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Interval Full-Order Switched Positive Observers for Uncertain Switched Positive Linear Systems

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Abstract: In this paper, we study observation problems for uncertain switched positive linear systems. Firstly, interval full-order switched positive observers for both continuous-time and discrete-time uncertain switched positive linear systems are studied. Further, it is shown that the obtained results are reduced to the results of non-switched positive linear systems. Finally, two illustrative numerical examples of interval full-order observes for both continuous-time and discrete-time uncertain switched positive linear systems are investigated.

Keywords: Switched linear systems, Positive linear systems, Uncertain linear systems, Interval switched positive observers, Full-order observes, Arbitrary switching

1. INTRODUCTION

Switched positive systems consist of a family of positive systems and a rule that orchestrates the switching among them. Stability and stabilization problems for switched positive systems have been extensively studied (e.g., Blanchini et al. (2012), Fornasini and Valcher (2010)). Further, some interesting mathematical models such as HIV virus therapy models, epidemiological models, thermal models and fluid models (e.g., Blanchini et al. (2015), Blanchini et al. (2012), Zorzan (2014)) were introduced as switched positive systems.

On the other hand, it is important to study the positive observation problem which gives the estimates of nonnegative states for positive systems. In such a situation, positive linear observers have been studied by Dautrebande and Basin (1999), Shu et al. (2008), Rami and Tadeo (2006) and Rami et al. (2011) and so on. As an interesting study among them, observer designs for both non-uncertain and uncertain positive linear systems were given by Rami et al. (2011). The obtained results give the conditions that linear programing problem (LP problem) is feasible in some variables and robust interval positive observers are designed. After that, full-order and reduced-order observer designs for both continuous-time and discrete-time switched positive linear systems under arbitrary switching have been studied by the present authors (Otsuka and Kakehi (2019), Otsuka and Kakehi (2019)) .

The objective of this paper is to study the interval full-order switched positive observer designs for both continuous-time and discrete-time uncertain switched positive linear systems under arbitrary switching. Firstly, interval full-order switched positive observers for continuous-time uncertain switched positive linear systems are studied. After that discrete-time case for uncertain switched positive linear systems are briefly studied. Finally, two illustrative numerical examples of interval full-order observes are investigated.

2. PRELIMINARIES

In this section, we introduce some notations and fundamental results which are used throughout this study.

· $\mathbf{1}_n$ denotes the n-dimensional vector whose elements are all equal to 1.

· I_n denotes the $n \times n$ identity matrix.

· $M > 0$ ($M \geq 0$) means that its components m_{ij} are all positive (nonnegative) for all i, j. In this case, M is said to be a Positive (Nonnegative) matrix.

· $M < 0$ ($M \leq 0$) means that $m_{ij} < 0$ ($m_{ij} \leq 0$) for all i, j. In this case, M is said to be a Negative (Nonpositive) matrix.

· $\mathbf{R}^n_+$ denotes the nonnegative orthant of the n-dimensional real vector space $\mathbf{R}^n$.

· $\mathbf{R}^{n\times m}_+$ denotes a set of $n \times m$ nonnegative matrix.

· M^{T} denotes the transpose of a matrix M.

· A matrix $M \in \mathbf{R}^{n\times n}$ is said to be Metzler if its off-diagonal elements are all nonnegative, i.e., $m_{ij} \geq 0$ ($i \neq j$).

· A matrix $M \in \mathbf{R}^{n\times n}$ is said to be Hurwitz if all eigenvalues of M are in the open left-region of the complex plane $\mathbf{C}$.

· A matrix $M \in \mathbf{R}^{n\times n}$ is said to be Schur if all eigenvalues of M are in the open unit-disc of the complex plane $\mathbf{C}$.

· diag($\lambda_1,\cdots,\lambda_n$) denotes the diagonal matrix whose diagonal elements are $\lambda_1,\cdots,\lambda_n$.

· $\|x\|$ denotes the Euclidean norm of a vector x.

Consider the following continuous-time switched linear system:

$$\Sigma_\sigma : \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t),\ x(0) = x_0, \quad (1)$$

where $A_i \in \mathbf{R}^{n\times n}$, $B_i \in \mathbf{R}^{n\times m}$, $x(t) \in \mathbf{R}^n$ is the state, $u(t) \in \mathbf{R}^m$ is the input and $\sigma(t) : [0, \infty) \to \{1, \cdots, N\}$ is the switching signal which is piecewise constant taking value from the index set $\{1, \cdots, N\}$ of subsystems.

The following lemma gives an useful result on the stability of general continuous-time switched linear systems.

Lemma 1. (Sun and Ge (2011)) The following statements are equivalent.

(i) The switched system (1) without input is asymptotically stable, i.e.,

$$\lim_{t\to\infty}\|x(t)\| = 0$$

under arbitrary switching.

(ii) The switched system (1) without input is exponentially stable, i.e., there exist $M > 0$ and $\alpha > 0$ such that

$$\|x(t)\| \le Me^{-\alpha t}\|x(0)\| \quad (\forall t \ge 0)$$

under arbitrary switching.

(iii) A_i $(i = 1, \cdots, N)$ share a common Lyapunov function.

The following Lemma can be used to prove the main theorem.

Lemma 2. If A_i $(i = 1, \cdots, N)$ share a common Lyapunov function and $u(t)$ is bounded for the switched system (1), then the trajectory $x(t)$ of (1) is bounded under arbitrary switching.

Proof. The proof follows from Lemma 1 that $x(t)$ is exponentially stable under arbitrary switching. Hence, since $u(t)$ is bounded, it can be easily obtained that $x(t)$ is also bounded.

