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Robust multi-agent differential games for general linear systems with model uncertainties ⋆

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Abstract: The problem of infinite-horizon multi-agent differential games is investigated, where the process can be modeled by a set of uncertain linear dynamics. The players are divided into two teams, one of which consists of a fixed number of follower agents while the other has one leader agent. The two teams constitute the adversaries. The multi-agent differential games can be transformed into a two-player game. The dynamics of the agents are subjected to norm-bound model uncertainties. Based on quadratic stabilization techniques, a set of saddle point strategies of the game is designed to stabilize the closed-loop multi-agent system, where the weighting matrices of the cost function are properly selected. For any given cost function, by modifying the solution of the linear quadratic differential game of the nominal model, the sufficient conditions are presented such that the stabilization of the system is guaranteed and the uncertainties are compensated. It is proved that the modified solution achieves optimality. A numerical example is given to verify the effectiveness of the theoretical results.

Keywords: multi-agent system, leader-following consensus, differential games, model uncertainties, Nash solution.

1. INTRODUCTION

The researches of multi-agent systems have received significant attention in the past several decades. The critical research motivation is that a large amount of simple agents can collectively perform massive and complicated missions, with extensive applications such as transportation, wireless communication and navigation, unmanned vehicles formation flight (Li et al. (2010)), Dong et al. (2015)). In the literature, the terms such as distributed control, cooperative control, consensus problems and collaborative control represent multi-agent control problems. The agents can be categorized as leaders and followers, where the followers try to achieve consensus with the states of the leaders. Many scenarios can be attributed leader-following problems. For example, in the formation tracking control, the followers are desired to track the trajectory generated by the leader or leaders (Dong and Hu (2017), Liu et al. (2019)).

In some applications, the agents try to collaborate to complete a task, however, sometimes the agents may have different goals, which are reflected in the cost function designed in the game theory or differential games framework. The agents in the game seek to attain best individual goals, while in the process there exists cooperation or completion (Mylvaganam et al. (2017)). Linear quadratic differential games have been extensively studied, which can be traced to the researches of Ho-Bryson-Baron in 1965 (Ho et al. (1965)) about the classical pursuit-evasion problem. Then, many results have been obtained, such as the existence of the saddle point under open-loop and closed-loop conditions, and the linear quadratic differential games in the finite or infinite horizon case, and so on (Engwerda (2009)). In the field of guidance, guidance laws based on linear quadratic differential games were applied effectively in target interception problem, variants of which are the state-dependent Riccati equation based differential games guidance (Ratnoo and Ghose (2009)) and cooperative linear quadratic differential games(Jha et al. (2019)).

In recent years, the research of multi-agent differential games has been a fast-emerging topic in control and electronic engineering. A promising and prevalent method to multi-agent differential games is adaptive dynamic programming, which applies the reinforcement learning algorithms and neural networks to solve the Hamilton-Jacobi-Isaacs equation (Vamvoudakis and Lewis (2011)). Under the circumstance that the Nash equilibrium may not exist, a neural network approximator was proposed to obtain a mixed optimal solution of game in (Zhang et al. (2011)), where a policy iteration method was designed to

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solve the game. Moreover, data-driven approaches based on adaptive dynamics programming were developed to solve the differential games online (Zhang et al. (2017), Zhang and Zhao (2018)), where the current data and history data can be concurrently used to train the neural network and relax the persistence of excitation condition (Modares et al. (2014)). Adaptive dynamic programming based methods are able to solve affine nonlinear differential games, however, it should be noted from a practical standpoint that it may take much time to converge and the required initial admissible control policies are usually difficult to be obtained (Wei et al. (2016)).

