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Distributed Planning in Mean-Field-Type Games ⋆

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Abstract: In this paper we study the problem of designing a collection of terminal payoff of mean-field type by interacting decision-makers to a specified terminal measure. We solve in a semi-explicit way a class of distributed planning in mean-field-type games with different objective functionals. We establish some relationships between the proposed framework, optimal transport theory, and distributed control of dynamical systems.

Keywords: Game theory, optimal control, stochastic control, planning, mean-field-type games.

1. INTRODUCTION

Mean-field-type game theory studies a class of games in which the payoffs and or state dynamics depend not only on the state-action pairs but also the distribution of them. In mean-field-type games, (i) a single decision-maker may have a strong impact on the mean-field terms, (ii) the instantaneous expected payoffs are not necessarily linear with respect to the measure of the state, (iii) the number of decision-makers is not necessarily infinite.

In this paper we study some probability distribution steering problems in a mean-field-type game framework by exploring the role of decentralized strategies. When steering an initial distribution to a terminal one one aims to minimize a certain cost functional of mean-field type. In particular, we are interested on how to design an added terminal cost so as to provide incentives for decision-makers, to move collectively as specified and meet the initial-terminal conditions.

Lions (2009) proposed a planning problem in mean-field games, in which a central planner would like to steer a population to a predetermined final configuration while still allowing individuals to choose their own strategies. This is a distributed planning problem from the perspective of the agents. Since then several studies (see Porretta (2013, 2014); Achdou et al. (2012); Orrieri et al. (2019) and the references therein) have been conducted in the context of mean-field games and optimal transport theory (see Chen et al. (2016, 2018)). However, it has not been addressed in the context of mean-field-type games in which expected objective functional has a non-linear dependence with respect to the measure of the state. The present work focuses also on computation in the latter in a semi-explicit way.

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Our contribution can be summarized as follows.

- We introduce distributed planning problems in mean-field-type games. We show that the existence can be re-stored in wide range of cases if the terminal function can be freely designed.
- We then establish relationships between the proposed scenario and distributed optimal transport theory with a more general cost functional
- Particular equilibrium systems are derived in a semi-explicit way for the case where there are moment constraints at the final time and non-quadratic costs.

The rest of the paper is structured as follows. Section 2 presents a background on the planning problem in a deterministic setting. The framework is extended to distributed planning in subsection 2.2. Section 3 presents the main problem statement and derives an equilibrium system. Solutions to the distributed planning problem under moment constraints are derived in Section 4.

2. BACKGROUND

2.1 Planning Problem

A basic planning problem of a decision-maker 0 is to move from the point $\bar{x}(t_0)$ at time instant t_0 to the point $\bar{x}(t_1)$ at time instant t_1 while minimizing the energy along the path with a careful terminal cost design. The goal of the planner, who is the decision-maker 0, is to choose the terminal objective function $\bar{\alpha}(t_1)\bar{x}^2(t_1)$ and a strategy $\bar{u}_0$ such that

$$\begin{aligned}&\inf_{\bar{u}_0} \bar{\alpha}(t_1)\bar{x}^2(t_1) + \int_{t_0}^{t_1} \bar{u}_0^2(t)dt\\ &\text{subject to}\\ &d\bar{x} = \bar{u}_0 dt,\\ &\bar{x}(t_0) \text{ fixed}\\ &\bar{x}(t_1) \text{ fixed,}\end{aligned}$$

where $\bar{x}(t_0)$ and $\bar{x}(t_1)$ are two non-zero real numbers with the same sign.

To solve this optimal problem control under initial-terminal point, decision-maker 0 uses a backward induction method.