Now, we give the definition of switched positive systems and fundamental results.

Definition 3. Switched system (1) is said to be positive if all subsystems are positive, that is, for each i, every nonnegative initial state $x_0 \in \mathbf{R}^n_+$ and every nonnegative input $u(t) \in \mathbf{R}^m_+$, the corresponding trajectory $x(t) \in \mathbf{R}^n_+$ for all $t \ge 0$.

The following Lemma follows directly from Farina and Rinaldi (2000).

Lemma 4. Switched system (1) is positive if and only if A_i $(i = 1, \cdots, N)$ are Metzler and $B_i \ge 0$ $(i = 1, \cdots, N)$.

The following lemma gives sufficient conditions for a common Lyapunov function on switched positive systems to exist.

Lemma 5. (Fornasini and Valcher (2010)) Consider the switched positive system (1) without input. If there exists a vector $\lambda > 0$ such that

$$A_i^{\mathrm{T}}\lambda < 0 \ (i = 1, \cdots, N),$$

then A_i $(i = 1, \cdots, N)$ share a common (linear copositive) Lyapunov function. In fact, if we define $V(x) := \lambda^{\mathrm{T}}x$ for the state $x(t)$ of (1), then $V(x)$ gives a common (linear copositive) Lyapunov function.

Lemma 6. (Rami et al. (2011))

(i) A matrix M is Metzler if and only if $e^{tM} \ge 0$ for all $t \ge 0$.

(ii) If two matrices M and N are Metzler satisfying $M \le N$, then $(0 \le)\ e^{tM} \le e^{tN}$ for all $t \ge 0$.

3. CONTINUOUS-TIME INTERVAL FULL-ORDER SWITCHED POSITIVE OBSERVERS

Consider the continuous-time switched linear systems:

$$\Sigma^c_\sigma : \begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t), \quad x(0) = x_0, \\ y(t) = C_{\sigma(t)}x(t), \end{cases}$$

where $A_{\sigma(t)} \in \mathbf{R}^{n\times n}$, $C_{\sigma(t)} \in \mathbf{R}^{p\times n}$, $x(t) \in \mathbf{R}^n$ is the state, $y(t) \in \mathbf{R}^p$ is the output and $\sigma(t) : [0, \infty) \to \{1, \cdots, N\}$ is the switching signal which is piecewise constant taking value from the index set $\{1, \cdots, N\}$ of subsystems.

Throughout this section, we assume that coefficient matrices A_i, C_i $(i = 1, \cdots, N)$ and the initial state x_0 are uncertain in the sense that those are between known interval matrices and known vectors, respectively, that is, we give the following assumption.

Assumption 7.

(i) $\underline{A}_i \le A_i \le \overline{A}_i$ $(i = 1, \cdots, N)$,

(ii) $\underline{A}_i$ $(i = 1, \cdots, N)$ are Metzler,

(iii) $\underline{C}_i \le C_i \le \overline{C}_i$ $(i = 1, \cdots, N)$,

(iv) $0 \le \underline{x}_0 \le x_0 \le \overline{x}_0$,

where $\underline{A}_i, \underline{C}_i$ $(1, \cdots, N)$ and $\underline{x}_0$ are lower known matrices and a vector, and $\overline{A}_i, \overline{C}_i$ $(1, \cdots, N)$ and $\overline{x}_0$ are upper known ones.

In order to consider switched positive systems, it is necessary that $\underline{A}_i$ $(i = 1, \cdots, N)$ are Metzler and $\underline{x}_0$ is nonnegative. However, we remark that $\underline{C}_i$ $(i = 1, \cdots, N)$ are not always nonnegative in this study.

Then, the following theorem can be obtained.

Theorem 8. Assume that the state trajectory $x(t)$ of the switched positive system Σ^c_σ is bounded. If there exist nonnegative observer gain L_i $(i = 1, \cdots, N) \in \mathbf{R}^{n\times p}_+$ such that

(i) $(\underline{A}_i - L_i\overline{C}_i)$ $(i = 1, \cdots, N)$ are Metzler,

(ii) $L_i\underline{C}_i \ge 0$ $(i = 1, \cdots, N)$,

(iii) $(\overline{A}_i - L_i\underline{C}_i)$ $(i = 1, \cdots, N)$ share a common Lyapunov function,

then there exists an interval switched positive observer $(\hat{\Sigma}^{c\ell}_\sigma, \hat{\Sigma}^{cu}_\sigma)$ for Σ^c_σ with bounded error $\xi(t) (:= \hat{x}^u(t) - \hat{x}^\ell(t))$ satisfying $0 \le \hat{x}^\ell(t) \le x(t) \le \hat{x}^u(t)$ under arbitrary switching, where

$$\hat{\Sigma}^{c\ell}_\sigma : \dot{\hat{x}}^\ell(t) = (\underline{A}_\sigma - L_\sigma\overline{C}_\sigma)\hat{x}^\ell(t) + L_\sigma y(t),\ \hat{x}^\ell(0) = \underline{x}_0 \ \text{ and}$$

$$\hat{\Sigma}^{cu}_\sigma : \dot{\hat{x}}^u(t) = (\overline{A}_\sigma - L_\sigma\underline{C}_\sigma)\hat{x}^u(t) + L_\sigma y(t),\ \hat{x}^u(0) = \overline{x}_0.$$

Proof. Suppose that there exist nonnegative observer gain $L_i \in \mathbf{R}^{n\times p}_+$ $(i = 1, \cdots, N)$ such that the conditions (i)-(iii) are satisfied.