Due to the uncertainties in reality, accurate system model will be complicated and can barely be established. Besides, a practical system always deviates from an ideal too-detailed model in the course of time because of the unpredicted bias and faults. Therefore, in the differential games, the players evaluate their optimal strategies on the basis of uncertain model and a robust strategy insensitive to the deviations could be more valuable. In robust H-infinity control theory, lots of results are related to systems with norm-bounded uncertainties, many of which were obtained from 1960s to 1980s (Jia (2007)). When considering the disturbances, the control and the disturbances constitute the adversaries. N-players non-coorperative differential games were considered in (Cruz and Jimenez-Lizarraga (2017)) and coupled Riccati differential equations were solved to find the robust equilibrium strategies. In (Amato et al. (2002)), two-player zero-sum linear quadratic differential games were studied with the system subject to norm-bounded, one-block form uncertainties, where the cost of the players are guaranteed to be limited. Sun et al. (2017) transformed the uncertainties in a redesigned cost function and the robust control was turned into a two-player zero-sum differential game problem. To the best of out knowledge, the multi-agent differential games with model uncertainties has not been studied extensively.

Motivated by the facts stated above, this paper studies multi-agent differential games where the followers attempt to achieve consensus with the leader, whereas the leader tries to avoid it. While the agents adopt the equilibrium point strategies, conditions that guarantee the achievement of leader-following consensus are obtained. Comparing with the existing results, the main contributions of this paper are twofold. Firstly, model uncertainties are considered in multi-agent differential games with the cost function in quadratic form. The uncertainties exist in the control distribution matrices, making it difficult to determine the optimal strategies. In (Sun et al. (2017)), the uncertainties influence the control input while the system dynamics remain intact. Although Amato et al. (2002) considered model uncertainties, only the circumstance of perturbation in the system matrix was dealt with. Secondly, based on robust control theory, uncertainties in the control distribution matrix are compensated by modifying the Nash solution of the nominal system. The strategies are proved to be optimal and guarantee the quadratic stability of the system.

The rest of the paper is organized as follows. The problem formulation is presented in Section 2. The main results about the differential games solution with uncertainties are given in Section 3. A simulation example and the results are provided in Section 4. Section 5 concludes the paper.

2. PROBLEM FORMULATION

In this section, the model of multi-agent differential games subjected to model uncertainties is introduced.

The agents are classified into N followers and one leader. Each agent can access the states of the rest. All of the followers are described by uncertain general linear dynamics, i.e.,

$$\dot{x}_i = Ax_i + (B_i + \Delta B_i)u_i, \;\; i = 1, 2, ..., N, \tag{1}$$

where $x_i \in \mathbb{R}^n$ is the state of the ith follower, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are constant matrices, $u_i \in \mathbb{R}^m$ is the control input of the ith follower. $\Delta B_i = E\Sigma(t)F_i$ is the uncertainty matrix influencing the control distribution. E and F_i are known predetermined matrices with proper dimensions, and the unknown matrix $\Sigma(t)$ is norm-bounded, belonging to the set

$$\Omega = \{\Sigma(t)|\Sigma^T(t)\Sigma(t) \le I, \forall t\}. \tag{2}$$

The state of the leader is denoted as $x_0 \in R^n$, with the dynamics given by

$$\dot{x}_0 = Ax_0 + (C + \Delta C)u_0 \tag{3}$$

where $C \in \mathbb{R}^{n\times m}$ is a constant matrix, $\Delta C = E\Sigma(t)F_0$. The tracking error of follower i is defined as $e_i = x_i - x_0$. From (1) and (3), the dynamics of the tracking error e_i is given by

$$\dot{e}_i = Ae_i + (B_i + \Delta B_i)u_i - (C + \Delta C)u_0. \tag{4}$$

Definition 1. The multi-agent system (1) and (3) are said to achieve leader-following consensus if for any given bounded initial states, $\lim_{t\to+\infty} e_i \to 0$, $i = 1, 2, ..., N$.

