Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Adaptive Control for Nonlinear Systems with Time-Varying Parameters and Control Coefficient ⋆

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Abstract: This paper exploits the so-called *congelation of variables* method to design an adaptive controller for nonlinear systems with time-varying parameters. Two motivating examples describing scalar systems are discussed to illustrate the flexibility of the *congelation of variables* method to deal with the cases in which the time-varying parameters are coupled with the state and with the input, respectively. Interpretations from a passivity perspective are also provided. Then design procedures are derived for general nonlinear systems in *parametric strict-feedback* form, and it is shown that the state of the underlying system converges to the origin and all signals of the closed-loop system remain bounded. Simulations show that, in the presence of parameter variations, the performance of the proposed controller is superior to that of the classical adaptive controller designed for time-invariant systems.

Keywords: Adaptive control, Backstepping, Time-varying systems, Passivity

1. INTRODUCTION

Since the 1980s, adaptive control has undergone extensive research (see *e.g.* Narendra and Annaswamy (1989); Krstic et al. (1995); Ioannou and Sun (1996); Tao (2003); Astolfi et al. (2007)), yet the works on systems with time-varying parameters appear not to be as voluminous as works that only consider time-invariant systems. Some pioneering works on adaptive control for time-varying systems (see *e.g.* Goodwin and Teoh (1983)) exploit *persistence of excitation* to guarantee stability by ensuring that parameter estimates converge to the true parameters. Subsequent works (see *e.g.* Kreisselmeier (1986), Middleton and Goodwin (1988)) have removed the restriction of *persistence of excitation* by requiring bounded and slow (in an average sense) parameter variations.

More recent works can be mainly categorized into two trends. One of them is based on the so-called *robust adaptive law* or *σ-modification* (see Ioannou and Sun (1996)), which adds leakage to the parameter update law when the parameter estimates drift out of a pre-specified reasonable region to guarantee the boundedness of the parameter estimates. This approach achieves asymptotic tracking when the parameters are constant, otherwise the tracking error is nonzero and related to the rates of the parameter variations, see Tsakalis and Ioannou (1987). In Zhang and Ioannou (1996) and Zhang et al. (2003) the parameter variations are modelled in two parts: known parameter variations and unknown variations, so that the residual tracking error only depends on the rates of the unknown parameter variations.

The other trend exploits the so-called *filtered transformation*, which is essentially an adaptive observer described via a change of coordinates, and the *projection operation*, which confines the parameter estimates within a pre-specified compact set to guarantee the boundedness of the parameter estimates, see Marino and Tomei (1993), Marino and Tomei (1999) and Marino and Tomei (2003). These methods can guarantee asymptotic tracking provided that the parameters are bounded in a compact set, their derivatives are $\mathcal{L}_1$ and the disturbance on the state evolution is additive and $\mathcal{L}_2$. Moreover, *a priori* knowledge on parameter variations is not needed and the residual tracking error is independent of the rates of the parameter variations.

The methods mentioned above cannot guarantee zero-error regulation when the unknown parameters are persistently varying. To achieve asymptotic state/output regulation when the time-varying parameters are neither known nor asymptotically constant, in Chen and Astolfi (2018a) and Chen and Astolfi (2018b) a method called the *congelation of variables* has been proposed and developed on the basis of the *adaptive backstepping* approach and the *adaptive immersion and invariance (I&I)* approach, respectively. In the spirit of the *congelation of variables* method each unknown time-varying parameter is treated as a nominal unknown constant parameter perturbed by the difference between the true parameter and the nominal parameter, which causes a time-varying perturbation term. The controller design is then divided into a classical

⋆ This work has been partially supported by the European Union's Horizon 2020 Research and Innovation Programme under grant agreement No 739551 (KIOS CoE).

adaptive control design, with constant unknown parameters, and a damping design via dominance to counteract the time-varying perturbation terms. This method is compatible with most adaptive control schemes using parameter estimates, as it does not change the original parameter update law designed for time-invariant systems.

Time-varying parameters coupled with the input needs special treatment in the *congelation of variables* scheme, as the perturbation terms depend on the input. The simple strengthening of damping terms in the controller alters the input (as well as the perturbation itself) and therefore causes a *chicken-and-egg* dilemma. The aim of this paper is to design a controller such that, in the spirit of the *congelation of variables* method, one can naturally guarantee stability properties in the presence of a time-varying control coefficient without incurring the above-mentioned *chicken-and-egg* problem.