Optimization step Given the coefficient $\bar{\alpha}(t_1)$, Decision-maker 0 first solves the optimal control problem given by

$$\begin{array}{l} L_0 := \bar{\alpha}(t_1)\bar{x}^2(t_1) + \int_{t_0}^{t_1} \bar{u}_0^2(t)dt \\ \inf_{\bar{u}_0} L_0 \\ \text{subject to} \quad \dot{\bar{x}} = \bar{u}_0, \ \bar{x}(t_0) = \bar{x}_0 \in \mathbb{R}, \\ \bar{u}_0(.) \in U_0 = \mathbb{R} \ \text{ unconstrained.} \end{array}$$

It follows that

$$\begin{array}{l} \inf_{\bar{u}_0 \in \mathcal{U}_0} L_0 = \bar{\alpha}(t_0)\bar{x}^2(t_0), \\ \text{state-feedback: } \bar{u}_0^* = -\bar{\alpha}\bar{x}, \\ \text{Riccati: } \dot{\bar{\alpha}} - \bar{\alpha}^2 = 0, \ \ \bar{\alpha}(t_1) \text{ fixed,} \\ \implies \ \bar{\alpha}(t) = \dfrac{1}{\frac{1}{\bar{\alpha}(t_1)} + t_1 - t}, \\ \bar{x}(t) = \bar{x}(t_0)e^{-\int_{t_0}^{t} \bar{\alpha}(t')dt'} = \bar{x}(t_0)\left(1 - \dfrac{t-t_0}{\frac{1}{\bar{\alpha}(t_1)} + t_1 - t_0}\right). \end{array}$$

Matching step In order to determine $\bar{\alpha}(t_1)$ one can match the ending point of the ordinary differential equation $x(t)$ with the required conditions at $t = t_1$. The planning problem of decision-maker 0 is then reduced to the problem of finding $\bar{\alpha}(t_1)$ such that

$$\bar{x}(0)\left(1 - \frac{t_1 - t_0}{\frac{1}{\bar{\alpha}(t_1)} + t_1 - t_0}\right) = \bar{x}(t_1),$$

which means, for $\bar{x}(0) \neq 0$

$$\bar{\alpha}(t_1) = \frac{\bar{x}(t_0) - \bar{x}(t_1)}{\bar{x}(t_1)(t_1 - t_0)},$$

satisfies the requirement.

As expected, the optimal velocity has the direction of $\bar{x}(t_1) - \bar{x}(t_0)$ with a proper scaling. To connect this result with the Euclidean distance between $\bar{x}(t_1)$ and $\bar{x}(t_0)$ we impose the terminal cost to be zero by adjusting the system as follows.

$$\begin{array}{l} \inf_{u_0} \int_{t_0}^{t_1} u_0^2(t)dt \\ \text{subject to} \\ dx = u_0 dt, \\ x(t_0) = \bar{x}(t_0) - \bar{x}(t_1), \end{array} \tag{1}$$

where we set $x(t) = \bar{x}(t) - \bar{x}(t_1)$. The optimal cost is

$$\alpha(t_0)x^2(t_0) = \alpha(t_0)[\bar{x}(t_0) - \bar{x}(t_1)]^2,$$

Next, we present the deterministic distributed planning problem.

2.2 Distributed Planning Problem

The horizon of the interaction is $[t_0, t_1], \ \ t_0 < t_1$. There are $I \geq 2$ decision-makers. The set of decision-makers is denoted by $\mathcal{I} = \{1, \ldots, I\}$. Decision-maker i has a control action $\bar{u}_i \in U_i = \mathbb{R}$. The state $\bar{x}$ is driven by an ordinary differential with two-point boundary constraint given by

$$d\bar{x} = bdt, \quad \bar{x}(t_0), \quad \bar{x}(t_1)$$

where

$$\text{Drift: } b: \ [t_0, t_1] \times \mathbb{R} \times \prod_i U_i \to \mathbb{R}.$$

The cost functional of decision-maker i is

$$L_i(\bar{x}, \bar{x}(t_0), \bar{x}(t_1)) = h_i(\bar{x}(t_1)) + \int_{t_0}^{t_1} l_i(t, \bar{x}, \bar{u})dt,$$

The goal of the decision-maker i is to choose the terminal objective function h_i and a strategy u_i such that

$$\begin{array}{l} \inf_{\bar{u}_i} L_i(\bar{u}, \bar{x}(t_0), \bar{x}(t_1)) \\ \text{subject to} \\ d\bar{x} = bdt, \\ \bar{x}(t_0) \text{ fixed} \\ \bar{x}(t_1) \text{ fixed} \end{array}$$

The two-point boundary conditions are in general challenging to meet in a first-order ordinary differential equation. However, here, each decision-maker has the freedom to design also its terminal cost h_i, which makes the problem feasible for a wider range of parameters. Notice that we are not optimizing over the best design of the function h_i, we are optimizing over the strategy u_i once the function h_i is chosen.