At first, we consider the following uncertain switched system given by

$$\underline{\hat{\Sigma}}^c_\sigma : \ \dot{\underline{\hat{x}}}(t) = (A_\sigma - L_\sigma C_\sigma)\underline{\hat{x}}(t) + L_\sigma y(t),$$
$$\underline{\hat{x}}(0) = \hat{x}^\ell(0) = \underline{x}_0 \ (\ge 0).$$

Then, since $\underline{A}_i - L_i\overline{C}_i \le A_i - L_iC_i$ $(i = 1, \cdots, N)$, it follows from (i), (ii), Lemma 4 and Lemma 6 that

$(0 \le)\ \hat{x}^\ell(t) \le \underline{\hat{x}}(t)$ $(t \ge 0)$. Further, since

$$\frac{d}{dt}(x(t) - \underline{\hat{x}}(t)) = (A_\sigma - L_\sigma C_\sigma)(x(t) - \underline{\hat{x}}(t)),$$

$x(0) - \underline{\hat{x}}(0) = x_0 - \underline{x}_0 \ge 0$ and $A_i - L_iC_i$ are Metzler, we have $\underline{\hat{x}}(t) \le x(t)$ $(t \ge 0)$. Thus, we have

$$(0 \le)\ \hat{x}^\ell(t) \le \underline{\hat{x}}(t) \le x(t) \ (t \ge 0). \quad (2)$$

Similarly, if we consider the following uncertain switched system given by

$$\hat{\overline{\Sigma}}^c_\sigma : \ \dot{\hat{\overline{x}}}(t) = (A_\sigma - L_\sigma C_\sigma)\hat{\overline{x}}(t) + L_\sigma y(t), \quad \hat{\overline{x}}(0) = \hat{x}^u(0) = \overline{x}_0 \ (\geq 0),$$

then we have

$$x(t) \leq \hat{\overline{x}}(t) \leq \hat{x}^u(t) \ (t \geq 0). \tag{3}$$

Thus, it follows from (2) and (3) that

$$(0 \leq) \ \hat{x}^\ell(t) \leq x(t) \leq \hat{x}^u(t) \ (t \geq 0). \tag{4}$$

Finally, we prove that $(\hat{x}^u(t) - \hat{x}^\ell(t))$ is bounded for all $t \geq 0$. If we define $\xi(t) := \hat{x}^u(t) - \hat{x}^\ell(t)$, then, we have

$$\dot{\xi}(t) = H_\sigma \xi(t) + J_\sigma \hat{x}^\ell(t), \tag{5}$$

where $H_\sigma := (\overline{A}_\sigma - L_\sigma \underline{C}_\sigma)$ and $J_\sigma := (\overline{A}_\sigma - \underline{A}_\sigma + L_\sigma(\overline{C}_\sigma - \underline{C}_\sigma))$. Since the state trajectory $x(t)$ of the switched positive system Σ^c_σ is bounded from the assumption, it follows from (4) that the state trajectory $\hat{x}^\ell(t)$ is also bounded. Then, since coefficient matrices $H_i := (\overline{A}_i - L_i\underline{C}_i)$ $(i = 1, \cdots, N)$ in (5) share a common Lyapunov function and $\hat{x}^\ell(t)$ is bounded, it follows from Lemma 2 that $\xi(t)$ is bounded under arbitrary switching. This completes the proof.

As we can see from the proof of Theorem 8, the following inequalities are necessary to get interval positive observers.

$$\underline{A}_i - L_i\overline{C}_i \leq A_i - L_iC_i \leq \overline{A}_i - L_i\underline{C}_i \ (i = 1, \cdots, N)$$

which implies essentially that observer gain L_i are nonnegative. Similarly, in the case of non-switched uncertain positive linear systems, it is assumed that observer gain is nonnegative to get interval positive observers (e.g., Rami et al. (2011)). Though it is not easy to find a common Lyapunov function of the condition (iii) in Theorem 8, the following theorem gives useful sufficient conditions in the sense to compute Λ and Z_i $(i = 1, \cdots, N)$ as LP problem.

Theorem 9. Assume that the state trajectory $x(t)$ of the switched positive system Σ^c_σ is bounded. If there exist a $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)(\lambda_i > 0)$ and $Z_i \in \mathbf{R}^{p\times n}_+$ $(i = 1, \cdots, N)$ such that

$$\begin{aligned} &\text{(i)} \ (\underline{A}_i^{\mathrm{T}}\Lambda - \overline{C}_i^{\mathrm{T}} Z_i) \ (i = 1, \cdots, N) \text{ are Metzler,} \\ &\text{(ii)} \ \underline{C}_i^{\mathrm{T}} Z_i \geq 0 \ (i = 1, \cdots, N), \\ &\text{(iii)} \ (\overline{A}_i^{\mathrm{T}}\Lambda - \underline{C}_i^{\mathrm{T}} Z_i)\mathbf{1}_n < 0 \ (i = 1, \cdots, N), \end{aligned} \tag{6}$$

then there exists an interval switched positive observer $(\hat{\Sigma}^{c\ell}_\sigma, \hat{\Sigma}^{cu}_\sigma)$ for Σ^c_σ with bounded error $\xi(t)(:= \hat{x}^u(t) - \hat{x}^l(t))$ satisfying $0 \leq \hat{x}^l(t) \leq x(t) \leq \hat{x}^u(t)$ under arbitrary switching, where $(\hat{\Sigma}^{c\ell}_\sigma, \hat{\Sigma}^{cu}_\sigma)$ is the same as Theorem 8. In this case, observer gain L_i can be given by $L_i := \Lambda^{-1}Z_i^{\mathrm{T}}$ (≥ 0) $(i = 1, \cdots, N)$.