Notation. This paper uses standard notation unless stated otherwise. For an n-dimensional vector $v \in \mathbb{R}^n$, $|v|$ denotes its Euclidean 2-norm, $|v|_M = \sqrt{v^\top M v}$, $M = M^\top \succ 0$, denotes the weighted 2-norm with weight M, $\underline{v_i} \in \mathbb{R}^i$, $1 \leq i \leq n$, denotes the vector composed of the first i elements of v. For a matrix M, $|M|_{\mathrm{F}}$ denotes its Frobenius norm. I and S denote the identity matrix and the upper-shift matrix with proper dimension, respectively. For an n-dimensional time-varying signal $s : \mathbb{R} \to \mathbb{R}^n$, with image contained in a compact set $\mathcal{S}$, $\Delta_s : \mathbb{R} \to \mathbb{R}^n$ denotes the deviation of s from a constant reference ℓ_s, *i.e.* $\Delta_s(t) = s(t) - \ell_s$, and $\delta_s \in \mathbb{R}$ denotes the supremum of the 2-norm of s, *i.e.* $\delta_s = \sup_{s\in\mathcal{S}, t\geq 0} |s(t)| \geq 0$. ⋄

In this paper the vector of unknown time-varying system parameters $\theta : \mathbb{R} \to \mathbb{R}^q$ may verify one of the assumptions below.

Assumption 1. (Bounded parameters). $\theta(t) \in \Theta_0$, $\forall t \geq 0$, where Θ_0 is a compact set. ⋄

Assumption 2. (Sign-definite parameter). The parameter $b_m : \mathbb{R} \to \mathbb{R}$ is bounded away from 0 in the sense that there exists a constant ℓ_{b_m} such that $\mathrm{sgn}(\ell_{b_m}) = \mathrm{sgn}(b_m(t)) \neq 0$ and $0 < |\ell_{b_m}| \leq |b_m(t)|$, $\forall t \geq 0$. The sign of ℓ_{b_m} and $b_m(t)$, $\forall t \geq 0$, is known[1] and does not change. ⋄

2. MOTIVATING EXAMPLES

2.1 Parameter in the Feedback Path

To begin with consider a scalar nonlinear system described by the equation

$$\dot{x} = \theta(t)x^2 + u, \tag{1}$$

where $x(t) \in \mathbb{R}$ is the state, $u(t) \in \mathbb{R}$ is the input, and $\theta(t) \in \mathbb{R}$ is an unknown time-varying parameter satisfying Assumption 1. Assuming that we have an "estimate" $\hat{\theta}$ of the parameter $\theta(t)$, we can rewrite (1) as

$$\dot{x} = \hat{\theta}x^2 + u + (\theta - \hat{\theta})x^2. \tag{2}$$

One way to find an update law for $\hat{\theta}$ is to consider a Lyapunov function candidate

$$V(x, \hat{\theta}, \theta) = \frac{1}{2}x^2 + \frac{1}{2\gamma_\theta}(\theta - \hat{\theta})^2. \tag{3}$$

Taking the time derivative of V along the solutions of (2) yields

$$\dot{V} = \hat{\theta}x^3 + ux + (\theta - \hat{\theta})x^3 - (\theta - \hat{\theta})\frac{\dot{\hat{\theta}}}{\gamma_\theta} + (\theta - \hat{\theta})\frac{\dot{\theta}}{\gamma_\theta}, \tag{4}$$

which means that the selection of the parameter update law

$$\dot{\hat{\theta}} = \gamma_\theta x^3 \tag{5}$$

cancels the effect of the unknown $(\theta - \hat{\theta})x^3$ term. The constant $\gamma_\theta > 0$ is known as adaptation gain. In classical adaptive control problems one assumes that θ is constant, that is $\dot{\theta} = 0$, for all $t \geq 0$, and selects the control law

$$u = -kx - \hat{\theta}x^2 \tag{6}$$

with $k > 0$, which yields $\dot{V} = -kx^2 \leq 0$. We can conclude from this that x and $\hat{\theta}$ are bounded, and x converges to 0 by invoking Barbalat lemma. When $\dot{\theta}$ is not identically zero, one has to deal with the indefinite term $(\theta - \hat{\theta})\frac{\dot{\theta}}{\gamma_\theta}$. One way to do this is to modify (5) with the so-called *projection operation* (see *e.g.* Goodwin and Mayne (1987), Pomet and Praly (1992)), which confines the parameter $\hat{\theta}$ inside a convex compact set and therefore guarantees the boundedness of $(\theta - \hat{\theta})$. It follows that the boundedness of $\dot{\theta}$ guarantees the boundedness of x (either exact boundedness, *e.g.* in Zhou and Wen (2008) or boundedness in an average sense, *e.g.* in Middleton and Goodwin (1988)), and $\dot{\theta} \in \mathcal{L}_1$ guarantees the convergence of x to 0 (*e.g.* in Marino and Tomei (1999), Marino and Tomei (2000), Marino and Tomei (2003)). In some other works (*e.g.* in Tsakalis and Ioannou (1987), Zhang and Ioannou (1996), Zhang et al. (2003)), the boundedness of $\hat{\theta}$ is guaranteed by the so-called *σ-modification*, which adds some leakage to the integrator (5) if the parameter estimate drifts outside a reasonable region, and it is often referred to as soft projection. All these schemes share the similarity that they treat $\dot{\theta}$ as a disturbance. As a result some disturbance attenuation effort is required to guarantee that bounded $\dot{\theta}$ causes bounded state/output regulation/tracking error, and sufficiently fast converging $\dot{\theta}$, which means that θ becomes constant eventually, guarantees the convergence of the error to 0. As a result, none of these methods can guarantee zero-error regulation/tracking when the unknown parameter is persistently time-varying, in which case $\dot{\theta}$ is non-vanishing.

