_2 & \delta_m N_m^\top P_3 & 0 & 0 & \dots & -\delta_m e^{-2\kappa D_{0,m}} Q_m \end{bmatrix}. \tag{3}$$

$$+ e^{2\kappa(\tau - t)} \dot{x}(\tau)^\top Q_k \dot{x}(\tau).$$

We deduce from (5-7) that

$$\dot{V}(t) + 2\kappa V(t) \leq \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^\top \Psi \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix},$$

where

$$\Psi \triangleq P^\top \begin{bmatrix} 0 & I \\ M & -I \end{bmatrix} + \begin{bmatrix} 0 & I \\ M & -I \end{bmatrix}^\top P + 2 \begin{bmatrix} \kappa P_1 & 0 \\ 0 & \sum_{k=1}^m \delta_k Q_k \end{bmatrix}$$
$$+ \sum_{k=1}^m \delta_k e^{2\kappa D_{0,k}} P^\top \begin{bmatrix} 0 \\ N_k \end{bmatrix} Q_k^{-1} \begin{bmatrix} 0 \\ N_k \end{bmatrix}^\top P.$$

From $\Theta(\Delta,\kappa) \preceq 0$, the use of the Schur complement yields $\dot{V}(t) + 2\kappa V(t) \leq 0$. The conclusion follows from the fact that $\lambda_{\mathrm{m}}(P_1)\|x(t)\|^2 \leq V(t) \leq \max\left(\lambda_{\mathrm{M}}(P_1), 2\sum_{k=1}^m \delta_k \lambda_{\mathrm{M}}(Q_k)\right) \|x(t+\cdot)\|_W^2$, $\forall t \geq 0$. □

Remark 1. Assume that there exist $\delta_1, \dots, \delta_m > 0$ such that the LMI $\Theta(\Delta, 0) \prec 0$ is feasible. By a continuity argument, there exists $\kappa > 0$ such that $\Theta(\Delta,\kappa) \preceq 0$. Then, the feasability of $\Theta(\Delta, 0) \prec 0$ implies the existence of $\kappa > 0$ such that the conclusions of Lemma 1 apply. ∘

The conclusions of Lemma 1 imply that M is Hurwitz. A form of converse result is stated in the next Lemma. The proof is analogous to (Lhachemi et al., 2019a, Lem. 2).

Lemma 2. *Let $M, N_k \in \mathbb{R}^{n\times n}$ with M Hurwitz and $D_{0,k} > 0$ be given. Then there exist $\delta_k \in (0, D_{0,k})$ and $\kappa > 0$ such that the LMI $\Theta(\Delta,\kappa) \prec 0$ is feasible.*

2.3 Robustness of constant-delay predictor feedback with respect to distinct time-varying input delays

We can now state the main result of this section.

Theorem 1. *Let $A \in \mathbb{R}^{n\times n}$ and $B_k \in \mathbb{R}^n$ be such that (A, B) is stabilizable. Let $D_{0,k} > 0$ be given nominal delays and let φ be a transition signal[1] over $[0, t_0]$ with $t_0 > 0$. Let feedback gains[2] $K_k \in \mathbb{R}^{1\times n}$ be such that $A_{\mathrm{cl}} \triangleq A + \sum_{k=1}^m e^{-D_{0,k}A} B_k K_k$ is Hurwitz. Then, there exist $\delta_k \in (0, D_{0,k})$ such that for any $D_k \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{R}_+)$ with $|D_k - D_{0,k}| \leq \delta_k$, the closed-loop system given by*

$$\dot{x}(t) = Ax(t) + \sum_{k=1}^m B_k u_k(t - D_k(t)), \tag{8a}$$

$$u(t) = \varphi(t) K x(t) \tag{8b}$$
$$+ \varphi(t) K \sum_{i=1}^m \int_{t-D_{0,i}}^t e^{(t-D_{0,i}-s)A} B_i u_i(s)\,\mathrm{d}s,$$

$$x(0) = x_0, \tag{8c}$$

$$u(\tau) = 0, \qquad -\max_{1\leq k\leq m}(D_{0,k} + \delta_k) \leq \tau \leq 0 \tag{8d}$$