Proof. At first we remark that Λ is nonsingular matrix. Define the following observer gain matrices

$$L_i := \Lambda^{-1}Z_i^{\mathrm{T}} \ (\geq 0) \ (i = 1, \cdots, N). \tag{7}$$

Then, it follows from (i) and (7) that

$$\begin{aligned} \underline{A}_i^{\mathrm{T}}\Lambda - \overline{C}_i^{\mathrm{T}} Z_i &= \underline{A}_i^{\mathrm{T}}\Lambda - \overline{C}_i^{\mathrm{T}} L_i^{\mathrm{T}}\Lambda \\ &= (\underline{A}_i^{\mathrm{T}} - \overline{C}_i^{\mathrm{T}} L_i^{\mathrm{T}})\Lambda \\ &= (\underline{A}_i - L_i\overline{C}_i)^{\mathrm{T}}\Lambda \end{aligned}$$

is Metzler. Since Λ is diagonal, $\underline{A}_i - L_i\overline{C}_i$ is Metzler which implies (i) of Theorem 8.

Next, it follows from (ii) and (7) that

$$\underline{C}_i^{\mathrm{T}} Z_i = \underline{C}_i^{\mathrm{T}} L_i^{\mathrm{T}}\Lambda \geq 0.$$

Since Λ is a nonsingular diagonal matrix whose diagonal elements are positive, we have $L_i\underline{C}_i \geq 0$ $(i = 1, \cdots, N)$ which implies (ii) of Theorem 8.

Further, it follows from (iii) that

$$\begin{aligned} (\overline{A}_i^{\mathrm{T}}\Lambda - \underline{C}_i^{\mathrm{T}} Z_i)\mathbf{1}_n &= (\overline{A}_i^{\mathrm{T}}\Lambda - \underline{C}_i^{\mathrm{T}} L_i^{\mathrm{T}}\Lambda)\mathbf{1}_n \\ &= (\overline{A}_i - L_i\underline{C}_i)^{\mathrm{T}}\Lambda\mathbf{1}_n \\ &< 0, \end{aligned} \tag{8}$$

and if we set the $\lambda := \Lambda\mathbf{1}_n = [\lambda_1 \ \cdots \ \lambda_n]^{\mathrm{T}}$ (> 0) in (8), then we have

$$(\overline{A}_i - L_i\underline{C}_i)^{\mathrm{T}}\lambda < 0.$$

Hence, it follows from Lemma 5 that $(\overline{A}_i - L_i\underline{C}_i)$ $(i = 1, \cdots, N)$ share a common (linear copositive) Lyapunov function which implies (iii) of Theorem 8. This completes the proof.

We remark that Theorems 8 and 9 are reduced to the following two corollaries for uncertain non-switched positive linear systems which were studied by Rami et al. (2011).

Corollary 10. Suppose that $\underline{A} \leq A \leq \overline{A}$ and $\underline{C} \leq C \leq \overline{C}$ such that $A := A_1 = \cdots = A_N$ and $C := C_1 = \cdots = C_N$, where $\underline{A}, \overline{A}, \underline{C}$ and $\overline{C}$ are given matrices and assume that the state trajectory $x(t)$ of uncertain non-switched positive linear system is bounded. If there exists a nonnegative observer gain $L \in \mathbf{R}^{n\times p}_+$ such that

(i) $(\underline{A} - L\overline{C})$ is Metzler,

(ii) $L\underline{C} \geq 0$,

(iii) $(\overline{A} - L\underline{C})$ is Hurwitz,

then there exists an interval positive observer $(\hat{\Sigma}^{c\ell}, \hat{\Sigma}^{cu})$ with bounded error $\xi(t)(:= \hat{x}^u(t) - \hat{x}^\ell(t))$ satisfying $0 \leq \hat{x}^\ell(t) \leq x(t) \leq \hat{x}^u(t)$ under arbitrary switching, where

$$\hat{\Sigma}^{c\ell} : \dot{\hat{x}}^\ell(t) = (\underline{A} - L_\sigma\overline{C})\hat{x}^\ell(t) + Ly(t), \ \hat{x}^\ell(0) = \underline{x}_0 \ \text{ and}$$

$$\hat{\Sigma}^{cu} : \dot{\hat{x}}^u(t) = (\overline{A} - L\underline{C})\hat{x}^u(t) + Ly(t), \ \hat{x}^u(0) = \overline{x}_0.$$

Corollary 11. Suppose that $\underline{A} \leq A \leq \overline{A}$ and $\underline{C} \leq C \leq \overline{C}$ such that $A := A_1 = \cdots = A_N$ and $C := C_1 = \cdots = C_N$, where $\underline{A}, \overline{A}, \underline{C}$ and $\overline{C}$ are given matrices and assume that the state trajectory $x(t)$ of uncertain non-switched positive linear system is bounded. If there exist a $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)(\lambda_i > 0)$ and a $Z \in \mathbf{R}^{p\times n}_+$ such that

(i) $(\underline{A}^{\mathrm{T}}\Lambda - \overline{C}^{\mathrm{T}} Z)$ is Metzler,

(ii) $\underline{C}^{\mathrm{T}} Z \geq 0$,

(iii) $(\overline{A}^{\mathrm{T}}\Lambda - \underline{C}^{\mathrm{T}} Z)\mathbf{1}_n < 0$,

then there exists an interval positive observer $(\hat{\Sigma}^{c\ell}, \hat{\Sigma}^{cu})$ with bounded error $\xi(t)(:= \hat{x}^u(t) - \hat{x}^l(t))$ satisfying $0 \leq \hat{x}^l(t) \leq x(t) \leq \hat{x}^u(t)$ under arbitrary switching, where $(\hat{\Sigma}^{c\ell}, \hat{\Sigma}^{cu})$ is the same as Corollary 10. In this case, observer gain L can be given by $L := \Lambda^{-1}Z^{\mathrm{T}}$.