Note that the reason why we cannot avoid $\dot{\theta}$ in the analysis is the $\theta - \hat{\theta}$ term in (3). This term is included only to guarantee the boundedness of $\hat{\theta}$, yet by no means guaranteeing the convergence of $\hat{\theta}$ to θ, no matter whether θ is time-varying or constant, thus replacing θ with a constant ℓ_θ, to be determined, can guarantee the same properties. ℓ_θ can be regarded as the average of $\theta(t)$, which

[1] If the sign of $b_m(t)$ is unknown yet constant and $b_m(t)$ is bounded away from 0, then a Nussbaum-type function (see Nussbaum (1983)) can be exploited to compensate for $b_m(t)$, see *e.g.* Ge and Wang (2002). This method is not discussed here because it trades robustness against an unknown sign with undesirable transient performance due to the oscillatory exploration of the Nussbaum gain (and half of the exploration is actually in the wrong direction), whereas determining an unknown yet constant sign is typically not a challenge in many applications.

is not necessarily known. In the light of this, consider the modified Lyapunov function candidate $V_\ell(x, \hat{\theta}, \ell_\theta) = \frac{1}{2}x^2 + \frac{1}{2\gamma_\theta}(\ell_\theta - \hat{\theta})^2$. Taking the time derivative of V_ℓ along the trajectories of (2) yields

$$\dot{V}_\ell = \hat{\theta}x^3 + ux + (\theta - \hat{\theta})x^3 - (\theta - \hat{\theta})\frac{\dot{\hat{\theta}}}{\gamma_\theta} + \Delta_\theta x^3, \quad (7)$$

where $\Delta_\theta = \theta - \ell_\theta$. Comparing (7) with (4) we can see that the substitution of ℓ_θ for θ eliminates the $\dot{\theta}$ term, at the cost of adding a perturbation term $\Delta_\theta x^3$ due to the inconsistency between θ and ℓ_θ. Considering the same parameter update law as in (5) and a new control law

$$u = -\left(k + \frac{1}{2\epsilon_{\Delta_\theta}}\delta_{\Delta_\theta}\right)x - \frac{1}{2}\epsilon_{\Delta_\theta}\delta_{\Delta_\theta}x^3 - \hat{\theta}x^2, \quad (8)$$

where $\epsilon_{\Delta_\theta} > 0$ is a constant, to balance the linear and the nonlinear terms, yields $\dot{V}_\ell = -\big(k + \frac{1}{2\epsilon_{\Delta_\theta}}\delta_{\Delta_\theta}\big)x^2 - \frac{1}{2}\epsilon_{\Delta_\theta}\delta_{\Delta_\theta}x^4 + \Delta_\theta x^3 \leq -kx^2 \leq 0$. Therefore we can conclude boundedness of all trajectories of the closed-loop system as well as convergence of x to 0 using the same argument as the one used in the classical constant parameter problem and without requiring a vanishing $\dot{\theta}$. The method of substituting the constant ℓ_θ for the time-varying θ to avoid the occurrence of undesired time derivatives is called *congelation of variables*, see Chen and Astolfi (2018a).

Remark 1. The control law (8) and the parameter update law (5) do not depend on ℓ_θ, in the same way as classical adaptive controllers do not depend on θ, which preserves the "adaptive" property. One can interpret the proposed controller as a combination of an adaptive controller, to cope with the unknown parameter ℓ_θ, and a robust controller, to cope with the time-varying perturbation $\Delta_\theta(t)$. This fact can also be revealed by noting that when θ is a constant one could select $\ell_\theta = \theta$, hence $\delta_{\Delta_\theta} = 0$, and the control law (8) is reduced to the classical control law (6). ⋄

Remark 2. The control law (8) depends on δ_{Δ_θ}, which is assumed to be known by Assumption 1. Even if δ_{Δ_θ} is unknown, one can easily overcome this by building an "estimate" for δ_{Δ_θ} via classical adaptive control techniques as δ_{Δ_θ} is a constant. For instance, in this example we can substitute a dynamically updated $\hat{\delta}$ for δ_{Δ_θ}, using the update law $\dot{\hat{\delta}} = \gamma_\delta(x^2 + x^4)$, with $\gamma_\delta > 0$. However, we will not discuss this method in detail as it over-complicates the problem without providing any significant contribution. ⋄