As an example, consider the distributed planning problem in the differential game with the following data:

$$\begin{aligned} l_i(t, \bar{x}, \bar{u}) = \bar{q}_i \frac{\bar{x}^{2\bar{k}_i}}{2\bar{k}_i} + \bar{r}_i \frac{\bar{u}_i^{2\bar{k}_i}}{2\bar{k}_i} + \bar{c}_i \bar{x}^{2\bar{k}_i - 1}\bar{u}_i \\ + \sum_{j \neq i} \bar{\epsilon}_{ij} \bar{x}^{2(\bar{k}_i - 1)} \bar{u}_i \bar{u}_j, \end{aligned} \tag{2a}$$

$$\begin{array}{l} h_i(\bar{x}) = \bar{\alpha}_i(t_1)\dfrac{\bar{x}^{2\bar{k}_i}}{2\bar{k}_i}, \\ b(t, \bar{x}, \bar{u}) = \bar{b}_1\bar{x} + \sum_{j \in \mathcal{I}} \bar{b}_{2j}\bar{u}_j, \\ \bar{x}(t_0) \text{ fixed,} \\ \bar{x}(t_1) \text{ fixed, } \bar{x}(t_1)\bar{x}(t_0) > 0 \end{array}$$

where $\bar{k}_i \geq 1$, are natural numbers, the coefficients are time dependent. The coefficient functions $\bar{q}_i, \bar{r}_i$, are nonnegative functions. We solve the distributed planning problem as follows:

- We first solve the following system

$$\begin{array}{l} i \in \mathcal{I}, \\ \inf_{\bar{u}_i} L_i(\bar{u}, \bar{x}(t_0), \bar{x}(t_1)) \\ \text{subject to} \\ d\bar{x} = bdt, \\ \bar{x}(t_0) \text{ fixed} \end{array}$$

  Its solution, if any, is given by

$$\bar{u}_i = -\bar{\eta}_i\bar{x}, \tag{3a}$$

$$0 = -\bar{r}_i\bar{\eta}_i^{2\bar{k}_i - 1} - \sum_{j \neq i} \bar{\epsilon}_{ij}\bar{\eta}_j + \bar{b}_{2i}\bar{\alpha}_i + \bar{c}_i, \tag{3b}$$

$$v_i(t, \bar{x}) = \bar{\alpha}_i(t)\frac{\bar{x}^{2\bar{k}_i}}{2\bar{k}_i}, \tag{3c}$$

$$0 = \dot{\bar{\alpha}}_i + \bar{q}_i + \bar{r}_i\bar{\eta}_i^{2\bar{k}_i} - 2\bar{k}_i\bar{c}_i\bar{\eta}_i + 2\bar{k}_i \sum_{j \neq i} \bar{\epsilon}_{ij}\bar{\eta}_i\bar{\eta}_j$$

$$+ 2\bar{k}_i\bar{\alpha}_i[\bar{b}_1 - \sum_j \bar{b}_{2j}\bar{\eta}_j], \tag{3d}$$

$$\bar{\alpha}_i(t_1) \text{ fixed}, \tag{3e}$$

for all $i \in \mathcal{I}$ with

$$\bar{x}(t) = [\bar{x}(t_0)]\, e^{\int_{t_0}^{t}\left[\bar{b}_1 - \sum_j \bar{b}_{2j}\bar{\eta}_j\right]dt'}, \tag{3f}$$

whenever the above coefficient system admits a unique solution. □

- Then, we compute the terminal point $\bar{x}(t_1)$ as a function of $\bar{\alpha}_i(t_1),\ i \in \mathcal{I}$ via the coefficients $\eta_i(t_1),\ i \in \mathcal{I}$
- The distributed planning problem is then to design $\bar{\alpha}_i(t_1)$ as degrees of freedom such that the constraints are met at the specified times:

$$\bar{x}(t_1) = \bar{x}(t_0)e^{\int_{t_0}^{t_1}\left[\bar{b}_1 - \sum_j \bar{b}_{2j}\bar{\eta}_j\right]dt}$$

The later equation offers several choices for the vector $\bar{\alpha}(t_1)$.