4. DISCRETE-TIME INTERVAL FULL-ORDER SWITCHED POSITIVE OBSERVERS

Consider the discrete-time switched linear systems:

$$\Sigma_\sigma^d : \begin{cases} x(k+1) = A_{\sigma(k)}x(k), \quad x(0) = x_0, \\ y(k) = C_{\sigma(k)}x(k), \end{cases}$$

where $A_{\sigma(k)} \in \mathbf{R}^{n\times n}$, $C_{\sigma(k)} \in \mathbf{R}^{p\times n}$, $x(k) \in \mathbf{R}^n$ is the state, $y(k) \in \mathbf{R}^p$ is the output and $\sigma(k) : [0, 1, 2, \cdots) \to \{1, \cdots, N\}$ is the switching signal which depends on the time k. Further, we also give the Assumption 7 of the previous section.

Then, the following theorems can be obtained.

Theorem 12. Assume that the state trajectory $x(k)$ of the switched positive system Σ_σ^d is bounded. If there exist nonnegative observer gain L_i $(i = 1, \cdots, N) \in \mathbf{R}_+^{n\times p}$ such that

(i) $(\underline{A}_i - L_i\overline{C}_i) \geq 0 \ \ (i = 1, \cdots, N)$,

(ii) $L_i\underline{C}_i \geq 0 \ \ (i = 1, \cdots, N)$,

(iii) $(\overline{A}_i - L_i\underline{C}_i) \ \ (i = 1, \cdots, N)$ share a common Lyapunov function,

then there exists an interval switched positive observer $(\hat{\Sigma}_\sigma^{d\ell}, \hat{\Sigma}_\sigma^{du})$ for Σ_σ^d with bounded error $\xi(k)(:= \hat{x}^u(k) - \hat{x}^\ell(k))$ satisfying $0 \leq \hat{x}^\ell(k) \leq x(k) \leq \hat{x}^u(k)$ under arbitrary switching, where

$$\hat{\Sigma}_\sigma^{d\ell} : \hat{x}^\ell(k+1) = (\underline{A}_\sigma - L_\sigma\overline{C}_\sigma)\hat{x}^\ell(k) + L_\sigma y(k), \hat{x}^\ell(0) = \underline{x}_0 \text{ and}$$

$$\hat{\Sigma}_\sigma^{du} : \hat{x}^u(k+1) = (\overline{A}_\sigma - L_\sigma\underline{C}_\sigma)\hat{x}^u(k) + L_\sigma y(k), \hat{x}^u(0) = \overline{x}_0.$$

Similarly, it is not easy to find a common Lyapunov function of the condition (iii) in Theorem 12. Therefore, the following theorem gives useful sufficient conditions as the same as Theorem 9.

Theorem 13. Assume that the state trajectory $x(k)$ of the switched positive system Σ_σ^d is bounded. If there exist a $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)(\lambda_i > 0)$ and $Z_i \in \mathbf{R}_+^{p\times n}$ $(i = 1, \cdots, N)$ such that

$$\begin{aligned} &\text{(i)} \ \ (\underline{A}_i^{\mathrm{T}}\Lambda - \overline{C}_i^{\mathrm{T}}Z_i) \geq 0 \ (i = 1, \cdots, N), \\ &\text{(ii)} \ \ \underline{C}_i^{\mathrm{T}}Z_i \geq 0 \ (i = 1, \cdots, N), \\ &\text{(iii)} \ \ ((\overline{A}_i - I_n)^{\mathrm{T}}\Lambda - \underline{C}_i^{\mathrm{T}}Z_i)\mathbf{1}_n < 0 \ (i = 1, \cdots, N), \end{aligned} \tag{9}$$

then there exists an interval switched positive observer $(\hat{\Sigma}_\sigma^{d\ell}, \hat{\Sigma}_\sigma^{du})$ for Σ_σ^d with bounded error $\xi(k)(:= \hat{x}^u(k) - \hat{x}^\ell(k))$ satisfying $0 \leq \hat{x}^\ell(k) \leq x(k) \leq \hat{x}^u(k)$ under arbitrary switching, where $(\hat{\Sigma}_\sigma^{d\ell}, \hat{\Sigma}_\sigma^{du})$ is the same as Theorem 12. In this case, observer gain L_i can be given by $L_i := \Lambda^{-1}Z_i^{\mathrm{T}} \ (\geq 0))$ $(i = 1, \cdots, N)$.

Similarly, we remark that Theorems 12 and 13 are reduced to the following two corollaries.