Remark 3. (Passivity interpretation). Consider the classical adaptive control problem in which θ is constant. The closed-loop dynamics can be described via a negative feedback loop consisting of two passive systems

$$\Sigma_1 : \begin{cases} \dot{x}_1 = -kx_1 + x_1^2 u_1, \\ y_1 = x_1^3, \end{cases} \quad (9)$$

$$\Sigma_2 : \begin{cases} \dot{x}_2 = \gamma_\theta u_2, \\ y_2 = x_2, \end{cases} \quad (10)$$

where $x_1 = x$, $x_2 = \hat{\theta} - \theta$, $u_1 = -y_2$, $u_2 = y_1$. The storage functions are $S_1 = \frac{1}{2}x_1^2$ and $S_2 = \frac{1}{2\gamma_\theta}x_2^2$, respectively. It is well-known that the parameter update law (5) is neither designed to guarantee the convergence of $\hat{\theta} - \theta$ to zero nor to make $\hat{\theta}$ estimate θ, though $\hat{\theta}$ is called the parameter estimate by convention, but to make $\hat{\theta} - \theta$ an input/output signal to form a passive interconnection. When $\theta(t)$ is time-varying, the dynamics of Σ_2 are described by

$$\Sigma_2 : \begin{cases} \dot{x}_2 = \gamma_\theta u_2 - \dot{\theta}, \\ y_2 = x_2, \end{cases} \quad (11)$$

which causes the loss of passivity from u_2 to y_2. The *congelation of variables* method can therefore be interpreted as selecting a new signal $\hat{\theta} - \ell_\theta$, which can yield a passive interconnection, and maintaining the passivity of Σ_1, by strengthened damping. The two passive systems are described by the equations

$$\Sigma_1 : \begin{cases} \dot{x}_1 = -a(x_1, t)x_1 + x_1^2 u_1, \\ y_1 = x_1^3, \end{cases} \quad (12)$$

$$\Sigma_2 : \begin{cases} \dot{x}_2 = \gamma_\theta u_2, \\ y_2 = x_2, \end{cases} \quad (13)$$

where $x_1 = x$, $x_2 = \hat{\theta} - \ell_\theta$, $u_1 = -y_2$, $u_2 = y_2$ and $a(x_1, t) = \big(k + \frac{1}{2\epsilon_{\Delta_\theta}}\delta_{\Delta_\theta}\big) + \frac{1}{2}\epsilon_{\Delta_\theta}\delta_{\Delta_\theta}x_1^2 - \Delta_\theta x_1 \geq k > 0$. ⋄

2.2 Parameter in the input path

In what follows we show how to extend the idea of *congelation of variables* to systems in which the time-varying parameter is coupled with the input by considering the nonlinear system

$$\dot{x} = \theta(t)x^2 + b(t)u, \quad (14)$$

where $\theta(t)$ satisfies Assumption 1 and $b(t) \in \mathbb{R}$ satisfies Assumption 1 and Assumption 2. Equation (14) can be re-written as

$$\begin{aligned} \dot{x} &= \hat{\theta}x^2 + \bar{u} + \Delta_\theta x^2 + \Delta_b \hat{\varrho}\bar{u} \\ &\quad + (\ell_\theta - \hat{\theta})x^2 - \ell_b\left(\frac{1}{\ell_b} - \hat{\varrho}\right)\bar{u}, \end{aligned} \quad (15)$$

where $\Delta_b(t) = b(t) - \ell_b$, $\hat{\varrho}$ is an "estimate" of $\frac{1}{\ell_b}$, and $u = \hat{\varrho}\bar{u}$. From classical adaptive control theory (see *e.g.* Krstic et al. (1995)) we know that the effect of the second line of (15) can be cancelled by selecting the parameter update laws (5) and

$$\dot{\hat{\varrho}} = -\gamma_\varrho \mathrm{sgn}(\ell_b)\bar{u}x, \quad (16)$$

and by considering the Lyapunov function candidate $V(x, \hat{\theta}, \hat{\varrho}) = \frac{1}{2}x^2 + \frac{1}{2\gamma_\theta}(\theta - \hat{\theta})^2 + \frac{|\ell_b|}{2\gamma_\varrho}\big(\frac{1}{\ell_b} - \hat{\varrho}\big)^2$, the time derivative of which along the trajectories of (15) is

$$\dot{V} = \hat{\theta}x^3 + \bar{u}x + \Delta_\theta x^3 + \Delta_b \hat{\varrho}\bar{u}x. \quad (17)$$