with initial condition $x_0 \in \mathbb{R}^n$ is exponentially stable in the sense that there exist constants $\kappa, C_1 > 0$, independent of x_0 and D_k, such that $\|x(t)\| + \|u(t)\| \leq C_1 e^{-\kappa t}\|x_0\|$ for all $t \geq 0$. In particular, this conclusion holds true (resp., with given decay rate $\kappa > 0$) for any $\delta_k \in (0, D_{0,k})$ such that there exist $P_1, Q_k \in \mathbb{S}_n^{+*}$ and $P_2, P_3 \in \mathbb{R}^{n\times n}$ for which the LMI $\Theta(\Delta, 0) \prec 0$ (resp., $\Theta(\Delta,\kappa) \preceq 0$) holds with $M = A_{\mathrm{cl}}$, $N_k = B_k K_k$, and $\Delta = (\delta_1, \dots, \delta_m)$.

Remark 2. The control input u is obtained as the solution of the fixed point equation (8b). The existence and uniqueness of u can be shown as in (Bresch-Pietri et al., 2018) by rewriting (8b) as

$$u(t) = \varphi(t) K x(t) \tag{9}$$
$$+ \varphi(t) K \int_{\max(t-\overline{D}_0, 0)}^t e^{(t-\overline{D}_0-s)A} \hat{B}(t,s) u(s)\,\mathrm{d}s$$

where $\overline{D}_0 = \max_{1\leq k\leq m} D_{0,k}$ and $\hat{B}(t,s) \in \mathbb{R}^{n\times m}$ with the k-th column given by $\hat{B}_k(t,s) = 1|_{[t-D_{0,k},t]}(s) e^{(\overline{D}_0 - D_{0,k})A} B_k$. Equation (9) was studied in (Bresch-Pietri et al., 2018, Eq. 5) in the case $\varphi = 1$ and $\hat{B}$ a constant matrix independent of s, t. However, noting that $0 \leq \varphi \leq 1$ and $\|\hat{B}(t,s)\| \leq \sum_{k=1}^m \|e^{(\overline{D}_0 - D_{0,k})A} B_k\|$, where the right hand side of the latter inequality is a constant, the developments of (Bresch-Pietri et al., 2018, Subsec. 4.1) can be reapplied in a straightforward manner to show the existence and uniqueness of a function u solution of (9). Finally, the existence and uniqueness of the system trajectories of (8) can be shown by an induction argument. ∘

Proof. Let $\delta_k \in (0, D_{0,k})$ be such that $\Theta(\Delta, 0) \prec 0$ is feasible (Lemma 2). By a continuity argument, let $\kappa > 0$ be such that $\Theta(\Delta,\kappa) \preceq 0$. We introduce (Artstein, 1982):

$$z(t) = x(t) + \sum_{k=1}^m \int_{t-D_{0,k}}^t e^{(t-D_{0,k}-s)A} B_k u_k(s)\,\mathrm{d}s \tag{10}$$

for all $t \geq 0$. In particular, $u = \varphi K z$ and we infer that

$$\dot{z}(t) = \left(A + \varphi(t) \sum_{k=1}^m e^{-D_{0,k}A} B_k K_k\right) z(t) \tag{11}$$
$$+ \sum_{k=1}^m B_k K_k \{[\varphi z](t - D_k(t)) - [\varphi z](t - D_{0,k})\}$$

for all $t \geq 0$. For $t \geq t_1 \triangleq t_0 + \max_{1\leq k\leq m}(D_{0,k} + \delta_k)$ we have

$$\dot{z}(t) = A_{\mathrm{cl}} z(t) + \sum_{k=1}^m B_k K_k \{z(t - D_k(t)) - z(t - D_{0,k})\} \tag{12}$$

[1] See notation section.
[2] Existence is ensured by the stabilizability property of the pair (A, B), see discussion in Subsection 2.1.