Corollary 14. Suppose that $\underline{A} \leq A \leq \overline{A}$ and $\underline{C} \leq C \leq \overline{C}$ such that $A := A_1 = \cdots = A_N$ and $C := C_1 = \cdots = C_N$, where $\underline{A}, \overline{A}, \underline{C}$ and $\overline{C}$ are given matrices and assume that the state trajectory $x(k)$ of uncertain non-switched positive linear system is bounded. If there exists a nonnegative observer gain $L \in \mathbf{R}_+^{n\times p}$ such that

(i) $(\underline{A} - L\overline{C}) \geq 0$,

(ii) $L\underline{C} \geq 0$,

(iii) $(\overline{A} - L\underline{C})$ is Schur,

then there exists an interval switched positive observer $(\hat{\Sigma}_\sigma^{d\ell}, \hat{\Sigma}_\sigma^{du})$ for Σ_σ^d with bounded error $\xi(k)(:= \hat{x}^u(k) - \hat{x}^\ell(k))$ satisfying $0 \leq \hat{x}^\ell(k) \leq x(k) \leq \hat{x}^u(k)$ under arbitrary switching, where

$$\hat{\Sigma}^{d\ell} : \hat{x}^\ell(k+1) = (\underline{A} - L\overline{C})\hat{x}^\ell(k) + Ly(k), \hat{x}^\ell(0) = \underline{x}_0 \text{ and}$$

$$\hat{\Sigma}^{du} : \hat{x}^u(k+1) = (\overline{A} - L\underline{C})\hat{x}^u(k) + Ly(k), \hat{x}^u(0) = \overline{x}_0.$$

Corollary 15. Suppose that $\underline{A} \leq A \leq \overline{A}$ and $\underline{C} \leq C \leq \overline{C}$ such that $A := A_1 = \cdots = A_N$ and $C := C_1 = \cdots = C_N$, where $\underline{A}, \overline{A}, \underline{C}$ and $\overline{C}$ are given matrices and assume that the state trajectory $x(k)$ of uncertain non-switched positive linear system is bounded. If there exist a $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)(\lambda_i > 0)$ and a $Z \in \mathbf{R}_+^{p\times n}$ such that

(i) $(\underline{A}^{\mathrm{T}}\Lambda - \overline{C}^{\mathrm{T}}Z) \geq 0$,

(ii) $\underline{C}^{\mathrm{T}}Z \geq 0$,

(iii) $((\overline{A} - I_n)^{\mathrm{T}}\Lambda - \underline{C}^{\mathrm{T}}Z)\mathbf{1}_n < 0$,

then there exists an interval switched positive observer $(\hat{\Sigma}_\sigma^{d\ell}, \hat{\Sigma}_\sigma^{du})$ for Σ_σ^d with bounded error $\xi(k)(:= \hat{x}^u(k) - \hat{x}^\ell(k))$ satisfying $0 \leq \hat{x}^\ell(k) \leq x(k) \leq \hat{x}^u(k)$ under arbitrary switching, where $(\hat{\Sigma}_\sigma^{d\ell}, \hat{\Sigma}_\sigma^{du})$ is the same as Corollary 14. In this case, observer gain L can be given by $L := \Lambda^{-1}Z^{\mathrm{T}}$.

5. EXAMPLES

In this section, two numerical examples are given.

5.1 Continuous-Time Full-Order Positive Observer

Consider the continuous-time switched positive system :

$$\Sigma_\sigma^c : \begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t), \ x(0) = x_0 \\ y(t) = C_{\sigma(t)}x(t) \end{cases} \quad (\sigma(t) \in \{1, 2, 3\}),$$

where coefficients matrices and an initial state are as follows.

$$\underline{A}_1 = \begin{bmatrix} -23 & 3 & 3 \\ 7 & -19 & 1 \\ 4 & 1 & -16 \end{bmatrix} \leq A_1 \leq \overline{A}_1 = \begin{bmatrix} -20 & 8 & 8 \\ 11 & -15 & 5 \\ 9 & 4 & -13 \end{bmatrix},$$

$$\underline{C}_1 = [9 \ 6 \ 3] \leq C_1 \leq \overline{C}_1 = [13 \ 10 \ 6],$$

$$\underline{A}_2 = \begin{bmatrix} -19 & 5 & 2 \\ 4 & -22 & 3 \\ 2 & 3 & -24 \end{bmatrix} \leq A_2 \leq \overline{A}_2 = \begin{bmatrix} -16 & 8 & 7 \\ 9 & -19 & 8 \\ 5 & 8 & -19 \end{bmatrix},$$

$$\underline{C}_2 = [5 \ 4 \ 9] \leq C_2 \leq \overline{C}_2 = [8 \ 9 \ 12],$$

$$\underline{A}_3 = \begin{bmatrix} -23 & 5 & 6 \\ 4 & -18 & 1 \\ 1 & 1 & -17 \end{bmatrix} \leq A_3 \leq \overline{A}_3 = \begin{bmatrix} -19 & 8 & 10 \\ 9 & -13 & 4 \\ 4 & 5 & -14 \end{bmatrix},$$

$$\underline{C}_3 = [7 \ 4 \ 8] \leq C_3 \leq \overline{C}_3 = [11 \ 9 \ 12],$$

$$\underline{x}_0 = \begin{bmatrix} 9 \\ 7 \\ 3 \end{bmatrix} \leq x_0 \leq \overline{x}_0 = \begin{bmatrix} 10 \\ 11 \\ 11 \end{bmatrix}.$$

In order to give a computer simulation, we introduce the following matrices and an initial state.

$$A_1 = \begin{bmatrix} -21.6072 & 7.5482 & 6.6053 \\ 7.6874 & -18.1763 & 1.5458 \\ 7.0082 & 3.5477 & -13.4464 \end{bmatrix},$$
$$C_1 = [12.8043\ 8.3339\ 4.3823],$$

$$A_2 = \begin{bmatrix} -18.0734 & 6.4602 & 3.8354 \\ 8.4810 & -19.5023 & 5.8583 \\ 4.9288 & 5.4178 & -20.7157 \end{bmatrix},$$
$$C_2 = [7.8906\ 5.9059\ 9.5219],$$

$$A_3 = \begin{bmatrix} -21.6569 & 6.6752 & 9.9075 \\ 7.1561 & -13.7389 & 2.2076 \\ 1.8863 & 2.9891 & -14.5232 \end{bmatrix},$$
$$C_3 = [9.4848\ 6.3424\ 8.5784],$$

$$x_0 = [9.7400\ 9.6099\ 7.8206]^{\mathrm{T}}.$$

Then, we can obtain the following matrices satisfying the conditions (i)-(iii) in Theorem 9.

