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An LMI-based algorithm for static output-feedback stabilization of continuous-time positive polytopic linear systems

Álvaro A. Lemaire [*], **Amanda Spagolla** [*], **Cecília F. Morais** [**], **Ricardo C. L. F. Oliveira** [*], **Pedro L. D. Peres** [*]

[*] *School of Electrical and Computer Engineering, University of Campinas – UNICAMP, 13083-852, Campinas, SP, Brazil (e-mails: {arnanz, spagolla, ricfow, peres}@dt.fee.unicamp.br.)*

[**] *Pontifical Catholic University of Campinas, Center of Exact, Environmental and Technological Sciences, 13086-900, Campinas, SP, Brazil (e-mail: cecilia.morais@puc-campinas.edu.br)*

Abstract: This paper presents a new technique to design static output-feedback controllers for continuous-time positive uncertain linear systems. The design is performed through an iterative algorithm based on parameter-dependent linear matrix inequality conditions, solved by means of relaxations, with local convergence guaranteed. A qualified feasible solution provides a stabilizing output-feedback controller that also assures the positivity of the closed-loop system. The main advantage of the proposed methodology is that the control gain is handled directly as an optimization variable, that is, no change of variables is needed to recover the gain and no particular structure (e.g., diagonal) is imposed on the Lyapunov or slack variable matrix to guarantee closed-loop positivity. This particular feature also facilitates the design of decentralized or element-wise bounded gains, as illustrated by numerical experiments.

Keywords: Robust Control; Output-feedback; Positive Uncertain Linear Systems; Continuous-time Systems; Linear Matrix Inequalities.

1. INTRODUCTION

The evolution of the numerical procedures and the great advance of the computational processing capacity have allowed to solve increasingly complex control problems regarding dynamic systems subject to uncertainties. In this scenario, convex optimization techniques based on semidefinite programming stand out, specially those formulated in terms of linear matrix inequalities (LMIs) [Boyd et al., 1994, El Ghaoui and Niculescu, 2000].

At the same time, a wide range of practical problems is concerned with systems that have non-negative states and outputs, whose variables are usually associated with physical parameters that can only assume positive or null values. In this context, the so called *positive systems* can suitably model industrial processes with chemical reactors, heat exchangers, water reservoirs, network flows, storage and communications systems [Berman and Plemmons, 1979, Luenberger, 1979, Farina and Rinaldi, 2000]. Other applications can be found in economics and sociology, with demographic and sociological population models, or in medicine and biology, analyzing and controlling bacterial or cell cultures [Berman and Plemmons, 1979, Luenberger, 1979, Farina and Rinaldi, 2000].

Beyond the practical appealing, another motivation to investigate the control of positive systems is that not all methods employed to handle linear systems can be directly extended to deal with positive systems [Caccetta and Rumchev, 2000]. Note that to assure the positiveness of the closed-loop continuous-time system is equivalent to verify if the closed-loop dynamic matrix is Metzler (that is, with nonnegative off diagonal elements). Recently, several approaches emerged in the control theory literature aiming to treat this problem [Briat, 2013, Ebihara et al., 2014, Ait-Rami et al., 2014, Shen and Lam, 2015, 2017, 2016, Tanaka and Langbort, 2011, Wang and Huang, 2013]. One of the challenges is how to extend the methods developed for single input single output systems, as those based on linear programming [Arneson and Langbort, 2012, Ait Rami, 2011, Roszak and Davison, 2009], to handle the multi-variable case. Besides that, although LMI synthesis conditions for state-feedback control of positive systems can be obtained as a direct extension of the techniques presented in the literature for traditional linear systems, the main source of conservativeness comes from the fact that the positivity constraint is handled by imposing a diagonal structure to the square matrix used to recover

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the control gain, as done in Tanaka and Langbort [2011], Ebihara et al. [2014]. Another alternative could be, for instance, to employ iterative approaches that work with the gain directly as optimization variable [Felipe, 2017, Felipe et al., 2016].