with $A_{\rm cl} = A + \sum_{k=1}^{m} e^{-D_{0,k}A} B_k K_k$ Hurwitz and the initial condition $z|_{[t_0,t_1]}$ which is of class $\mathcal{C}^1$. The application of Lemma 1 shows that $\|z(t)\| \leq C_0 e^{-\kappa(t-t_1)} \|z(t_1+\cdot)\|_W$ for all $t \geq t_1$. Now, based on (11), classical estimations (using e.g. Grönwall's inequality) show the existence of a constant $c_1 > 0$, independent of x_0 and D_k, such that $\|z(t)\| \leq c_1 \|x_0\|$ for all $0 \leq t \leq t_1$. The later estimate, combined with (11), yields the existence of $\tilde{c}_0 > 0$, independent of x_0 and D_k, such that $\|\dot{z}(t)\| \leq \tilde{c}_0 \|x_0\|$ for all $0 \leq t \leq t_1$. Then, we infer that $\|z(t_1+\cdot)\|_W \leq \tilde{c}_1 \|x_0\|$ with $\tilde{c}_1 = \sqrt{c_1^2 + \max_{1\leq k\leq m} (D_{0,k}+\delta_k)\tilde{c}_0^2}$ and thus $\|z(t)\| \leq \tilde{C}_0 e^{-\kappa t} \|x_0\|$ for all $t \geq 0$ with $\tilde{C}_0 = e^{\kappa t_1} \max(C_0 \tilde{c}_1, c_1) > 0$. The conclusion follows from $u = \varphi K z$ and (10). □

3. APPLICATION TO A CLASS OF DIAGONAL INFINITE-DIMENSIONAL SYSTEMS

3.1 Problem setting

Let $D_{0,k} > 0$ and $\delta_k \in (0, D_{0,k})$ be given. We consider:

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}X(t), & t \geq 0 \\ \mathcal{B}X(t) = \tilde{u}(t), & t \geq 0 \\ X(0) = X_0 \end{cases} \tag{13}$$

on the separable Hilbert space $\mathcal{H}$ with $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ a linear (unbounded) operator and $\mathcal{B} : D(\mathcal{B}) \subset \mathcal{H} \to \mathbb{K}^m$ with $D(\mathcal{A}) \subset D(\mathcal{B})$ a linear boundary operator. The control input takes the form

$$\tilde{u}(t) = (u_1(t - D_1(t)), \ldots, u_m(t - D_m(t))) \tag{14}$$

with $u_i(\tau) = 0$ for $\tau \leq 0$ and $D_k(t) \in (D_{0,k} - \delta_k, D_{0,k} + \delta_k)$. Following the terminology of (Curtain and Zwart, 2012, Def. 3.3.2), we assume that $(\mathcal{A}, \mathcal{B})$ is a boundary control system; we denote by $\mathcal{A}_0$ the associated disturbance-free operator and by $B \in \mathcal{L}(\mathbb{K}^m, \mathcal{H})$ an associated lifting operator.

Assumption 1. Operator $\mathcal{A}_0$ is a Riesz spectral operator (Curtain and Zwart, 2012, Def. 2.3.4), i.e. is a linear and closed operator with simple eigenvalues λ_n and corresponding eigenvectors $\phi_n \in D(\mathcal{A}_0)$, $n \in \mathbb{N}^*$, that satisfy: (1) $\{\phi_n,\ n \in \mathbb{N}^*\}$ is a Riesz basis; (2) for any distinct $a, b \in \overline{\{\lambda_n,\ n \in \mathbb{N}^*\}}$, $[a, b] \not\subset \overline{\{\lambda_n,\ n \in \mathbb{N}^*\}}$.

We introduce $\{\psi_n,\ n \in \mathbb{N}^*\}$ the biorthogonal sequence associated with the Riesz basis $\{\phi_n,\ n \in \mathbb{N}^*\}$, i.e. $\langle \phi_k, \psi_l \rangle = \delta_{k,l} \in \{0, 1\}$ with $\delta_{k,l} = 1$ if and only if $k = l$. Then, there exist constants $m_R, M_R > 0$ such that, for any $x \in \mathcal{H}$, $x = \sum_{n\geq 1} \langle x, \psi_n \rangle \phi_n$ and

$$m_R \sum_{n\geq 1} |\langle x, \psi_n \rangle|^2 \leq \|x\|^2 \leq M_R \sum_{n\geq 1} |\langle x, \psi_n \rangle|^2. \tag{15}$$

Assumption 2. There exist $N_0 \in \mathbb{N}^*$ and $\alpha \in \mathbb{R}_+^*$ such that $\mathrm{Re}\,\lambda_n \leq -\alpha$ for all $n \geq N_0 + 1$.