Note that the perturbation term $\Delta_b \hat{\varrho}\bar{u}x$ depends on $\bar{u}$ explicitly, which means that we cannot dominate this term by simply adding damping terms to $\bar{u}$, as doing this also alters the perturbation term itself. Instead, we need to make $\Delta_b \hat{\varrho}\bar{u}x$ non-positive by designing $\bar{u}$ and selecting ℓ_b. Consider $\bar{u}$ as a feedback control law with a non-positive nonlinear gain, that is

$$\begin{aligned} \bar{u} &= -\left(k + \frac{1}{2}\left(\frac{\delta_{\Delta_\theta}}{\epsilon_{\Delta_\theta}} + \frac{1}{\epsilon_{\hat{\theta}}}\right) + \frac{1}{2}(\epsilon_{\Delta_\theta}\delta_{\Delta_\theta} + \epsilon_{\hat{\theta}}\hat{\theta}^2)x^2\right)x, \\ &= -\kappa(x, \hat{\theta})x, \end{aligned} \quad (18)$$

where $\epsilon_{\hat{\theta}} > 0$. It is obvious that $\kappa(x, \hat{\theta}) > 0$ by definition. Substituting (18) into (16) yields $\dot{\hat{\varrho}} = \gamma_\varrho \mathrm{sgn}(\ell_b)\kappa x^2$. When $b(t) > 0$, due to Assumption 2, there exists a constant ℓ_b

such that $0 < \ell_b \leq b(t)$, $\Delta_b > 0$, $\dot{\hat{\varrho}} \geq 0$, for all $t \geq 0$, which means that any initialization such that $\hat{\varrho}(0) > 0$ guarantees $\hat{\varrho}(t) > 0$, and therefore $\Delta_b \hat{\varrho} \bar{u} x = -\Delta_b \hat{\varrho} \kappa x^2 \leq 0$, for all $t \geq 0$. When $b(t) < 0$, similarly, there exists ℓ_b such that $b(t) \leq \ell_b < 0$, $\Delta_b < 0$, $\dot{\hat{\varrho}} \leq 0$, for all $t \geq 0$. Then selecting $\hat{\varrho}(0) < 0$ guarantees $\hat{\varrho}(t) < 0$ and $\Delta_b \hat{\varrho} \bar{u} x \leq 0$. Recalling (17), (18), and noting that $\Delta_b \hat{\varrho} \bar{u} x \leq 0$ yields $\dot{V} \leq -kx^2 - \left(\frac{\epsilon_{\hat{\theta}}}{2}\hat{\theta}^2 x^4 + \frac{1}{2\epsilon_{\hat{\theta}}} x - \hat{\theta} x^3\right) - \left(\frac{\epsilon_{\Delta_\theta}\delta_{\Delta_\theta}}{2} x^4 + \frac{\delta_{\Delta_\theta}}{2\epsilon_{\Delta_\theta}} x^2 + \Delta_\theta x^3\right) \leq -kx^2 \leq 0$. With the same stability argument as before, the boundedness of the system trajectories and the convergence of x to zero follows.

Remark 4. From this example we see the flexibility of the *congelation of variables* method: the congealed parameter $\ell_{(\cdot)}$ can be selected according to the specific usage. It can be a nominal value for robust design, or an "extreme" value to create sign-definiteness, as long as the resulting perturbation $\Delta_{(\cdot)}$ is also considered consistently. One can even make $\ell_{(\cdot)}$ a time-varying parameter subject to some of the assumptions used in the literature (*e.g.* $\dot{\theta} \in \mathcal{L}_1$, $\dot{\theta} \in \mathcal{L}_\infty$), and use the *congelation of variables* method to relax these assumptions. This is the reason why the proposed method is named "congelation"[2] not "freeze". ⋄

Remark 5. (Passivity interpretation). Similarly to the effect of the selection of ℓ_θ in Remark 3, the selection of ℓ_b makes $\hat{\varrho} - \frac{1}{\ell_b}$ a passifying input/output signal (see Fig. 1). In addition, note that the overall system is passive from $-\Delta_b \hat{\varrho} \kappa x$ to x and our selection of ℓ_b always guarantees that $-\Delta_b \hat{\varrho} \kappa$ is negative, which makes use of the fact that a negative feedback control law possesses an arbitrarily large gain margin in a passive system. ⋄

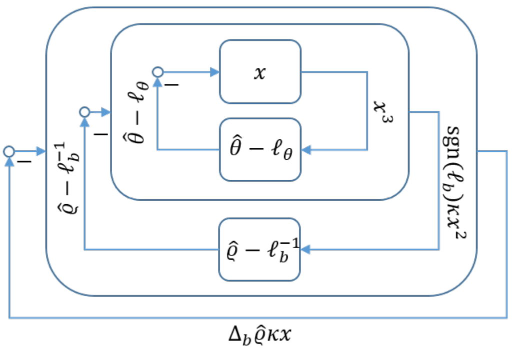

Fig. 1. Schematic representation of system (14), (5) and (16) as the interconnection of passive subsystems.