In this paper, an LMI-based iterative procedure is adopted to solve the problem of robust stabilization for continuous-time positive uncertain linear systems. For this purpose, firstly, a parameter-dependent LMI condition inspired in Felipe et al. [2016] is developed for the synthesis of static output-feedback (or state-feedback) controllers, in which the dynamic matrix does not multiply any decision variable of the problem, meaning that the gain can be dealt with as an optimization variable. In this sense, the conservativeness inherent to other LMI-based methods adapted to control design of positive continuous-time systems (that impose a diagonal structure on the Lyapunov matrix or on the square matrix used to recover the gain) disappears. Moreover, the method can handle output feedback control with uncertainties in the output matrix and bounds on the gain entries can be imposed by simply adding linear constraints to the optimization problem.

Notation: $M \in \mathbb{R}^{m\times p}$ denotes a real matrix M with m rows and p columns. The transpose of M is designated by M' and $\mathrm{He}(M)$ stands for $M + M'$. For symmetric matrices ($M = M'$), $M \succ 0$ ($M \prec 0$) denotes that M is positive (negative) definite, while $M \geq 0$ means that for $M = [m_{ij}]$, $m_{ij} \geq 0$, $\forall i, j$. The symbol $\star$ represents a symmetric block in a square matrix.

2. FUNDAMENTALS OF POSITIVE SYSTEMS

This section gathers the main definitions and fundamentals of positive systems necessary to develop the results of this paper.

Definition 1. (Farina and Rinaldi [2000]). A linear system is said to be positive if and only if for every nonnegative initial state and for every nonnegative input its state and output are nonnegative.

Definition 2. (Farina and Rinaldi [2000]). A square matrix $A \in \mathbb{R}^{n\times n}$ is called Metzler if all the elements that are not in the diagonal are positive or zero (nonnegative), that is, $A_{ij} \geq 0$ for all $i \neq j$.

Consider the continuous-time linear system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bw(t), \\ y(t) = Cx(t) + Dw(t) \end{cases} \tag{1}$$

where $x(t)$, $y(t)$ and $w(t)$ are the vectors of states, outputs and exogenous inputs, respectively. A condition equivalent to Definition 1 for the positivity of system (1) is presented next.

Definition 3. (Farina and Rinaldi [2000]). System (1) is positive if and only if matrix A is a Metzler matrix, and $B \geq 0$, $C \geq 0$ and $D \geq 0$, that is, all elements of B, C, and D are nonnegative.

Additionally, a condition for the stability of (1) is given in the next lemma.

Lemma 1. (Farina and Rinaldi [2000]). The positive system (1) is asymptotically stable if and only if there exists a positive definite diagonal matrix P such that

$$A'P + PA \prec 0. \tag{2}$$

Note that, although the stability of (1) can also be assured by a *symmetric* positive definite matrix P, the existence of a *diagonal* Lyapunov matrix P satisfying (2) also ensures properties of robustness with respect to perturbations in the state vector $x(t)$, a result known in the literature as D-stability [Kaszkurewicz and Bhaya, 1999, Oliveira and Peres, 2005].

3. PROBLEM STATEMENT

Consider the uncertain continuous-time positive linear system

$$\begin{cases} \dot{x}(t) = A(\alpha)x(t) + B(\alpha)u(t), \\ y(t) = C(\alpha)x(t), \end{cases} \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector and $y(t) \in \mathbb{R}^p$ is the measured output vector. Each one of the state-space matrices from (3) can be described as a convex combination of N matrices (known as *vertices*), that is,

$$(A(\alpha), B(\alpha), C(\alpha)) = \sum_{i=1}^{N} \alpha_i (A_i, B_i, C_i), \quad \alpha \in \Lambda \tag{4}$$

where $\alpha = [\alpha_1\ \alpha_2\ \cdots\ \alpha_N]'$ is a time-invariant vector belonging to the unit simplex

$$\Lambda = \Big\{ \alpha \in \mathbb{R}^N : \sum_{i=1}^{N} \alpha_i = 1,\ \alpha_i \geq 0,\ i = 1, \ldots, N \Big\}. \tag{5}$$

This paper addresses the problem of designing a static output-feedback robust control law given by

$$u(t) = Ky(t) \tag{6}$$

for system (3) such that the closed-loop system

$$\dot{x}(t) = \big(A(\alpha) + B(\alpha)KC(\alpha)\big)x(t) \tag{7}$$

is asymptotically stable and positive.