3.2 Spectral reduction

Under the regularity $\tilde{u} \in \mathcal{C}^2([0, +\infty); \mathbb{K}^m)$ with $\tilde{u}(0) = 0$ and $X_0 \in D(\mathcal{A}_0)$, there exists a unique classical solution $X \in \mathcal{C}^0(\mathbb{R}_+; D(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$ of (13) (Curtain and Zwart, 2012, Th. 3.3.3). Introducing the coefficient of projection $c_n(t) \triangleq \langle X(t), \psi_n \rangle$, we have for all $t \geq 0$ that (Lhachemi and Shorten, 2019):

$$\dot{c}_n(t) = \lambda_n c_n(t) + \langle (\mathcal{A} - \lambda_n I) B \tilde{u}(t), \psi_n \rangle . \tag{16}$$

Let $\mathcal{E} = (e_1, e_2, \ldots, e_m)$ be the canonical basis of $\mathbb{K}^m$ and let $b_{n,k} \triangleq \langle (\mathcal{A} - \lambda_n I) B e_k, \psi_n \rangle$. Then (16) yields

$$\dot{Y}(t) = A_{N_0} Y(t) + \sum_{k=1}^{m} B_{N_0,k} u_k(t - D_k(t)), \tag{17}$$

for all $t \geq 0$, where $A_{N_0} = \mathrm{diag}(\lambda_1, \ldots, \lambda_{N_0}) \in \mathbb{K}^{N_0 \times N_0}$, $B_{N_0,k} = (b_{n,k})_{1\leq n\leq N_0} \in \mathbb{K}^{N_0}$, and

$$Y(t) = [c_1(t)\ \ldots\ c_{N_0}(t)]^\top \in \mathbb{K}^{N_0}. \tag{18}$$

Introducing the matrix $B_{N_0} = [B_{N_0,1}\ \ldots\ B_{N_0,m}]$, we assume that the following holds.

Assumption 3. (A_{N_0}, B_{N_0}) is stabilizable.

With $\tilde{B}_{N_0} = \left[e^{-D_{0,1}A_{N_0}} B_{N_0,1}\ \ldots\ e^{-D_{0,m}A_{N_0}} B_{N_0,m}\right]$, the above assumption ensures that[3] the pair $(A_{N_0}, \tilde{B}_{N_0})$ is stabilizable and thus the existence of a feedback gain $K = \left[K_1^\top\ K_2^\top\ \ldots\ K_m^\top\right]^\top \in \mathbb{K}^{m \times N_0}$ such that $A_{\rm cl} \triangleq A_{N_0} + \tilde{B}_{N_0} K = A_{N_0} + \sum_{k=1}^{m} e^{-D_{0,k}A_{N_0}} B_{N_0,k} K_k$ is Hurwitz.

3.3 Dynamics of the closed-loop system

Let $t_0, D_{0,k} > 0$ and $\delta_k \in (0, D_{0,k})$ be given. Let $\varphi \in \mathcal{C}^2(\mathbb{R}; \mathbb{R})$ be a transition signal over $[0, t_0]$ and $D_k \in \mathcal{C}^2(\mathbb{R}_+; \mathbb{R})$ be a time-varying delay such that $|D_k - D_{0,k}| \leq \delta_k$. The dynamics of the closed-loop system is given by:

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}X(t), \tag{19a}$$

$$\mathcal{B}X(t) = \tilde{u}(t), \tag{19b}$$

$$u(t) = \varphi(t) K Y(t) \tag{19c}$$

$$\qquad + \varphi(t) K \sum_{i=1}^{m} \int_{t-D_{0,i}}^{t} e^{(t-D_{0,i}-s)A_{N_0}} B_{N_0,i} u_i(s)\,\mathrm{d}s,$$

$$X(0) = X_0, \tag{19d}$$

$$u(\tau) = 0, \qquad -\max_{1\leq k\leq m} (D_{0,k} + \delta_k) \leq \tau \leq 0, \tag{19e}$$

for any $t \geq 0$ with $\tilde{u}$ and Y given by (14) and (18), respectively. The gain $K \in \mathbb{K}^{m \times N_0}$ is selected such that $A_{\rm cl} = A_{N_0} + \tilde{B}_{N_0} K$ is Hurwitz.

The well-posedness of (19) in terms of classical solutions associated with initial conditions $X_0 \in D(\mathcal{A}_0)$ can be shown similarly to (Lhachemi et al., 2019a, Lem. 3).

3.4 Exponential stability of the closed-loop system

We can now state the main result of this section.