Let $\bar{E} = \mathrm{diag}\{E, E, ..., E\}$, $\bar{\Sigma}(t) = \mathrm{diag}\{\Sigma(t), \Sigma(t), ..., \Sigma(t)\}$, $\bar{F} = \mathrm{diag}\{F_1, F_2, ..., F_N\}$, $\tilde{F} = [F_0^T, F_0^T, ..., F_0^T]^T$. The dynamics of tracking error in a compact form is

$$\dot{e} = \bar{A}e + (\bar{B} + \Delta\bar{B})u - (\bar{C} + \Delta\bar{C})u_0, \tag{5}$$

where $e = [e_1^T, e_2^T, ..., e_N^T]^T$ is the tracking error vector, $\bar{A} = \mathrm{diag}\{A, A, ..., A\}$, $\bar{B} = \mathrm{diag}\{B_1, B_2, ..., B_n\}$, $\Delta\bar{B} = \bar{E}\bar{\Sigma}(t)\bar{F}$, $\bar{C} = [C^T, C^T, ..., C^T]^T$, $\Delta\bar{C} = \bar{E}\bar{\Sigma}(t)\tilde{F}$ are dynamic matrices, $u = [u_1^T, u_2^T, ..., u_N^T]^T$ is the control input of the followers.

The nominal system related to (5) without considering the uncertainties is obtained as

$$\dot{e} = \bar{A}e + \bar{B}u - \bar{C}u_0. \tag{6}$$

For the nominal system, considering the uncertainties as defined in (5), one objective is to find the control strategies of the followers u^* that minimizes the infinite-horizon cost function and the control strategy of the leader u_0^* that maximizes the cost function

$$J(u, u_0) = \frac{1}{2}\int_0^\infty (Q(e) + u^T R_F u - u_0^T R_L u_0)dt, \tag{7}$$

with $Q(e) = e^T Q_0 e$, where R_F, R_L Q_0 are positive definite matrices.

Such policy pair u^* and u_0^* are saddle point strategies of the game, and comparing with any other strategies u and u_0, the following relation holds: $J(u^*, u_0) \le J(u^*, u_0^*) \le J(u, u_0^*)$.

This paper mainly focuses on two aspects of the multi-agent problem. One is to find the control strategies of agents that stabilize the system and achieve consensus, since the dynamics is of infinite horizon. The other is to solve the differential games and ensure the optimality of the strategies under uncertainties.

3. MAIN RESULTS

In this section, inspired by results in robust control theory, the sufficient conditions are derived to achieve quadratic stability of the system. Control strategies of the agents are proposed and the proof of optimality is presented.

Lemma 2. (Jia (2007)) Let $G(s) = C(sI - A)^{-1}B + D$, then A is stable and $\|G\|_\infty < \gamma$ if and only if $\|D\| < \gamma$ and the following Riccati inequality has a positive definite solution P:

$$\begin{aligned}(A + B\bar{D}^{-1}D^TC)^TP + P(A + B\bar{D}^{-1}D^TC)\\ +PB\bar{D}^{-1}B^TP + C^T(I + D\bar{D}^{-1}D^T)C < 0,\end{aligned} \tag{8}$$

with $\bar{D} = \gamma^2 I - D^TD$.

Consider the uncertain system

$$\dot{x} = (A + \Delta A)x, x(0) = x_0, \tag{9}$$

where $\Delta A = E\Sigma(t)F$, $\Sigma(t) \in \Omega$.

Lemma 3. (Jia (2007)) The necessary and sufficient condition for the quadratic stabilization of system (9) is that A is stable and $\left\|F(sI - A)^{-1}E\right\|_\infty < 1$.

3.1 Quadratic stability of the system

When the competition between the followers and the leader is neglected, the control input u and u_0 that ensures the quadratic stability of the tracking error system (5) can be obtained based on Lemma 2 and Lemma 3. The control inputs are designed in feedback form as

$$\begin{cases} u = K_1 e, \\ u_0 = K_2 e, \end{cases} \tag{10}$$

where $K_1 \in \mathbb{R}^{Nm\times Nn}$, $K_2 \in \mathbb{R}^{m\times Nn}$. Substituting (10) into (5), the closed-loop dynamics of the tracking error becomes

$$\dot{e} = (\bar{A} + \bar{B}K_1 - \bar{C}K_2 + \bar{E}\bar{\Sigma}(t)(\bar{F}K_1 - \tilde{F}K_2))e. \tag{11}$$

From Lemma 2 and the fact that $\bar{\Sigma}(t) \in \Omega$, system (11) achieves quadratic stabilization if and only if $\bar{A} + \bar{B}K_1 - \bar{C}K_2$ is stable and the condition $\left\|(\bar{F}K_1 - \tilde{F}K_2)(sI - \bar{A} - \bar{B}K_1 + \bar{C}K_2)^{-1}\bar{E}\right\|_\infty < 1$ is satisfied. The design of feedback gains K_1 and K_2 can be obtained in the next lemma.