4. MAIN RESULT

This section presents the main contributions of this paper. First, a sufficient parameter-dependent LMI condition for the existence of a static output-feedback control gain assuring a relaxed condition for stability and positivity of the closed-loop uncertain system, as presented in Theorem 2. Since Theorem 2 requires some matrices to be pre-specified, to avoid bilinear matrix inequality terms, Theorem 3 proposes a choice for those matrices that always provides a feasible solution for Theorem 2. Finally, an algorithm is proposed to iteratively solve the conditions of Theorem 2, with local convergence guaranteed.

Theorem 2. For given matrices $Y_i(\alpha) \in \mathbb{R}^{n\times n}$, $i = 1, 2, 3$, if there exist parameter-dependent matrices $0 \prec P(\alpha) = P(\alpha)' \in \mathbb{R}^{n\times n}$, $X_i(\alpha) \in \mathbb{R}^{n\times n}$, $i = 1, 2, 3$, a parameter-independent matrix $K \in \mathbb{R}^{m\times p}$ and real scalars δ and r such that

$$A_{cl}(\alpha) + \delta I \geq 0, \tag{8}$$

$$\mathcal{Q}(\alpha) + \mathcal{X}(\alpha)\mathcal{B}(\alpha) + \mathcal{B}(\alpha)'\mathcal{X}(\alpha)' \prec 0 \tag{9}$$

are verified $\forall \alpha \in \Lambda$, where $\mathcal{B}(\alpha) = [Y_1(\alpha)\ Y_2(\alpha)\ Y_3(\alpha)]$,

$$\mathcal{Q}(\alpha) = \begin{bmatrix} 0 & \star & \star \\ P(\alpha) & 0 & \star \\ (A_{cl}(\alpha) - rI) & -I & 0 \end{bmatrix},\ \mathcal{X}(\alpha) = \begin{bmatrix} X_1(\alpha) \\ X_2(\alpha) \\ X_3(\alpha) \end{bmatrix},$$

with $A_{cl}(\alpha) = A(\alpha) + B(\alpha)KC_y(\alpha)$, then K is a static output-feedback robust control gain such that the real part of the eigenvalues of $A_{cl}(\alpha)$, $\forall\alpha \in \Lambda$, is smaller than r and the positivity of closed-loop system (7) is assured.

Proof. First observe that (8) certifies the positivity of the closed-loop system (all off diagonal terms of $A_{cl}(\alpha)$ are nonnegative), that is, $(A(\alpha) + B(\alpha)KC(\alpha))$ is Metzler. Additionally, note that, if (9) is feasible, block (3,3) from (9), given by $X_3(\alpha)Y_3(\alpha) + Y_3(\alpha)'X_3(\alpha)' \prec 0$, guarantees that $Y_3(\alpha)$ is a full-rank matrix and, therefore, $\mathcal{B}(\alpha)$ can be rewritten as $\mathcal{B}(\alpha) = Y_3(\alpha)\,[-F(\alpha)\ -G(\alpha)\ I]$. Thus, pre- and post-multiplying (9) respectively by

$$\mathcal{B}^{\perp}(\alpha)' = \begin{bmatrix} I & 0 & F(\alpha) \\ 0 & I & G(\alpha) \end{bmatrix}$$

and $\mathcal{B}^{\perp}(\alpha)$ yields

$$\begin{bmatrix} \mathrm{He}\,(F(\alpha)(A_{cl}(\alpha) - rI)) & \star \\ P(\alpha) - F(\alpha) + G(\alpha)(A_{cl}(\alpha) - rI) & \mathrm{He}\,(G(\alpha)) \end{bmatrix} \prec 0. \tag{10}$$