Theorem 2. *Let $(\mathcal{A}, \mathcal{B})$ be an abstract boundary control system such that Assumptions 1, 2, and 3 hold true. There exist $\delta_k \in (0, D_{0,k})$ and $\eta > 0$ such that, for any given $\delta_r > 0$, we have the existence of a constant $C_2 > 0$ such that, for any $X_0 \in D(\mathcal{A}_0)$ and $D_k \in \mathcal{C}^2(\mathbb{R}_+; \mathbb{R})$ with $|D_k - D_{0,k}| \leq \delta_k$ and $\sup_{t\in\mathbb{R}_+} \left|\dot{D}_k(t)\right| \leq \delta_r$, the trajectory X and the control input u of the closed-loop dynamics (19) satisfy $\|X(t)\| + \|u(t)\| \leq C_2 e^{-\eta t} \|X_0\|$ for all $t \geq 0$. In*

[3] See discussion in Subsection 2.1.

particular, this conclusion holds true for any $\delta_k \in (0, D_{0,k})$ such that $\Theta(\Delta, 0) \prec 0$ is feasible with

- *in the case $\mathbb{K} = \mathbb{R}$, $M = A_{N_0} + \tilde{B}_{N_0} K$, $N_k = B_{N_0,k} K_k$, $P_1, Q_k \in \mathbb{S}_n^{+*}$, and $P_2, P_3 \in \mathbb{R}^{n \times n}$;*
- *in the case $\mathbb{K} = \mathbb{C}$, $M = \mathcal{R}(A_{N_0} + \tilde{B}_{N_0} K)$, $N_k = \mathcal{R}(B_{N_0,k} K_k)$, $P_1, Q_k \in \mathbb{S}_{2n}^{+*}$, and $P_2, P_3 \in \mathbb{R}^{2n \times 2n}$.*

Furthermore, if $\kappa > 0$ is such that $\Theta(\Delta, \kappa) \preceq 0$ is feasible, then the decay rate η can be selected as any element of $(0, \kappa]$ if $\alpha > \kappa$ or $(0, \alpha)$ if $\alpha \leq \kappa$.

Proof. Let $\delta_k \in (0, D_{0,k})$ and $\kappa > 0$ be such that $\Theta(\Delta, \kappa) \preceq 0$ is feasible (see Lemma 2). We introduce $\eta \in (0, \kappa]$ if $\alpha > \kappa$ or $\eta \in (0, \alpha)$ if $\alpha \leq \kappa$ and we select $\epsilon \in (0, 1)$ such that $\alpha_\epsilon \triangleq \alpha(1 - \epsilon) > \eta$. Let $\delta_r > 0$ be arbitrarily given. The key point of the proof relies on the introduction of the functional (which is finite for any $t \geq 0$, see (15)): $V(t) = \frac{1}{2} \sum_{k \geq N_0+1} |\langle X(t) - B\tilde{u}(t), \psi_k \rangle|^2$ for $t \geq 0$. As shown in (Lhachemi et al., 2019a, Proof of Thm. 3), we have

$$\|X(t)\| \leq \|B\tilde{u}(t)\| + \sqrt{2M_R \left(V(t) + \|Y(t)\|^2 + \frac{1}{m_R} \|B\tilde{u}(t)\|^2 \right)}$$

for all $t \geq 0$. Introducing

$$Z(t) = Y(t) + \sum_{i=1}^{m} \int_{t-D_{0,i}}^{t} e^{(t-D_{0,i}-s)A_{N_0}} B_{N_0,i} u_i(s) \,\mathrm{d}s,$$

we have that $u = \varphi K Z$. As Y satisfies (17) with $A_{\mathrm{cl}} = A_{N_0} + \tilde{B}_{N_0} K$ Hurwitz, Theorem 1 shows that $\|Y(t)\| + \|u(t)\| \leq C_1 e^{-\kappa t} \|Y(0)\| \leq C_1 e^{-\eta t} \|X_0\| / \sqrt{m_R}$ and $\|Z(t)\| \leq \tilde{C}_0 e^{-\kappa t} \|Y(0)\| \leq \tilde{C}_0 e^{-\eta t} \|X_0\| / \sqrt{m_R}$ for all $t \geq 0$, and thus