Lemma 4. If $F_c = [\bar{F}, -\tilde{F}]$ is of full column rank, for a given system (5), the state feedback controller (10) makes the system quadratically stable if and only if the Riccati inequality (12) has a positive definite solution P

$$\bar{A}^TP + P\bar{A}^T + P\bar{E}\bar{E}^TP - PB_c(F_c^TF_c)^{-1}B_c^TP < 0. \tag{12}$$

If such a solution P exists, then the controller feedback gain can be chosen as

$$K_c = -(F_c^TF_c)^{-1}B_c^TP, \tag{13}$$

with $K_c = [K_1^T, K_2^T]^T$, $B_c = [\bar{B}, -\bar{C}]$.

Proof. Let $A_c = \bar{A} + \bar{B}K_1 - \bar{C}K_2$ and $C_c = \bar{F}K_1 - \tilde{F}K_2$. The quadratic stability of the uncertain system requires that A_c is stable and $\left\|C_c(sI - A_c)^{-1}\bar{E}\right\|_\infty < 1$. From Lemma 2, the necessary and sufficient condition is that there exists a positive definite matrix P satisfying

$$A_c^TP + PA_c + P\bar{E}\bar{E}^TP + C_c^TC_c < 0. \tag{14}$$

Note that $A_c = \bar{A} + B_cK_c$ and $C_c = F_cK_c$. It follows from (14) that

$$\bar{A}^TP + P\bar{A}^T + P\bar{E}\bar{E}P - PB_c(F_c^TF_c)^{-1}B_c^TP + H_c^TH_c < 0, \tag{15}$$

where $H_c = F_c(K_c + (F_c^TF_c)^{-1}B_c^TP)$.

Therefore, if Riccati inequality (14) holds with a solution P, then (12) also holds because of the equivalence of (14) and (15). Conversely, if Riccati inequality (12) has a solution P and K is given as (15), then $H_c = 0$. Consequently, inequality (14) holds, and A_c is stable and $\left\|C_c(sI - A_c)^{-1}\bar{E}\right\|_\infty < 1$ in view of Lemma 2. This completes the proof of the lemma.

Necessary and sufficient condition for the quadratic stability of the tracking error system is given in Lemma 4. However, the differential games between the followers and the leader are not considered. The designed controller feedback gain in (13) only ensures the stability of the system without minimizing or maximizing the cost function (7). Moreover, the quadratic stability of the system requires the existence of the full rank of the matrix F_c. When including the differential games between the agents, the condition (12) would be too strict. The results will be shown in the next section.

3.2 Quadratic stability of the system with differential games solution

In this subsection, a set of controllers for the agents will be given. Besides, a theorem will be presented to show that the controllers are solutions of some quadratic differential games and guarantee the quadratic stability of the uncertain system (5).

The feedback gains of the proposed controllers are given as

$$\begin{cases} K_1 = -\dfrac{1}{\beta_1^2}(\bar{F}^T\bar{F})^{-1}\bar{B}^TP, \\ K_2 = -\dfrac{1}{\beta_2^2}(\tilde{F}^T\tilde{F})^{-1}\bar{C}^TP, \end{cases} \tag{16}$$

where P satisfies the following matrix inequality

$$\begin{aligned}\bar{A}^TP + P\bar{A} + P\bar{E}\bar{E}^TP - \frac{1}{\beta_1^2}PB(\bar{F}^T\bar{F})^{-1}B^TP\\ +\frac{1}{\beta_2^2}P\bar{C}(\tilde{F}^T\tilde{F})^{-1}\bar{C}^TP + J_c^TJ_c < 0,\end{aligned} \tag{17}$$

with $0 < \eta < 1$, $\beta_1^2 = 1 - \eta^2$, $\beta_2 = -(1 - \frac{1}{\eta^2})$ and $J_c = \eta\bar{F}K_1 + \frac{1}{\eta}\tilde{F}K_2$.