3. PLANNING IN MEAN-FIELD-TYPE GAMES

The horizon of the interaction is $[t_0, t_1],\quad t_0 < t_1$. There are $I \geq 2$ number of decision-makers. The set of decision-makers is denoted by $\mathcal{I} = \{1, 2, \ldots, I\}$. Decision-maker $i \in \mathcal{I}$ has a control action $u_i \in U_i = \mathbb{R}$. The state x is driven by Drift-Jump-Diffusion of mean-field type given by

$$dx = bdt + \sigma dB + \int_\Theta \mu(.,\theta)\tilde{N}(dt, d\theta),$$
$$x(t_0) \sim m(t_0, .),\ x(t_1) \sim m(t_1, .),$$

where

Drift: b,
Diffusion: Brownian motion B,
Jump process: $N(dt, d\theta)$,
Compensated Jump: $\tilde{N}(dt, d\theta) = N(dt, d\theta) - \nu(d\theta)dt$,

$$b, \sigma, \mu(.,\theta), :\ [t_0, t_1] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \prod_{j=1}^{I} U_j \to \mathbb{R}.$$

The performance functional of decision-maker i is

$$L_i(u, m_0) = h_i(x(t_1), m(t_1)) + \int_{t_0}^{t_1} l_i(t, x, u, m)dt,$$

where $m(t, dy) = \mathbb{P}_{x(t)}(dy)$ is the probability measure of the state $x(t)$ at time t.

The risk-neutral mean-field-type game is given by

$$(\mathcal{I}, U_i, \mathcal{U}_i, \mathbb{E}[L_i])_{i\in\mathcal{I}}.$$

A risk-neutral Nash Mean-Field-Type Equilibrium is a solution of the following fixed-point problem:

$$i \in \mathcal{I},$$
$$\mathbb{E}[L_i(u^*, m_0)] = \inf_{u_i \in \mathcal{U}_i} \mathbb{E}[L_i(u_1^*, \ldots, u_{i-1}^*, u_i, u_{i+1}^*, \ldots, u_I^*, m_0)].$$

Let $\hat{V}_i(t, m)$ be the optimal cost-to-go from m at time $t \in (t_0, t_1)$ given the strategies of the others.

$$\hat{V}_i(t, m) = \inf_{u_i} \mathbb{E}[h_i(x(t_1), m(t_1)) + \int_t^{t_1} l_i(t, x, u, m)dt' | m(t) = m].$$

Let $\hat{V}_{i,m}$ be the Gâteaux derivative of $\hat{V}_i(t, .)$ with the respect to the measure m. Introduce the integrand Hamiltonian as

$$\hat{H}_i(x, m, \hat{V}_m, \hat{V}_{xm}, \hat{V}_{xxm}) = \inf_{u_i \in U_i}\Big\{ l_i + b\, \hat{V}_{i,xm} + \frac{\sigma^2}{2}\hat{V}_{i,xxm} + \int_\Theta [\hat{V}_{i,m}(t_-, x+\mu) - \hat{V}_{i,m} - \mu\hat{V}_{i,xm}]\nu(d\theta)\Big\}.$$