$$\|\tilde{u}(t)\| \leq \sqrt{m} \max_{1 \leq k \leq m} |u_k(t - D_k(t))| \leq \sqrt{m} \max_{1 \leq k \leq m} \|u(t - D_k(t))\| \leq \frac{\sqrt{m} C_1 e^{\eta \hat{D}}}{\sqrt{m_R}} e^{-\eta t} \|X_0\|$$

with $\hat{D} = \max_{1 \leq k \leq m} (D_{0,k} + \delta_k)$. Recalling that B is bounded, the proof will be complete if we can show the existence of a constant $\tilde{C}_1 > 0$, independent of X_0 and D_k, such that $V(t) \leq \tilde{C}_1 e^{-2\eta t} \|X_0\|^2$. Following (Lhachemi et al., 2019a, Proof of Thm. 3), the computation of $\dot{V}$ and the use of both (16) and Young's inequality yield

$$\dot{V}(t) \leq -2\alpha_\epsilon V(t) + \frac{1}{2\epsilon\alpha} \sum_{k \geq N_0+1} \left(|\langle \mathcal{A} B\tilde{u}(t), \psi_k \rangle|^2 + \left| \langle B\dot{\tilde{u}}(t), \psi_k \rangle \right|^2 \right). \tag{20}$$

The estimation of the right hand side of the above inequality slightly differs from (Lhachemi et al., 2019a, Proof of Thm. 3) due to the presence of distinct delays. First, we have for all $t \geq \hat{D} + t_0$ that

$$\sum_{k \geq N_0+1} |\langle \mathcal{A} B\tilde{u}(t), \psi_k \rangle|^2 \leq m \sum_{i=1}^{m} \sum_{k \geq 1} |\langle \mathcal{A} B e_i, \psi_k \rangle|^2 |K_i Z(t - D_i(t))|^2 \leq \frac{m}{m_R} \sum_{i=1}^{m} \|\mathcal{A} B e_i\|^2 \|K_i\|^2 \|Z(t - D_i(t))\|^2.$$

Similarly, we have that

$$\sum_{k \geq N_0+1} \left| \langle B\dot{\tilde{u}}(t), \psi_k \rangle \right|^2 \leq m \sum_{i=1}^{m} |1 - \dot{D}_i(t)|^2 \sum_{k \geq 1} |\langle B e_i, \psi_k \rangle|^2 |\dot{u}_i(t - D_i(t))|^2 \leq \frac{\beta m}{m_R} \sum_{i=1}^{m} \|B e_i\|^2 |\dot{u}_i(t - D_i(t))|^2.$$

with $\beta = (1 + \delta_r)^2$. For $t \geq t_1 \triangleq 2\hat{D} + t_0$ we have [4]

$$\begin{aligned} \dot{u}_i(t - D_i(t)) &= K_i \dot{Z}(t - D_i(t)) \\ &\overset{(12)}{=} K_i A_{\mathrm{cl}} Z(t - D_i(t)) \\ &\quad + K_i \sum_{k=1}^{m} B_{N_0,k} K_k \{ Z(t - D_i(t) - D_k(t - D_i(t))) \\ &\qquad - Z(t - D_i(t) - D_{0,k}) \} \end{aligned}$$

and we deduce that

$$\begin{aligned} \sum_{k \geq N_0+1} \left| \langle B\dot{\tilde{u}}(t), \psi_k \rangle \right|^2 &\leq \frac{\beta m(m+1)}{m_R} \sum_{i=1}^{m} \|B e_i\|^2 \|K_i A_{\mathrm{cl}}\|^2 \|Z(t - D_i(t))\|^2 \\ &\quad + \frac{\beta m(m+1)}{m_R} \sum_{i=1}^{m} \sum_{k=1}^{m} \|B e_i\|^2 \|K_i B_{N_0,k} K_k\|^2 \\ &\quad \times \|Z(t - D_i(t) - D_k(t - D_i(t))) - Z(t - D_i(t) - D_{0,k})\|^2 \end{aligned}$$

for all $t \geq t_1$. Recalling that $\|Z(t)\| \leq \tilde{C}_0 e^{-\eta t} \|X_0\| / \sqrt{m_R}$ for all $t \geq 0$, we obtain that, for all $t \geq t_1$, $\dot{V}(t) \leq -2\alpha_\epsilon V(t) + \omega(t)$ where $\omega(t) \leq k_1 e^{-2\eta t} \|X_0\|^2$ with $k_1 > 0$ a constant that is independent of X_0 and D_k. The rest of the proof is identical to (Lhachemi et al., 2019a, Thm. 3). □