Theorem 5. If the Riccati inequality (17) holds and the feedback gains for the agents are designed as (16), then the system (5) is quadratic stabilized and the controllers are solutions to some linear quadratic differential games of the nominal system.

Proof. From Lemma 4, if (14) is satisfied, then the quadratic stability of the closed-loop uncertain system (11) is achieved.

Substituting the explicit definitions of A_c and C_c into (14), one gets

$$\begin{aligned}&\bar{A}^TP+P\bar{A}+P\bar{E}\bar{E}^TP-\frac{1}{\beta_1^2}PB(\bar{F}^T\bar{F})^{-1}B^TP+\frac{1}{\beta_2^2}\\&\times P\bar{C}(\tilde{F}^T\tilde{F})^{-1}\bar{C}^TP+J_c^TJ_c+M_c^TM_c+N_c^TN_c<0.\end{aligned} \tag{18}$$

When the feedback gains are given as (16), then $M_c = 0$, $N_c = 0$. Consequently, (17) is obtained and the uncertain system achieves quadratic stability. Considering a multi-agent differential game with a quadratic cost function as (7) and constant weighting matrices, the min-max cost is as follows

$$\begin{aligned}&\min_u \max_{u_0} J(u,u_0)\\&=\min_u \max_{u_0} \frac{1}{2}\int_0^\infty (Q(e)+u^TR_Fu-u_0^TR_Lu_0)dt.\end{aligned} \tag{19}$$

The Hamiltonian associated with the min-max problem of the nominal system (6) is

$$\begin{aligned}H(e,u,u_0,\lambda) = \;&\frac{1}{2}(Q(e)+u^TR_Fu-u_0^TR_Lu_0)\\&+\lambda^T(\bar{A}e+\bar{B}u-\bar{C}u_0),\end{aligned} \tag{20}$$

where $\lambda \in \mathbb{R}^{Nn}$ is the costate vector.

Define the value function

$$V(e,u,u_0)=\frac{1}{2}\int_t^\infty (Q(e)+u^TR_Fu-u_0^TR_Lu_0)dt. \tag{21}$$

The differential game has a unique solution if a game saddle point exists, which means the value function yields

$$V^*(e_0)=\min_u \max_{u_0} J(u,u_0)=\max_{u_0}\min_u J(u,u_0). \tag{22}$$

According to Bellman's principle of optimality, the optimality condition can be derived as

$$\min_u \max_{u_0} H(e,u,u_0,\lambda)=0. \tag{23}$$

The optimal solutions satisfies the stationarity conditions, which are

$$\frac{\partial H(e,u,u_0,\lambda)}{u}=0, \tag{24}$$

$$\frac{\partial H(e,u,u_0,\lambda)}{u_0}=0, \tag{25}$$

where the costate vector is defined as $\lambda = \frac{\partial V^*}{\partial e}$.

The optimal value function $V^*(e)$ for the linear quadratic differential game is of the form $V^*(e) = \frac{1}{2}e^TPe$ for some symmetric definite positive matrix P.

Then in (24) and (25), the costate vector is $\lambda = Pe$, and the optimal controllers are obtained as

$$u^* = -R_F^{-1}\bar{B}^TPe, \tag{26}$$

$$u_0^* = -R_L^{-1}\bar{C}^TPe. \tag{27}$$

Substituting (26) and (27) into (23), the Hamilton-Jaccabi-Isaacs equation is given as

$$\begin{aligned}&e^T(\bar{A}^TP+P\bar{A}-P\bar{B}R_F^{-1}\bar{B}^TP+P\bar{C}R_L^{-1}\bar{C}^TP\\&+Q_0)e=0.\end{aligned} \tag{28}$$