Denote the jump operator J as

$$J[\phi_i] := \int_\Theta [\phi_{i,m}(t_-, x+\mu) - \phi_{i,m} - \mu\phi_{i,xm}]\nu(d\theta),$$

and J^* be the adjoint operator of J:

$$\langle J[\phi], m\rangle = \langle \phi, J^*[m]\rangle.$$

A sufficiency condition for a risk-neutral Nash equilibrium system is given by the following (backward-forward) partial integro-differential system

$$i \in \mathcal{I}, \tag{4a}$$
$$0 = \hat{V}_{i,t}(t, m) + \int \hat{H}_i(t, x, m, \hat{V}_m, \hat{V}_{xm}, \hat{V}_{xxm})m(dx), \tag{4b}$$
$$\hat{V}_i(t_1, m) = \int m(dy)h_i(y, m), \tag{4c}$$
$$\tag{4d}$$

We refer the reader to Bensoussan et al. (2020) for a derivation of the equilibrium system. The system (4) is an infinite PIDE system in m and it provides the Nash equilibrium values of the mean-field-type game. Notice that the PIDE system satisfied by $\{\hat{V}_{i,m}\}_{i\in\mathcal{I}}$ is not the equilibrium value $\{\hat{V}_i\}_{i\in\mathcal{I}}$ in (4). We use (4) to find risk-neutral Nash mean-field-type equilibrium.

In the distributed planning problem, the goal of the decision-maker i is to choose the terminal objective function h_i and a strategy u_i such that

$$\begin{aligned} &\inf_{u_i} \mathbb{E}L_i(u, m(t_0), m(t_1)) \\ &\text{subject to} \\ &dx = bdt + \sigma dB + \int_\Theta \mu(.,\theta)\tilde{N}(dt, d\theta), \\ &x(t_0) \sim m(t_0, .),\ \text{fixed} \\ &x(t_1) \sim m(t_1, .)\ \text{fixed} \end{aligned} \tag{5}$$

The distributed planning problem (5) extends the classical optimal transport problem and also the classical Schrödinger bridge problem (see Chen et al. (2016, 2018) and the references therein). That is, for a distribution of mass $m(t_0, dy)$, we wish to transport the mass in such a way that it is transformed into the distribution $m(t_1, dy)$ on the same space. The optimal transport theory is the study of optimal transportation and allocation between measures. The optimal transport problem was first introduced by Monge (1781) and formalized by Kantorovich (1942), leading to the so called Monge-Kantorovitch transportation problem.

In the distributed planning the equilibrium terminal cost of decision-maker i must satisfy

$$\hat{V}_i(t_1, m) = \int m(t_1, dy)h_i(y, m(t_1)),$$

where the equality is in a functional sense. The distributed planning problem is then to choose h_i such that the terminal distribution matches with the prescribed distribution $m(t_1, dy)$.

$$i \in \mathcal{I}, \tag{6a}$$
$$0 = \hat{V}_{i,t}(t,m) \tag{6b}$$
$$+ \int_x H_i(x, m, \hat{V}_m, \hat{V}_{xm}, \hat{V}_{xxm}) m(dx),$$
$$\hat{V}_i(t_1, m) = \int m(t_1, dy) h_i(y, m), \tag{6c}$$

In some problems, the planning is to satisfy some variance or moment constraints. This is illustrated in the next section.

4. GAMES WITH MOMENT CONSTRAINTS

We will investigate the mean-field-type game with the following data:

$$\begin{aligned} l_i(t,x,u,m) &= q_i \frac{(x-\bar{x})^{2k_i}}{2k_i} + r_i \frac{(u_i-\bar{u}_i)^{2k_i}}{2k_i} \\ &+ c_i (x-\bar{x})^{2k_i-1}(u_i-\bar{u}_i) \\ &+ \sum_{j\in\mathcal{I}\backslash\{i\}} \epsilon_{ij}(x-\bar{x})^{2(k_i-1)}(u_i-\bar{u}_i)(u_j-\bar{u}_j) \\ &+ \bar{q}_i \frac{\bar{x}^{2\bar{k}_i}}{2\bar{k}_i} + \bar{r}_i \frac{\bar{u}_i^{2\bar{k}_i}}{2\bar{k}_i} + \bar{c}_i \bar{x}^{2\bar{k}_i-1}\bar{u}_i \\ &+ \sum_{j\neq i} \bar{\epsilon}_{ij} \bar{x}^{2(\bar{k}_i-1)} \bar{u}_i \bar{u}_j, \end{aligned} \tag{7a}$$