By multiplying (10) on the left by $\Gamma(\alpha)'$ and on the right by $\Gamma(\alpha) = \left[I\ (A_{cl}(\alpha) - rI)'\right]'$ one has

$$P(\alpha)(A_{cl}(\alpha) - rI) + (A_{cl}(\alpha) - rI)'P(\alpha) \prec 0, \tag{11}$$

that assures the asymptotic stability of a system with dynamic matrix given by $A(\alpha)+B(\alpha)KC(\alpha)-rI$ (reproducing an extension of Lemma 1 for the polytopic case). Therefore, K is a static output-feedback robust controller that allocates the real part of the eigenvalues of the closed-loop system (7) on the left of r. Furthermore, if $r \le 0$, K is a stabilizing robust gain for (3). □

Theorem 2 can also be used to provide state-feedback robust gains for system (3) by making $C_y(\alpha) = I$.

Note that condition (9) in Theorem 2 is not a bilinear matrix inequality (BMI) only because matrices $Y_i(\alpha)$, $i = 1, 2, 3$, are given. The introduction of variable r in the conditions of Theorem 2 represents a *relaxation* in the stability condition, in the sense that the eigenvalues of $A_{cl}(\alpha)$ must be located in the half-plain on the left of r, and r is a free optimization variable. Therefore, Theorem 2 can be recurrently used, by minimizing r until $r \le 0$, in order to assure the stability of the original closed-loop system. The iterative procedure, described in Algorithm 1, always provide feasible solutions for the conditions of Theorem 2 such that r is non-increasing, whenever a suitable initial condition for $Y_i(\alpha)$, $i = 1, 2, 3$ is applied, as described in next theorem.

Theorem 3. The choice $\mathcal{B}(\alpha) = \mathcal{B}_0 = [I\ I\ -I]$ ensures the existence of a feasible solution for Theorem 2.

Proof. Choose $\mathcal{X}(\alpha) = (-r/2)\mathcal{B}_0$, then rewrite (9) as

$$\begin{bmatrix} -rI & \star & \star \\ P(\alpha) - rI & -rI & \star \\ A_{cl}(\alpha) & (r-1)I & -rI \end{bmatrix} \prec 0. \tag{12}$$

For the particular choice $P(\alpha) = rI$, $r > 0$, and applying a Schur complement in (12), the resulting inequality is

$$-rI - [A_{cl}(\alpha)\ \ (r-1)I]\,(-rI)^{-1}\,[A_{cl}(\alpha)\ \ (r-1)I]' \prec 0,$$

which is equivalent to

$$-rI + \frac{1}{r}A_{cl}(\alpha)A_{cl}(\alpha)' + \frac{(r-1)^2}{r}I \prec 0. \tag{13}$$

By multiplying (13) by r on both sides, one has

$$(1-2r)I + A_{cl}(\alpha)A_{cl}(\alpha)' \prec 0,$$

which can be satisfied with any $r > 0$ such that $(2r-1)$ is greater than the spectral radius of $A_{cl}(\alpha)A_{cl}(\alpha)'$, $\forall\alpha \in \Lambda$. Finally, the choice $K = 0$ guarantees that (8) is also satisfied, since $A(\alpha)$ is, by hypothesis, Metzler. □

4.1 Iterative Procedure

This section presents a detailed description of the iterative procedure proposed to solve the conditions of Theorem 2 and, possibly, to provide a stabilizing static output-feedback robust controller for system (3). It is well known that the closed-loop stability of linear positive systems can be certified by means of a diagonal Lyapunov matrix without loss of generality. However, numerical experiments showed that the iterative procedure proposed in this paper produces better results when the Lyapunov matrix is allowed to be full, symmetric and positive definite, as presented in the statement of Theorem 2. The algorithm can be divided in two steps, sequentially executed at each iteration.