$$\Lambda = I_3, Z_1 = [0.1705\ 0.1058\ 0.0481],$$

$$Z_2 = [0.0552\ 0.1215\ 0.0743], Z_3 = [0.0675\ 0.0671\ 0.0289].$$

Thus, it follows from Theorem 9 under assumption that we obtain the interval full-order switched positive observer $(\hat{\Sigma}_\sigma^{c\ell}, \hat{\Sigma}_\sigma^{cu})$ under arbitrary switching, where the observer gains are given by

$$L_1 = \Lambda^{-1} Z_1^{\mathrm{T}} = \begin{bmatrix} 0.1705 \\ 0.1058 \\ 0.0481 \end{bmatrix}, L_2 = \Lambda^{-1} Z_2^{\mathrm{T}} = \begin{bmatrix} 0.0552 \\ 0.1215 \\ 0.0743 \end{bmatrix},$$

$$L_3 = \Lambda^{-1} Z_3^{\mathrm{T}} = \begin{bmatrix} 0.0675 \\ 0.0671 \\ 0.0289 \end{bmatrix}.$$

Fig. 1 and Fig.2 show simulation result by using MATLAB. In fact, we can see that $0 \le \hat{x}^\ell(t) \le x(t) \le \hat{x}^u(t)$ under a random switching.

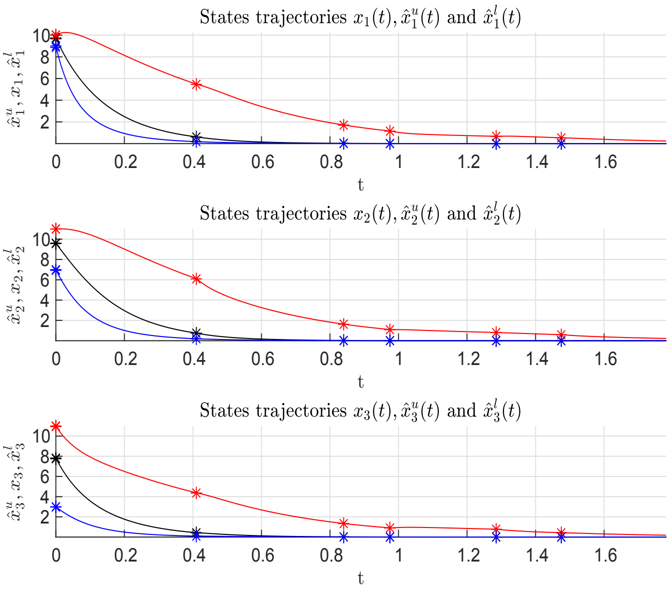

Fig. 1. Trajectories of State $x(t)$ and estimated states $(\hat{x}^\ell(t), \hat{x}^u(t))$

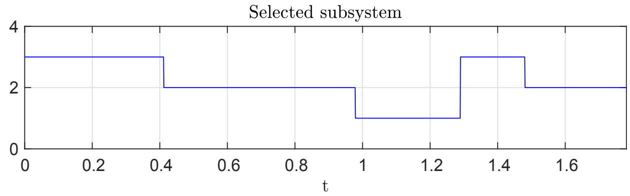

Fig. 2. Selected subsystems

5.2 Discrete-Time Full-Order Positive Observer

Consider the discrete-time switched positive system :

$$\Sigma_\sigma^d : \begin{cases} x(k+1) = A_{\sigma(k)} x(k), \ x(0) = x_0 \\ y(k) = C_{\sigma(k)} x(k) \end{cases} \quad (\sigma(k) \in \{1, 2, 3\}),$$

where coefficients matrices and an initial state are as follows.

$$\underline{A}_1 = \begin{bmatrix} 0.0369 & 0.3776 & 0.2886 \\ 0.0233 & 0.1444 & 0.0382 \\ 0.0427 & 0.0210 & 0.0646 \end{bmatrix} \le A_1$$
$$\le \overline{A}_1 = \begin{bmatrix} 0.3657 & 0.5673 & 0.3449 \\ 0.1472 & 0.2121 & 0.2367 \\ 0.0658 & 0.1570 & 0.3932 \end{bmatrix},$$
$$\underline{C}_1 = [0.9779\ 0.0703\ 0.3518] \le C_1$$
$$\le \overline{C}_1 = [1.0589\ 0.3774\ 0.7118],$$

$$\underline{A}_2 = \begin{bmatrix} 0.1000 & 0.1055 & 0.0340 \\ 0.1139 & 0.0006 & 0.0710 \\ 0.3110 & 0.1760 & 0.1711 \end{bmatrix} \le A_2$$
$$\le \overline{A}_2 = \begin{bmatrix} 0.1610 & 0.1921 & 0.5346 \\ 0.1387 & 0.3815 & 0.1715 \\ 0.5543 & 0.3749 & 0.2820 \end{bmatrix},$$
$$\underline{C}_2 = [0.1232\ 0.0999\ 0.4152] \le C_2$$
$$\le \overline{C}_2 = [0.7451\ 0.9238\ 0.4480],$$

$$\underline{A}_3 = \begin{bmatrix} 0.2905 & 0.2235 & 0.0473 \\ 0.0303 & 0.0564 & 0.0114 \\ 0.2328 & 0.0676 & 0.5282 \end{bmatrix} \le A_3$$
$$\le \overline{A}_3 = \begin{bmatrix} 0.4275 & 0.3783 & 0.2737 \\ 0.1986 & 0.2662 & 0.0133 \\ 0.3432 & 0.1070 & 0.5837 \end{bmatrix},$$
$$\underline{C}_3 = [0.5071\ 0.2773\ 0.3017] \le C_3$$
$$\le \overline{C}_3 = [1.2570\ 0.7175\ 0.5549],$$

$$\underline{x}_0 = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \le x_0 \le \overline{x}_0 = \begin{bmatrix} 4 \\ 13 \\ 4 \end{bmatrix}.$$