Since $e \neq 0$, then from (28), the Riccati equation is obtained as

$$\bar{A}^TP+P\bar{A}-P\bar{B}R_F^{-1}\bar{B}^TP+P\bar{C}R_L^{-1}\bar{C}^TP+Q_0=0 \tag{29}$$

Since (17) is satisfied, a positive definite matrix Δ can be found such that

$$\begin{aligned}\bar{A}^TP\;&+P\bar{A}+P\bar{E}\bar{E}^TP-\frac{1}{\beta_1^2}PB(\bar{F}^T\bar{F})^{-1}B^TP\\&+\frac{1}{\beta_2^2}P\bar{C}(\tilde{F}^T\tilde{F})^{-1}\bar{C}^TP+J_c^TJ_c+\Delta=0.\end{aligned} \tag{30}$$

Let $Q_0 = \Delta + J_c^TJ_c$, $R_F = \beta_1^2\bar{F}^T\bar{F}$, $R_L = \beta_2^2\tilde{F}^T\tilde{F}$, then the controllers (16) satisfying (17) are solutions of the differential game (19) regarding to the nominal system (6). This completes the proof of the theorem.

Remark 6. From the proof of Theorem 5, the policy pair (u, u_0) is a saddle point solution to a differential game. However, the cost function is dependent on the information of the system uncertainty matrices, which limits the application of the results. It is necessary to find the solution of a differential game with any given cost function.

With regards to a determined cost function (7), the optimal controllers are calculated as (26) and (27) by Theorem 5. The controllers are the unique saddle point strategies of the nominal system. In the presence of uncertainties, the players are unable to evaluate their optimal saddle point strategies because of the stochastic influences on the model dynamics. In this case, the optimal controllers (26) and (27) of the nominal system are modified as follows

$$\bar{u} = -\gamma_1R_F^{-1}\bar{B}^TPe, \tag{31}$$

$$\bar{u}_0 = -\gamma_2R_L^{-1}\bar{C}^TPe, \tag{32}$$

where $\gamma_1 > 0$ and $\gamma_2 > 0$.

The next theorem gives the sufficient conditions that the modified controllers (31) and (32) stabilize the system and posses the optimality. For system (5), a new cost function transformed from (7) is defined as

$$J(u,u_0)=\frac{1}{2}\int_0^\infty (\bar{Q}(e)+\frac{1}{\gamma_1}u^TR_Fu-\frac{1}{\gamma_2}u_0^TR_Lu_0)dt. \tag{33}$$

where P is the solution of (29) and $\bar{Q}(e) = e^T(Q_0 + (\gamma_1 - 1)P\bar{B}\bar{B}^TP + (1-\gamma_2)P\bar{C}\bar{C}^TP)e$,

The scalars γ_1 and γ_2 should be chosen such that $Q_0 + (\gamma_1-1)P\bar{B}\bar{B}^TP + (1-\gamma_2)P\bar{C}\bar{C}^TP$ is positive definite.

Theorem 7. Consider system (5) with cost function (33), the controllers (31) and (32) are stabilizing Nash solutions of the differential game problem if the following inequality holds

$$-\bar{Q}+P(\bar{E}\bar{E}^T+YZY^T)P<0. \tag{34}$$

where $Y = [\bar{B}R_F^{-1}, -\bar{C}R_L^{-1}]$ and

$$Z=\begin{bmatrix}\gamma_1^2\bar{F}^T\bar{F}+(1-2\gamma_1)R_F & -\gamma_1\gamma_2\bar{F}^T\tilde{F}\\ -\gamma_1\gamma_2\tilde{F}^T\bar{F} & \gamma_2^2\tilde{F}^T\tilde{F}-(1-2\gamma_2)R_L\end{bmatrix}.$$

Proof. Firstly, the stability of the system will be proved. The quadratic stability of the uncertain system is achieved if (14) holds with a positive definite matrix P.