3. STATE-FEEDBACK DESIGN FOR UNMATCHED PARAMETERS

In the previous simple examples the unknown parameter $\theta(t)$ enters the system dynamics from the same integrator from which the input u enters, that is, the so-called *matching condition* holds. For a more general class of systems in which the unknown parameters are separated from the input by integrators, the *adaptive backstepping* design of Krstic et al. (1995) is needed. Consider an n-dimensional nonlinear system in the so-called *parametric strict-feedback* form

$$\begin{aligned} \dot{x}_1 &= \phi_1^\top(x_1)\theta(t) + x_2, \\ &\vdots \\ \dot{x}_i &= \phi_i^\top(\underline{x_i})\theta(t) + x_{i+1}, \\ &\vdots \\ \dot{x}_n &= \phi_n^\top(x)\theta(t) + b(t)u, \end{aligned} \tag{19}$$

where $i = 2, \ldots, n-1$, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}$ is the input, $\theta(t) \in \mathbb{R}^q$ is the vector of unknown parameters satisfying Assumption 1, and $b(t) \in \mathbb{R}$ is an unknown parameter satisfying Assumption 1 and Assumption 2. The regressors $\phi_i : \mathbb{R}^i \to \mathbb{R}^q$, $i = 1, \ldots, n$, are smooth mappings, and satisfy $\phi_i(0) = 0$.

Remark 6. The condition $\phi_i(0) = 0$ implies that $\phi_i^\top(0)\theta(t) = 0$, which allows zero control effort at $x = 0$. One can easily see that if $\phi_i(0) \neq 0$, $\phi_i^\top(0)\theta(t)$ becomes an unknown time-varying disturbance, yielding a disturbance rejection/attenuation problem not discussed here. By the smoothness of $\phi(\cdot)$ one can express the regressors as $\phi_i(\underline{x_i}) = \bar{\Phi}_i(\underline{x_i})\underline{x_i}$, where $\bar{\Phi}_i$ are smooth mappings, due to Hadamard's lemma (see Nestruev (2006)). ⋄

We directly give the results below and omit the step-by-step procedures[3]. For each step i, $i = 1, \ldots, n$, define the error variables

$$z_0 = 0, \tag{20}$$

$$z_i = x_i - \alpha_{i-1}, \tag{21}$$

the new regressor vectors

$$w_i(\underline{x_i}, \hat{\theta}) = \phi_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \phi_j, \tag{22}$$

the tuning functions

$$\tau_i(\underline{x_i}, \hat{\theta}) = \tau_{i-1} + w_i z_i = \sum_{j=1}^{i} w_i z_i, \tag{23}$$

and the virtual control laws

$$\alpha_0 = 0, \tag{24}$$

$$\begin{aligned} \alpha_i(\underline{x_i}, \hat{\theta}) = &- z_{i-1} - (c_i + \zeta_i) z_i - w_i^\top \hat{\theta} \\ &+ \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} x_{j+1} + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \Gamma_\theta \tau_i \\ &+ \sum_{j=2}^{i-1} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma_\theta w_i z_j, \ i = 1, \ldots, n-1, \end{aligned} \tag{25}$$

$$\alpha_n = \hat{\varrho} \bar{\alpha}_n = -\hat{\varrho} \kappa(x, \hat{\theta}) z_n, \tag{26}$$

where $c_i > 0$ are constant feedback gains, $\zeta_i(\underline{x_i}, \theta)$ are nonlinear feedback gains to be defined, $\Gamma_\theta = \Gamma_\theta^\top \succ 0$ is the adaptation gain, $\kappa(x, \hat{\theta})$ is a positive nonlinear feedback gain to be defined, and similar to the one in Section 2.2. To proceed with the analysis, select the control law and the parameter update laws as

[2] The word "congelation" is polysemous: it means both "coagulation" and "freeze/solidification".