$$h_i(x,m) = \alpha_i(t_1)\frac{(x-\bar{x}(t_1))^{2k_i}}{2k_i} + \bar{\alpha}_i(t_1)\frac{\bar{x}(t_1)^{2\bar{k}_i}}{2\bar{k}_i}, \tag{7b}$$

$$\begin{aligned} b(t,x,u,m) &= b_1(x-\bar{x}) + \bar{b}_1\bar{x} \\ &+ \sum_{j\in\mathcal{I}} \left[b_{2j}(u_j-\bar{u}_j) + \bar{b}_{2j}\bar{u}_j\right], \end{aligned} \tag{7c}$$

$$\sigma(t,x,u,m) = (x-\bar{x})\tilde{\sigma}, \tag{7d}$$
$$\mu(t,x,u,m,\theta) = (x-\bar{x})\tilde{\mu}(.,\theta), \tag{7e}$$
$$\bar{x}(t) = \int y m(t,dy), \tag{7f}$$
$$\bar{u}(t) = \int u(t,y,m) m(t,dy), \tag{7g}$$

where $k_i \geq 1,\ \bar{k}_i \geq 1$, are natural numbers, the coefficients are time dependent. The coefficient functions $q_i, r_i, \bar{q}_i, \bar{r}_i$, are nonnegative functions. $\alpha_i(t_1), \bar{\alpha}_i(t_1)$ are to be determined below.

In the distributed planning problem, the goal of decision-maker i is to choose the terminal weights $\alpha_i(t_1), \bar{\alpha}_i(t_1)$ and a strategy u_i such that

$$\begin{aligned} &\inf_{u_i} \mathbb{E} L_i(u, m(t_0), d, \bar{x}(t_1)) \\ &\text{subject to} \\ &dx = bdt + \sigma dB + \int_\Theta \mu(.,\theta)\tilde{N}(dt,d\theta), \\ &x(t_0) \sim m(t_0,.),\ \text{fixed} \\ &\mathbb{E}[(x(t_1) - \int y m(t_1,dy))^{2k_i}] = \bar{c}_i,\ \text{fixed} \\ &\int y m(t_1,dy) = \bar{x}(t_1),\ \text{fixed} \end{aligned} \tag{8}$$

Proposition 1. A risk-neutral Nash mean-field-type equilibrium is given in a semi-explicit way as follows:

$$u_i^{ne} = -\eta_i\left(x - \int y m(dy)\right) - \bar{\eta}_i \int y m(dy), \tag{9a}$$
$$0 = -r_i \eta_i^{2k_i-1} - \sum_{j\neq i} \epsilon_{ij}\eta_j + b_{2i}\alpha_i + c_i, \tag{9b}$$
$$0 = -\bar{r}_i \bar{\eta}_i^{2\bar{k}_i-1} - \sum_{j\neq i} \bar{\epsilon}_{ij}\bar{\eta}_j + \bar{b}_{2i}\bar{\alpha}_i + \bar{c}_i, \tag{9c}$$
$$\begin{aligned} \hat{V}_i(t,m) &= \alpha_i \int_x \frac{(x-\int y m(dy))^{2k_i}}{2k_i} m(dx) \\ &+ \bar{\alpha}_i \frac{(\int y m(dy))^{2\bar{k}_i}}{2\bar{k}_i}, \end{aligned} \tag{9d}$$
$$\begin{aligned} 0 &= \dot{\alpha}_i + q_i + r_i\eta_i^{2k_i} - 2k_i c_i \eta_i + 2k_i \sum_{j\neq i}\epsilon_{ij}\eta_i\eta_j \\ &+ 2k_i\alpha_i[b_1 - \sum_{j\in\mathcal{I}} b_{2j}\eta_j] + 2k_i(2k_i-1)\alpha_i\frac{1}{2}\tilde{\sigma}^2 \\ &+ \alpha_i \int_\Theta [(1+\tilde{\mu})^{2k_i} - 1 - 2k_i\tilde{\mu}]\nu(d\theta), \end{aligned} \tag{9e}$$
$$\alpha_i(t_1) \text{ to be determined}, \tag{9f}$$
$$\begin{aligned} 0 &= \dot{\bar{\alpha}}_i + \bar{q}_i + \bar{r}_i\bar{\eta}_i^{2\bar{k}_i} - 2\bar{k}_i\bar{c}_i\bar{\eta}_i + 2\bar{k}_i\sum_{j\neq i}\bar{\epsilon}_{ij}\bar{\eta}_i\bar{\eta}_j \\ &+ 2\bar{k}_i\bar{\alpha}_i[\bar{b}_1 - \sum_j \bar{b}_{2j}\bar{\eta}_j], \end{aligned} \tag{9g}$$
$$\bar{\alpha}_i(t_1) \text{ to be determined}, \tag{9h}$$