First, matrix $\mathcal{B}(\alpha) = [Y_1(\alpha)\ Y_2(\alpha)\ Y_3(\alpha)]$ in Theorem 2 is initialized with a value defined by the designer (a possible option is $\mathcal{B}_0 = [I\ I\ -I]$ presented in Theorem 3). Then, start Phase 1 of Algorithm 1, which searches for a feasible solution for Theorem 2. If feasible, the computed values of r, K and δ are stored and the algorithm proceeds to Phase 2. Otherwise, the algorithm is ended without providing a feasible solution (observe that the inequalities in Theorem 2 can be unfeasible for initial conditions different from those proposed in Theorem 3). Since r is an upper bound for the real part of the eigenvalues of the closed-loop dynamic matrix $A_{cl}(\alpha)$, if $r \le 0$, the eigenvalues of $A_{cl}(\alpha)$ are located in the left half-plan of the complex plan, meaning that the algorithm has found a robustly stabilizing gain K. On the other hand, when $r > 0$, it is not possible to assure the system stability. Nevertheless, since r is only an upper bound for the real part of the eigenvalues of $A_{cl}(\alpha)$, the eigenvalues could be negative even with $r > 0$. For this reason, when $r > 0$, the stability and positivity of the closed-loop system are tested and, if both conditions are verified, Algorithm 1 is interrupted, returning the stabilizing solution. To reduce the computational effort of this evaluation, the test first computes the eigenvalues associated to the vertices $A_{cl\,i}$, $i = 1, \ldots, N$, of the polytopic closed-loop dynamic matrix (making $\alpha_i = 1$, $\alpha_j = 0$, $j \ne i$). Only if all eigenvalues of the vertices have negative real part (necessary condition for the stability of the polytope), an LMI condition is used to certify the stability and an element-wise inequality condition is employed to assure the positivity of the closed-loop system. If both stability and positivity are confirmed, Phase 1 of Algorithm 1 has found a stabilizing gain. Otherwise, the value of $\mathcal{X}(\alpha)'$ provided by Algorithm 1 in this iteration replaces the value of $\mathcal{B}(\alpha)$ used as initial condition $\mathcal{B}_0$ of the next iteration. Note that, with this update, the feasibility is always assured because $\mathrm{He}(\mathcal{X}(\alpha)\mathcal{B}(\alpha)) = \mathrm{He}(\mathcal{B}(\alpha)'\mathcal{X}(\alpha)')$ in (9), repeating, in the worst case scenario, the previous solution in terms of the value of r. With this strategy, it is possible to assure that the value of r always decreases or, at least, remains the same computed in the previous iteration. In other words, Algorithm 1 is a locally convergent iterative procedure.

Those steps continue until Algorithm 1 provides a gain that simultaneously assures the positivity and asymptotic stability of the closed-loop system ($r \leq 0$, or feasible stability analysis and positivity tests in Phase 2), or when the maximum number of iterations it_{max} is achieved.

Algorithm 1

Initialize: $\mathcal{B}_0(\alpha) \leftarrow [I\ I\ -I]$, $k \leftarrow 0$, $it_{\max}$;
while $k < it_{\max}$ **do**
  $k \leftarrow k+1$;
  minimize r subject to (8)–(9) with $\mathcal{B}_0(\alpha)$;
  if feasible **then**
    $r_k \leftarrow r$; $K_{aux} \leftarrow K$; $\delta_k \leftarrow \delta$;
    $A_{cl}(\alpha) \leftarrow A(\alpha) + B(\alpha)K_{aux}C(\alpha)$;
  else
    quit (no solution);
  end if
  if $r \leq 0$ **then**
    $K_{sol} \leftarrow K_{aux}$;
    return K_{sol};
  else if $\max_{j=1\cdots n} \mathrm{Re}(\lambda_j(A_{cl}^i)) < 0$, $A_{cl}^i \geq 0$, $i = 1, \ldots, N$
then Apply stability and positivity tests on $A_{cl}(\alpha)$;
    if feasible **then**
      $K_{sol} \leftarrow K_{aux}$;
      return K_{sol};
    end if
  end if
  $\mathcal{B}_0(\alpha) \leftarrow \mathcal{X}(\alpha)'$;
end while