[3] The classical procedures of *adaptive backstepping*, on which the following procedures are based, can be found in Chapter 4 of Krstic et al. (1995).

$$u = \alpha_n, \tag{27}$$
$$\dot{\hat{\theta}} = \Gamma_\theta \tau_n, \tag{28}$$
$$\dot{\hat{\varrho}} = -\gamma_\varrho \mathrm{sgn}(\ell_b)\bar{\alpha}_n z_n, \tag{29}$$

and consider the Lyapunov function candidate $V(z,\hat{\theta},\hat{\varrho}) = \frac{1}{2}|z|^2 + \frac{1}{2}|\ell_\theta - \hat{\theta}|^2_{\Gamma^{-1}} + \frac{|\ell_b|}{2\gamma_\varrho}|\frac{1}{\ell_b} - \hat{\varrho}|^2$, in which $z = [z_1, \ldots, z_n]^\top$. Taking the time-derivative of V yields

$$\begin{aligned}\dot{V} = &-\sum_{i=1}^{n}(c_i+\zeta_i)z_i + z_n\bar{\alpha}_n + \Delta + z_n\psi \\ &+ (\ell_\theta - \hat{\theta})^\top\left(\sum_{i=1}^{n-1} w_i z_i - \Gamma_\theta^{-1}\dot{\hat{\theta}}\right) \\ &+ \ell_b\left(\frac{1}{\ell_b} - \hat{\varrho}\right)\left(\bar{\alpha}_n z_n - \frac{\dot{\hat{\varrho}}}{\gamma_\varrho}\right),\end{aligned} \tag{30}$$

where $\Delta = \sum_{i=1}^{n-1} z_i w_i^\top \Delta_\theta + \Delta_b \hat{\varrho}\bar{\alpha}_n z_n$ and $\psi = z_{n-1} + w_n^\top\hat{\theta} - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}x_{j+1} - \frac{\partial\alpha_{n-1}}{\partial\hat{\theta}}\Gamma_\theta\tau_n - \sum_{j=2}^{n-1}\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\Gamma_\theta w_n z_j$.

Remark 7. Recalling Remark 6 and implementing (20) to (26) recursively, it is not hard to see that $z_i(\underline{x_i},\hat{\theta})$, $w_i(\underline{x_i},\hat{\theta})$, $\tau_i(\underline{x_i},\hat{\theta})$, $\alpha_i(\underline{x_i},\hat{\theta})$ are smooth and $z_i(0,\hat{\theta}) = 0$, $w_i(0,\hat{\theta}) = 0$, $\tau_i(0,\hat{\theta}) = 0$, $\alpha_i(0,\hat{\theta}) = 0$. Also note that the $\hat{\theta}$-dependent change of coordinates between $\underline{z_i}$ and $\underline{x_i}$ is smooth, invertible, and $\underline{x_i} = 0 \Leftrightarrow \underline{z_i} = 0$, thus we can directly express w_i as $w_i = \bar{W}_i(\underline{x_i},\hat{\theta})\underline{z_i}$ with W_i smooth and, similarly, ψ as $\psi = \bar{\psi}^\top(x,\hat{\theta})z$ with $\bar{\psi}$ smooth. ◇

The last two lines of (30) are eliminated by the parameter update laws (28) and (29), and the non-positivity of $\Delta_b\hat{\varrho}\bar{\alpha}_n z_n$ can be established in the same way as in Section 2.2, thanks to the form[4] of $\bar{\alpha}_n$. The rest of the problem is to determine the nonlinear damping gains $\zeta_i(\underline{x_i},\hat{\theta})$ and $\kappa(x,\hat{\theta})$ to dominate the Δ_θ-terms.

Proposition 1. Consider system (19) and the control law (27) with the nonlinear damping gains

$$\zeta_i(\underline{x_i},\hat{\theta}) = \frac{1}{2}\left((n-i+1)\frac{\delta_{\Delta_\theta}}{\epsilon_{\Delta_\theta}} + \epsilon_{\Delta_\theta}\delta_{\Delta_\theta}|\bar{W}_i|^2_{\mathrm{F}} + \frac{1}{\epsilon_{\bar{\psi}}}\right), \tag{31}$$

$$\kappa(x,\hat{\theta}) = c_n + \zeta_n + \frac{1}{2}\epsilon_{\bar{\psi}}|\bar{\psi}|^2, \tag{32}$$

where $c_n > 0$, $\epsilon_{(\cdot)} > 0$, and the parameter update laws (28) and (29), with $\mathrm{sgn}(\hat{\varrho}(0)) = \mathrm{sgn}(b)$. Then all closed-loop signals are bounded and $\lim_{t\to+\infty} x(t) = 0$. ◇

4. SIMULATIONS

Consider the benchmark nonlinear system in *parametric strict-feedback* form described by

$$\begin{aligned}\dot{x}_1 &= x_2 + \theta_1(t)x_1^2, \\ \dot{x}_2 &= \theta_2(t)x_1^2 + \theta_3(t)x_2^2 + b(t)u,\end{aligned} \tag{33}$$

where $\theta(t) = [\theta_1,\theta_2,\theta_3]^\top$ and $b(t)$ are given by

$$\theta(t) = \theta_{\mathrm{const}} + R_\theta\frac{w_1z_1 + w_2z_2}{|w_1z_1 + w_2z_2|}(t), \tag{34}$$