for all $i \in \mathcal{I}$ with

$$\int y m(t,dy) = \left[\int y m(0,dy)\right] e^{\int_0^t \left[\bar{b}_1 - \sum_j \bar{b}_{2j}\bar{\eta}_j\right]dt}, \tag{9i}$$

whenever the above coefficient system admits a unique solution. □

Proof. This proof is presented in the Appendix. ■

The uniqueness of the coefficient system (9) in η requires a stronger condition. For example for $k = 1$, the determinant must be non-zero. When the determinant is zero, the resulting control strategies become non-admissible and the costs become infinite.

To solve the distributed planning problem it remains to determine $\alpha_i(t_1), \bar{\alpha}_i(t_1), i \in \mathcal{I}$. $\bar{\alpha}_i(t_1)$ is determined from the evolution equation of the expected value and the constraint $\int y m(t_1,dy) = \bar{x}(t_1)$. The function $z_1(t) = \int y m(t,dy)$ satisfies

$$\begin{aligned} \dot{z}_1 &= z_1\left\{\bar{b}_1 - \sum_{j\in\mathcal{I}} \bar{b}_{2j}\bar{\eta}_j)\right\} \\ z_1(t_1) &= \bar{x}(t_1),\ \text{fixed}, \end{aligned} \tag{10}$$

from which we deduce a system satisfied by $\bar{\alpha}_i(t_1), i \in \mathcal{I}$.

It remains to determine $\alpha_i(t_1)$. To do so, we establish the dynamics of $\mathbb{E}[(x(t) - \int y m(t,dy))^{2k_i}]$ as time varies. Let $y_{k_i}(t) = \mathbb{E}[(x(t) - \int y m(t,dy))^{2k_i}]$.

$$
\begin{aligned}
&\dot{y}_{k_i} = y_{k_i}\Bigg\{2k_i(b_1 - \sum_{j\in\mathcal{I}} b_{2j}\eta_j)\\
&+k_i(2k_i-1)\tilde{\sigma}^2 + \int_\Theta \nu(d\theta)[(1+\tilde{\mu})^{2k_i} - 1 - 2k_i\tilde{\mu}]\Bigg\}\\
&y_{k_i}(t_1) = \bar{c}_i \geq 0, \text{ fixed}
\end{aligned}
\tag{11}
$$

from which we deduce the elements $\alpha_i(t_1), i \in \mathcal{I}$. This system is underdetermined when the size of ranges of k_i is smaller than the cardinality of the number of decision-makers. Another way is to consider several moment constraints.