Assumed that P is obtained from (29). Under the modified controllers (31) and (32), it follows from (14) that

$$\begin{aligned}&\bar{A}^TP+P\bar{A}^T+P(\bar{E}\bar{E}^T-2\gamma_1\bar{B}R_F^{-1}B^T\\&\quad+2\gamma_2\bar{C}R_L^{-1}\bar{C}^T+(-\gamma_1\bar{F}R_F^{-1}\bar{B}^T-\gamma_2\tilde{F}R_L^{-1}\bar{C}^T)^T\\&\quad\times(-\gamma_1\bar{F}R_F^{-1}\bar{B}^T-\gamma_2\tilde{F}R_L^{-1}\bar{C}^T))P<0.\end{aligned} \tag{35}$$

Considering (29), equation (35) becomes

$$-Q_0+P(\bar{E}\bar{E}^T+(1-2\gamma_1)\bar{B}R_F^{-1}B^T-(1-2\gamma_2)\bar{C}R_L^{-1}\bar{C}^T + (-\gamma_1\bar{F}R_F^{-1}\bar{B}^T - \gamma_2\tilde{F}R_L^{-1}\bar{C}^T)^T \times(-\gamma_1\bar{F}R_F^{-1}\bar{B}^T - \gamma_2\tilde{F}R_L^{-1}\bar{C}^T))P < 0. \quad (36)$$

The second term of the left hand of (36) can be presented as

$$P(\bar{E}\bar{E}^T + (1-2\gamma_1)\bar{B}R_F^{-1}B^T - (1-2\gamma_2)\bar{C}R_L^{-1}\bar{C}^T + (-\gamma_1\bar{F}R_F^{-1}\bar{B}^T - \gamma_2\tilde{F}R_L^{-1}\bar{C}^T)^T \times(-\gamma_1\bar{F}R_F^{-1}\bar{B}^T - \gamma_2\tilde{F}R_L^{-1}\bar{C}^T))P = P(\bar{E}\bar{E}^T + Y^TZY^T)P. \quad (37)$$

Recalling that (34) holds, we can obtain (36). Therefore, the matrix P is a solution of (14) and the quadratic stability of the system is proved. The Hamiltonian function of the nominal system associated with the cost function (33) is

$$H(e,\bar{u},\bar{u}_0,\lambda) = \frac{1}{2}(\bar{Q}(e) + \frac{1}{\gamma_1}\bar{u}^TR_F\bar{u} - \frac{1}{\gamma_2}\bar{u}_0^TR_L\bar{u}_0) + \lambda^T(\bar{A}e + \bar{B}\bar{u} - \bar{C}\bar{u}_0). \quad (38)$$

Using (29), (31) and (32), it can be further obtained that $H(e,\bar{u}^*,\bar{u}_0^*,\lambda) = 0$, which indicates that (31) and (32) achieve optimality with cost function (33) of the differential game. And the optimality is about the nominal system. For the uncertain system, the quadratic stability is ensured if the conditions in the theorem is satisfied. The proof of the theorem completes.

4. NUMERIAL EXAMPLE

In this section, a numerical example is presented to show the effectiveness of the theoretical results.

Consider a second-order system of three followers and one leader, with dynamics described by (1) and (3), respectively. The nominal system dynamics of the agents are described by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -4 & -1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 2 & -1 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}. \quad (39)$$

And the uncertainties in the system dynamics are described by the following matrices

$$E = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}, \Sigma(t) = \begin{bmatrix} \sin(t) & 0 & 0 \\ 0 & \cos(2t) & 0 \\ 0 & \sin(t) & \cos(t) \end{bmatrix}, F_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}, F_i = \begin{bmatrix} 0 & 0.5 \\ 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix} (i = 1,2,...,N). \quad (40)$$

The weighting matrices in the cost function are chosen as $Q_0 = 2 \times I_{6\times6}$, $R_F = I_{6\times6}$,$R_L = 20 \times I_{2\times2}$. It can be verified that (29) has a solution $P > 0$. To ensure that (34) holds and $\bar{Q}$ is positive definite, the feedback gains in (31) and (32) are selected as $\gamma_1 = 2$, $\gamma_2 = 0.7$.