5. COMPUTATIONAL ASPECTS

The conditions proposed in Theorem 2 are infinite dimensional problems, because they must be solved for all $\alpha \in \Lambda$. To circumvent this obstacle, this paper employs the so-called relaxations: sufficient LMI conditions constructed from the imposition of homogeneous polynomial structures of fixed degree to the optimization variables of the problem [Oliveira and Peres, 2007]. This finite set of LMIs can be automatically generated by the Robust LMI Parser (ROLMIP) [Agulhari et al., 2019] toolbox. Particularly, the examples of the next section employ degree one for the variables of Theorem 2 that depend on α and also for $P(\alpha)$ in the stability analysis conditions given by $\mathrm{He}(A_{cl}(\alpha)'P(\alpha)) \prec 0$, $P(\alpha) \succ 0$. Clearly, when evaluating systems not affected by uncertainties (Section 6.1), all optimization variables are α-independent. All the computational scripts were programmed in Matlab (R2014a) employing the SDP solver SeDuMi [Sturm, 1999] and parsers Yalmip [Löfberg, 2004] and ROLMIP [Agulhari et al., 2019].

6. NUMERICAL EXPERIMENTS

This section presents numerical examples to evaluate the performance of the approach developed in this paper regarding the synthesis of robust controllers that assure the positivity and stability of closed-loop uncertain continuous-time linear systems. A second aim of this section is to compare the proposed technique with the methods presented in Bhattacharyya and Patra [2018], (BP18) and Feng et al. [2011], (FLLS11). It is important to emphasize that, since the literature conditions do not handle the uncertain case, the comparative examples investigate only precisely known positive systems.

6.1 Precisely known systems

In this section, the proposed method is compared with three techniques from the literature (FLLS and the two methods presented in BP18: BP-1 and BP-2) concerning static output-feedback (SOF) stabilization of precisely known continuous-time positive linear systems. The first four examples from BP18 (Ex 1, 2, 3 and 4) consist in providing a stabilizing controller without imposing structures on the control gain. The fifth example of BP18 proposes the stabilization of the system investigated in Ex 1 by imposing four different structure constraints in the gain K (Ex 5.1, 5.2, 5.3, 5.4). Method BP-1 is capable of providing stabilizing solutions only for three of the first four examples and for none of the cases with constrained structure. On the other hand, BP-2 provides stabilizing solutions for all the proposed examples. Finally, while the iterative procedure proposed in this paper provides stabilizing SOF gains for all examples, none of the other methods investigated in BP18 (FLLS, Shen and Lam [2015], Ait Rami [2011], Wang and Huang [2013]) succeeds on all the cases.

To better evaluate the conservativeness of the methods, consider the design of an SOF controller for the systems presented in the examples of BP18 by replacing the open-loop dynamic matrix A by $A + \rho I$. The aim is to verify which approach guarantees solutions for the greater range of positive values of ρ. Algorithm 1 (A1, employing arbitrary structure in $X_{1,2}(\alpha)$ and symmetric structure in both $X_3(\alpha)$ and $P(\alpha)$ in Theorem 2) proposed in this paper, the two algorithms from BP18 (BP-1 and BP-2) and the algorithm from FLLS were tested. The maximum number of iterations imposed for all algorithms is $it_{max} = 20$. Figure 1 presents the results for the unconstrained synthesis examples while Figure 2 shows the results for the tests where structure constraints are imposed on the gain K. Note that the proposed iterative procedure has the best results in all cases, never producing results worser than the compared methods.