$$b(t) = b_{\mathrm{const}} + R_b\frac{uz_2}{|uz_2|}(t), \tag{35}$$

with $\theta_{\mathrm{const}} = [1,1,1]^\top$, $b_{\mathrm{const}} = 1$, $R_\theta = 1.4$, $R_b = 0.5$, and z_i, w_i given by (21), (22), respectively. These parameters are intentionally designed to destabilize the system, as they cause non-negative terms in the time derivative of the Lyapunov function candidate along the system trajectories. Now consider three controllers: Controller 1, the classical backstepping controller proposed in Chapter 4 of Krstic et al. (1995) (essentially (25) with $i = 2$, $\zeta_2 = 0$, and the corresponding parameter update laws); Controller 2, a modified version of Controller 1 with *projection operation* in the parameter update laws; and Controller 3, as proposed in this paper and described by (27)–(29), (31), and (32). Set the common parameters as $c_1 = 1$, $c_2 = 1$, $\Gamma_\theta = I$, $\gamma_\varrho = 1$. For Controller 2, let the *projection operation* confine the parameter estimates within two balls such that $|\hat{\theta}(t)| \le 10$ and $|\hat{\varrho}(t)| \le 2$, for all $t \ge 0$. For Controller 3, set the discounted radius of parameter variations[5] $\delta_{\Delta_\theta} = 0.3R_\theta$ and all balancing constants $\epsilon_{(\cdot)} = 1$. The simulations are performed with all states/parameter estimates initialized at 0: the results are shown in Fig. 2. The "Baseline" results are the system

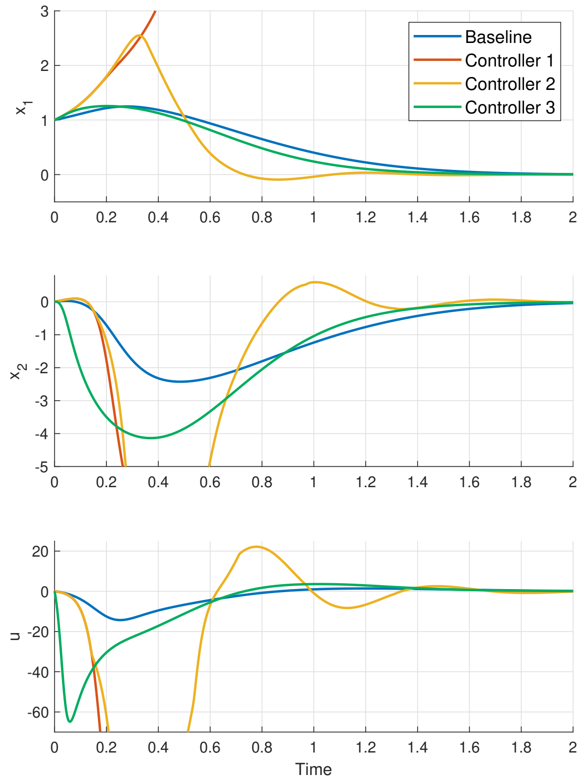

Fig. 2. Time history of the system state and the control signal driven by different controllers.

[4] This form of $\bar{\alpha}_n$ is inspired by Li and Krstic (1997), which also designs a control law with a nonlinear negative feedback gain, albeit to achieve inverse optimality.

[5] In practice, we do not directly treat R_θ as the radius of parameter variations to avoid an over-conservative controller design, see Remark 7 of Chen and Astolfi (2018b).

response driven by Controller 1 when the parameters are constant, *i.e.* $\theta(t) = \theta_{\text{const}}$ and $b(t) = b_{\text{const}}$. It can be seen that under the given parameter variations the trajectories resulting from the use of Controller 1 diverge (finite escape-time can be observed in the simulation); the trajectories resulting from Controller 2 are bounded yet significant overshoots and oscillations can be observed, indicating a degradation of transient performance; while Controller 3 restores the performance of the Baseline case.

5. CONCLUSIONS AND FUTURE WORK

In this paper we have discussed how to apply the *congelation of variables* method to a class of nonlinear systems with time-varying parameters. Motivating examples, in which the time-varying parameters are in the feedback path and in the input path, are discussed, and interpretations from a passivity perspective are given. Then, design procedures for nonlinear systems in *parametric strict-feedback* form are proposed. These guarantee the convergence of the state to the origin and the boundedness of the closed-loop system trajectories. Simulations show that the proposed controller is robust under the parameter variations, whereas a classical controller may not prevent the occurrence of finite escape-time.

Although the method proposed requires state-feedback, it could be a crucial stepping stone to solve the remaining restriction in Chen and Astolfi (2019a) and Chen and Astolfi (2019b), that is, to allow a time-varying "high-frequency gain" in the adaptive output-feedback regulation problem, a problem which is worthy of further investigation.

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