When all of the agents choose their optimal controls, the results about state trajectories are given from Fig. 1 to Fig. 4. The first, second and third state component are shown in Fig. 1, Fig. 2 and Fig. 3, respectively, both of which illustrate that the followers achieve leader-following consensus with the leader despite of the existence of uncertainties in the dynamics. The 3-D phase plane plot is given in Fig. 4, where the initial positions of the followers are denoted by circles and that of the leader is denoted by a pentagram.

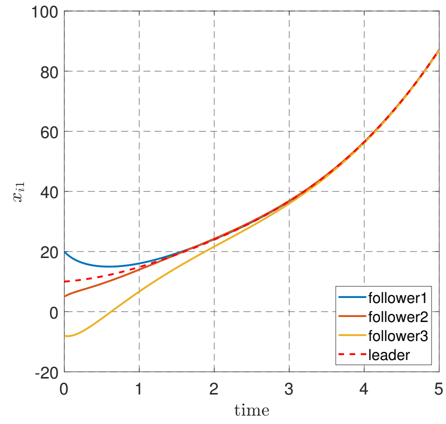

Fig. 1. Plot of trajectories of x_{i1}

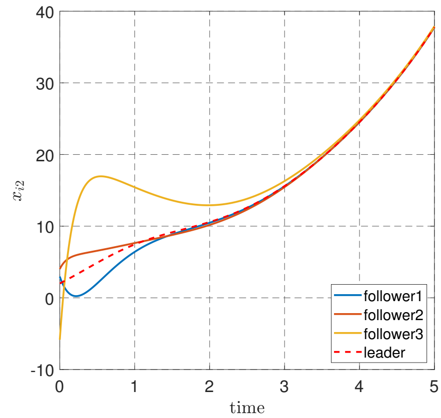

Fig. 2. Plot of the trajectories of x_{i2}

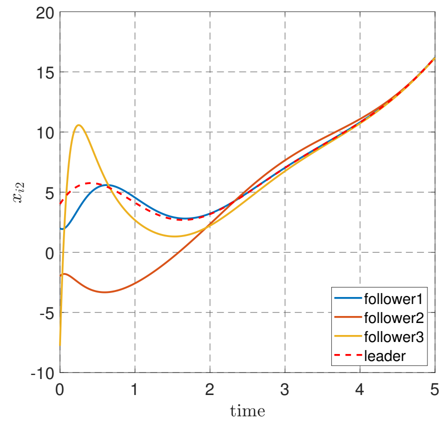

Fig. 3. Plot of the trajectories of x_{i3}

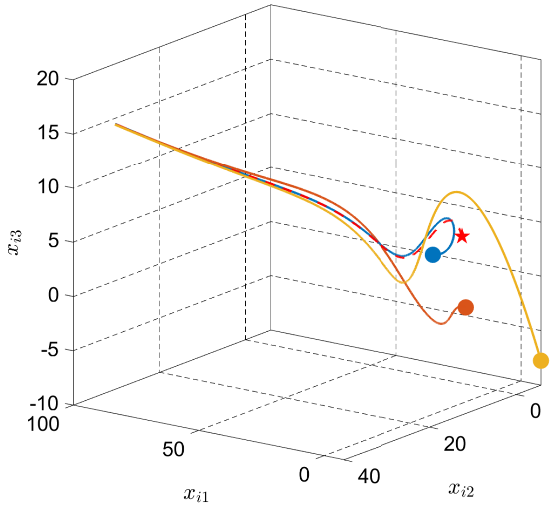

Fig. 4. 3-D plot of trajectories of agents in the phase plane

5. CONCLUSION

An approach to the linear quadratic differential games about centralized multi-agent system leader-following consensus problem was investigated in the presence of model uncertainties. The quadratic stabilization of the multi-agent system was firstly discussed without considering the differential games. Then the proposed control strategies were proved to be the optimal strategies of a certain differential games with the weighting matrices related to the system uncertainties. To solve a linear quadratic differential games with given cost function and deal with the uncertainties, the optimal solutions of the nominal system were modified and the modified solution was proved to be a Nash solution to a differential game. The conditions on the parameters of the modified solution were given to ensure the quadratic stability of the system. A numerical example was finally presented.

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