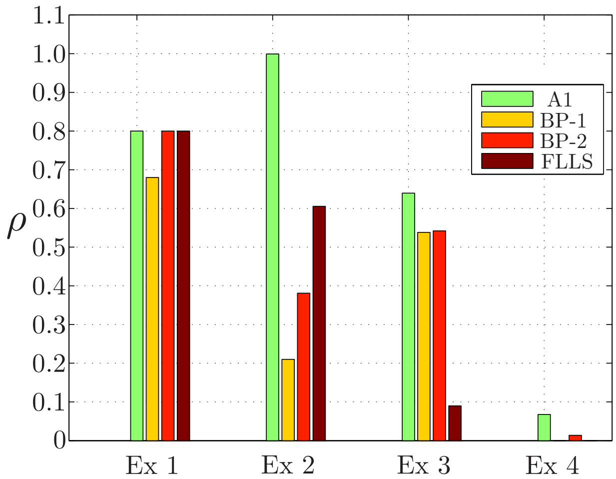

Figure 1. Stabilizing range of ρ obtained for examples 1 to 4 from BP18.

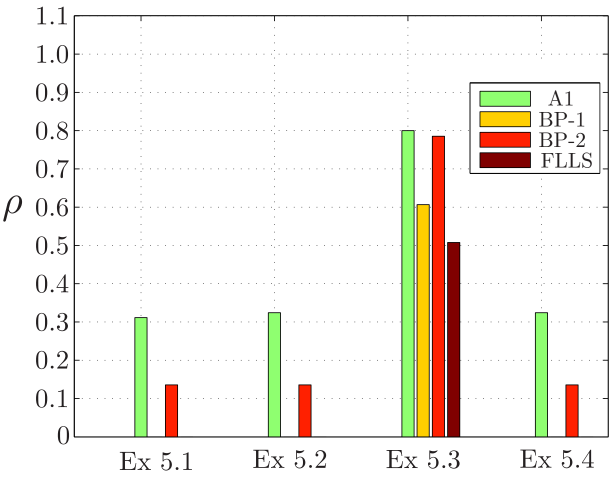

Figure 2. Stabilizing range of ρ obtained for examples 5.1 to 5.4 from BP18 (structured gain).

6.2 Uncertain System

To illustrate the applicability of Algorithm 1 in robust control design for an uncertain continuous-time positive linear system, consider the state-space matrices borrowed from [Ait Rami, 2011, Section 4]

$$A = \begin{bmatrix} -0.15 & 1.90 & 1.55 \\ 0.50 & -0.3 & 0.10 \\ 0.20 & 0.50 & -2.55 \end{bmatrix}, \quad B = \begin{bmatrix} 0.55 & -0.64 \\ 1.69 & 0.38 \\ 0.59 & -1.50 \end{bmatrix}. \quad (14)$$

The output matrix is considered to be uncertain, belonging to a polytope, aiming to simulate a failure of, at most, 5% in the sensor power of the first state and up to 100% in the sensor power measuring the second state

$$C(\beta) = \beta\,[1\ 1\ 0] + (1-\beta)\,[0.95\ 0\ 0], \quad \beta \in [0,1]. \quad (15)$$

Algorithm 1 (with $it_{\max} = 13$, employing arbitrary structure in $X_{1,2}(\alpha)$ and symmetric structure in both $X_3(\alpha)$ and $P(\alpha)$ in Theorem 2) provides the following stabilizing controller (truncated with 4 decimal digits):

$$K = [-0.2994\ 0.0156]'. \quad (16)$$

As mentioned before, the proposed method is appropriate to synthesize gains with particular structures. Thus, imposing the constrained structure to the control matrix $K = [k_1\ 0]$, the following stabilizing solution is obtained

$$K = [-0.2959\ 0]'. \quad (17)$$

To further illustrate that both controllers gains, in (16) and (17), assure the positivity and stability of the closed-loop system, Figure 3 (a) presents the minimum values of the non-diagonal elements of $A_{cl}(\alpha)$ $(\min(a_{ij}), i \neq j)$, while Figure 3 (b) shows the maximum value of the real part of the eigenvalues of $A_{cl}(\alpha)$ $(\lambda_{\max} = \max_{i=1,\ldots,n} \mathrm{Re}(\lambda_i(A_{cl}(\alpha))))$, for all $\alpha \in [0,1]$ (a fine grid was employed). Note in Figure 3 (a) that the closed-loop matrix is Metzler for the entire uncertainty domain (all the non-diagonal elements of $A_{cl}(\alpha)$ are nonnegative). Additionally, observe in Figure 3 (b) that $A_{cl}(\alpha)$ is Hurwitz since $\lambda_{\max} < 0$, $\forall \alpha \in \Lambda$.