In order to give a computer simulation, we introduce the following matrices and an initial state.

$$A_1 = \begin{bmatrix} 0.3341 & 0.4853 & 0.2945 \\ 0.0361 & 0.1861 & 0.0927 \\ 0.0522 & 0.0964 & 0.3746 \end{bmatrix}, C_1 = [0.9875\ 0.2174\ 0.6110],$$

$$A_2 = \begin{bmatrix} 0.1409 & 0.1169 & 0.4317 \\ 0.1276 & 0.0371 & 0.1246 \\ 0.3120 & 0.1764 & 0.2191 \end{bmatrix}, C_2 = [0.2559\ 0.6491\ 0.4230],$$

$$A_3=\begin{bmatrix}0.4252 & 0.3498 & 0.1819\\ 0.1053 & 0.2454 & 0.0115\\ 0.2875 & 0.0805 & 0.5818\end{bmatrix}, C_3=[0.7237\ 0.6300\ 0.4633],$$

$$x_0=[3.7329\ 9.1489\ 2.6650]^{\mathrm{T}}.$$

Then, we can obtain the following matrices satisfying the conditions (i)-(iii) in Theorem 13.

$$\Lambda = I_3, Z_1=[0.0204\ 0.0294\ 0.0049],$$

$$Z_2=[0.0076\ 0.0001\ 0.0133], Z_3=[0.0223\ 0.0001\ 0.0230].$$

Thus, it follows from Theorem 13 under assumption that we obtain the interval full-order switched positive observer $(\hat{\Sigma}^{d\ell}_{\sigma}, \hat{\Sigma}^{du}_{\sigma})$ under arbitrary switching, where the observer gains are given by

$$L_1=\Lambda^{-1}Z_1^{\mathrm{T}}=\begin{bmatrix}0.0204\\0.0294\\0.0049\end{bmatrix}, L_2=\Lambda^{-1}Z_2^{\mathrm{T}}=\begin{bmatrix}0.0076\\0.0001\\0.0133\end{bmatrix},$$

$$L_3=\Lambda^{-1}Z_3^{\mathrm{T}}=\begin{bmatrix}0.0223\\0.0001\\0.0230\end{bmatrix}.$$

Fig. 3 and Fig. 4 show simulation result by using MATLAB. In fact, we can see that $0 \le \hat{x}^{\ell}(k) \le x(k) \le \hat{x}^{u}(k)$ under a random switching.

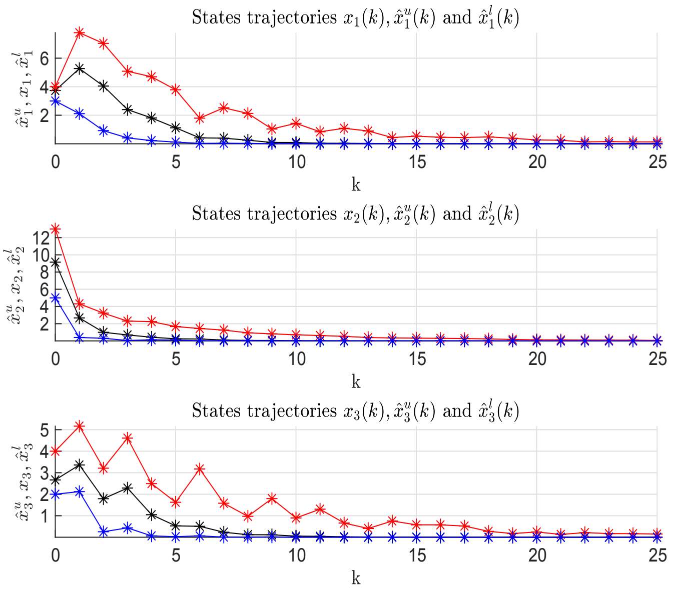

Fig. 3. Trajectories of State $x(t)$ and estimated states $(\hat{x}^{\ell}(t), \hat{x}^{u}(t))$

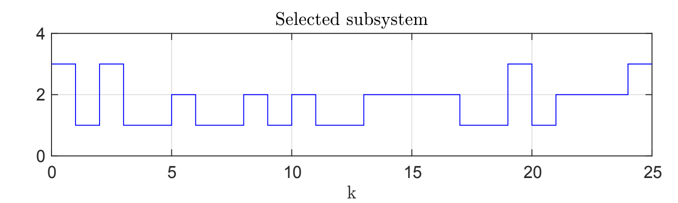

Fig. 4. Selected subsystems

6. CONCLUSION

In this paper, sufficient conditions for interval full-order switched positive observers for both continuous-time and discrete-time uncertain switched positive linear systems to exist were studied, respectively. A part of the provided results are reduced to the results of continuous-time uncertain non-switched positive linear system which was studied by Rami et al. (2011). Finally, two illustrative numerical examples of interval full-order switched positive observes were investigated. As a future study, it is important to obtain an effective algorithm to compute Λ and Z_i $(i=1,\cdots,N)$ of LP problem in the conditions of Theorems 9 and 13.

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