4. ILLUSTRATIVE EXAMPLE

We illustrate Thm. 2 via the following reaction-diffusion equation with delayed Dirichlet boundary controls:

$$\begin{cases} y_t(t,x) = a y_{xx}(t,x) + c y(t,x), & (t,x) \in \mathbb{R}_+ \times (0, L) \\ \begin{bmatrix} y(t,0) \\ y(t,L) \end{bmatrix} = \begin{bmatrix} u_1(t - D_1(t)) \\ u_2(t - D_2(t)) \end{bmatrix}, & t > 0 \end{cases}$$

where $a, c > 0$, $y(t,x) \in \mathbb{R}$, and $u(t) \in \mathbb{R}^2$. Introducing the real state-space $\mathcal{H} = L^2(0, L)$ endowed with $\langle f, g \rangle_{\mathcal{H}} = \int_0^L f g \,\mathrm{d}x$, it can be shown similarly to (Lhachemi et al., 2019a) that the assumptions, hence the conclusions, of Theorem 2 apply. For numerical simulations we set $a = c = 0.5$, $L = 2\pi$, $D_{0,1} = 1\,\mathrm{s}$, $D_{0,2} = 0.5\,\mathrm{s}$, and $t_0 = 0.5\,\mathrm{s}$. We have two unstable modes $\lambda_1 = 0.375$ and $\lambda_2 = 0$ while the two first stable modes are such that $\lambda_3 = -0.625$ and $\lambda_4 = -1.5$. Setting $N_0 = 3$, the feedback gain $K \in \mathbb{R}^{2 \times 3}$ is computed to place the poles of the closed-loop truncated model at -0.75, -1, and -1.25. Theorem 2 ensures the exponential stability of the closed-loop system for $\delta_1 = 0.450$ and $\delta_2 = 0.308$. The time domain evolution of the closed-loop system, obtained based on the 30 dominant modes of the system, is depicted in Figs. 1-3. As expected from Theorem 2, both the system state and the control input converge to zero.

[4] In the corresponding computation in (Lhachemi et al., 2019a, p7), the four occurrences of $Z(t-2D(t))$ must be replaced by $Z(t-D(t)-D(t-D(t)))$. The remainder of the proof remains unchanged.

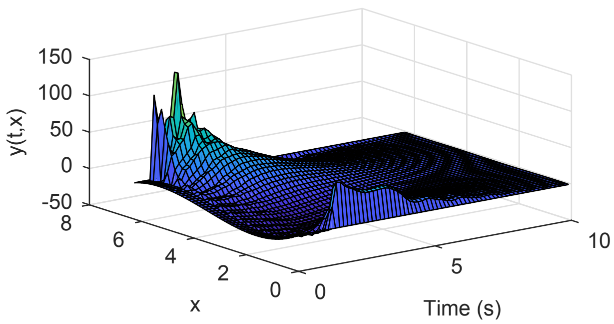

Fig. 1. Time evolution of $y(t)$ for the closed-loop system

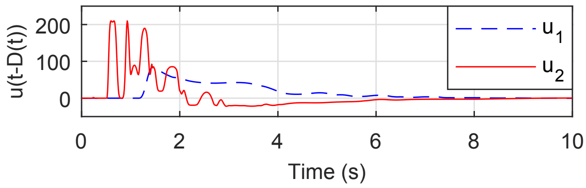

Fig. 2. Delayed command effort $\tilde{u}(t)$

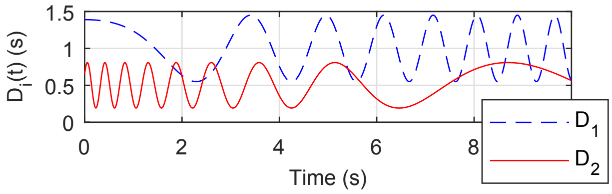

Fig. 3. Delays $D_k(t)$

5. CONCLUSION

This paper assessed the robustness of the predictor feedback for the stabilization of LTI systems in the presence distinct and uncertain time-varying input delays. This result has been extended to the stabilization of a class of diagonal infinite-dimensional boundary control systems and was illustrated with a reaction-diffusion equation.

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