To conclude, consider the following uncertain output matrix

$$C(\beta) = \beta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + (1-\beta) \begin{bmatrix} 0.95 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}, \quad \beta \in [0,1]. \quad (18)$$

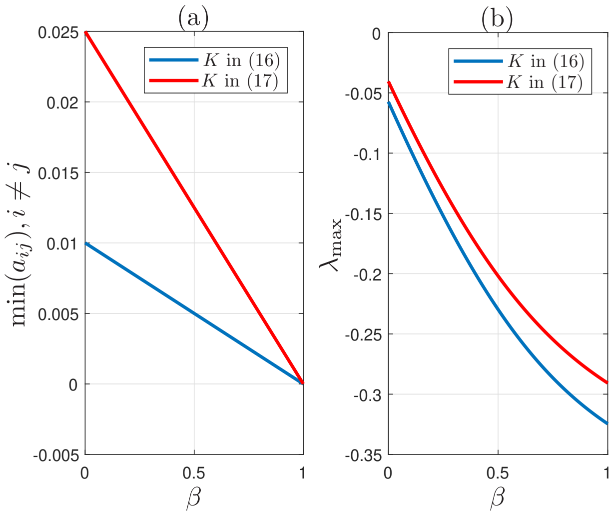

Figure 3. (a) Minimum values of the non-diagonal elements of $A_{cl}(\alpha)$ $(\min(a_{ij}), i \neq j)$ and (b) maximum value of the real part of the eigenvalues of $A_{cl}(\alpha)$ $(\lambda_{\max})$ versus $\alpha \in [0,1]$.

The robust stabilizing SOF gain provided by Algorithm 1 without imposing any restriction is

$$K = \begin{bmatrix} -0.2994 & -5.6549 \\ 0.0156 & -1.8909 \end{bmatrix}. \quad (19)$$

Suppose that the application requires that the stabilizing gain is subject to magnitude constraints in each one of its entries, that is, the aim is to determine the minimum value M, such that for $|k_{ij}| \leq M$, the closed-loop system (14), (18) with the gain $K = [k_{ij}]$, $i = 1,2$, $j = 1,2$, is stable and positive. Applying Algorithm 1, the minimum value obtained for M is $M = 0.271$ associated with the following control matrix

$$K = \begin{bmatrix} -0.2710 & -0.2710 \\ 0.0267 & 0.2267 \end{bmatrix} \quad (20)$$

which corresponds to a reduction of 95% of the largest entry of the control gain (5.6249 in (19) and 0.2710 in (20)). Note that this type of magnitude constraint on the entries of the control gain could not be imposed directly in other LMI-based strategies for multiple-input-multiple-output systems based on change of variables without introducing additional conservativeness (as, for instance, constraints in some of the other optimization variables), while using the proposed method it suffices to add mp linear constraints in the problem, since the gain is an optimization variable.

7. CONCLUSION

This paper presented an iterative method based on parameter-dependent LMIs to provide static output-feedback controllers that robustly stabilize continuous-time positive polytopic linear systems. The existence of initial conditions that always assure feasible solutions for a particular relaxation strategy and the local convergence of the proposed algorithm were demonstrated. One of the advantages of the technique is the possibility of designing controllers with particular structures without restricting other variables of the problem, due to the fact that the gain is handled directly as an optimization variable. The low level of conservatism of the proposed iterative procedure when compared with other literature methods and the applicability of the method in robust control considering magnitude or structure constraints in the gain were illustrated by means of numerical examples from the literature.

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