MULTI-MARGINAL DISTRIBUTED PLANNING

The methodology presented here can be used to plan a path at multiple time instance $t_0 < t_{\frac{1}{n}} < t_{\frac{2}{n}} < \ldots < t_{\frac{n}{n}} = t_1$ such that $x(t_{\frac{k}{n}}) \sim m(t_{\frac{k}{n}}, .)$, where the $n+1$ marginal measures $m(t_{\frac{k}{n}}, .)$ are prescribed. The multi-marginal distributed planning problem becomes the design of a terminal objective functions h_i and a strategy u_i such that

$$
\begin{aligned}
&\inf_{u_i} \mathbb{E}L_i(u, (m(t_{\frac{k}{n}}, .))_k)\\
&\text{subject to}\\
&dx = bdt + \sigma dB + \int_\Theta \mu(., \theta)\tilde{N}(dt, d\theta),\\
&x(t_0) \sim m(t_0, .), \text{ fixed}\\
&x(t_{\frac{k}{n}}) \sim m(t_{\frac{k}{n}}, .), \quad k \in \{1, \ldots, n-1\}, \text{ fixed}\\
&x(t_1) \sim m(t_1, .) \text{ fixed}
\end{aligned}
\tag{12}
$$

We solve the multi-marginal problem, we decompose into subproblems as follows.

$$
\begin{aligned}
&\inf_{u_i} \mathbb{E}h_i(\bar{x}(t_{\frac{k}{n}})) + \int_{t_{\frac{k-1}{n}}}^{t_{\frac{k}{n}}} l_i(t, \bar{x}, \bar{u})dt,\\
&\text{subject to}\\
&dx = bdt + \sigma dB + \int_\Theta \mu(., \theta)\tilde{N}(dt, d\theta), \quad t \in (t_{\frac{k-1}{n}}, t_{\frac{k}{n}})\\
&x(t_{\frac{k-1}{n}}) \sim m(t_{\frac{k-1}{n}}, .), \text{ fixed}\\
&x(t_{\frac{k}{n}}) \sim m(t_{\frac{k}{n}}, .), \text{ fixed}
\end{aligned}
\tag{13}
$$

Each of the subproblems is solved using the same method as above.

CONCLUSION

In this paper we have studied planning problems in mean-field-type games with perfect and complete information. We have provided semi-explicit solutions for a class of mean-field-type games with polynomial cost and moment constraints. In our future work we plan to investigate planning problems under incomplete information and or imperfect measurement.

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Appendix A. PROOF OF PROPOSITION 1

We aim to solve the PIDE system (4): We start with the following guess functional of decision-maker i as

$$
\hat{V}_i(t, m) = \alpha_i(t)\int_x \frac{(x - \int y m(dy))^{2k_i}}{2k_i} m(dx) + \bar{\alpha}_i(t)\frac{(\int y m(dy))^{2\bar{k}_i}}{2\bar{k}_i},
$$

where the coefficient functions $\alpha_i, \bar{\alpha}_i$ need to be determined. We compute the key terms $\hat{V}_{i,m}(t, m)$, $\hat{V}_{i,xm}(t, m)$, $\hat{V}_{i,xxm}(t, m)$.

The Integrand Hamiltonian is strictly convex in $(u_i - \bar{u}_i, \bar{u}_i)$ and the best-response strategy is the unique minimizer of

$$
\begin{aligned}
&r_i\frac{(u_i - \bar{u}_i)^{2k_i}}{2k_i} + c_i(x - \bar{x})^{2k_i - 1}(u_i - \bar{u}_i)\\
&+ \sum_{j\neq i} \epsilon_{ij}(x - \bar{x})^{2(k_i - 1)}(u_i - \bar{u}_i)(u_j - \bar{u}_j)\\
&+ \left[\hat{V}_{i,xm}(t, m) - \int \hat{V}_{i,xm}(t, m)(x)m(dx)\right]\sum_{j\in\mathcal{I}} b_{2j}(u_j - \bar{u}_j)\\
&+ \bar{r}_i\frac{\bar{u}_i^{2\bar{k}_i}}{2\bar{k}_i} + \bar{c}_i\bar{x}^{2\bar{k}_i - 1}\bar{u}_i + \sum_{j\neq i}\bar{\epsilon}_{ij}\bar{x}^{2(\bar{k}_i - 1)}\bar{u}_i\bar{u}_j\\
&+ \left[\int \hat{V}_{i,xm}(t, m)(x)m(dx)\right]\sum_j \bar{b}_{2j}\bar{u}_j.
\end{aligned}
\tag{A.1}
$$

By strictly convexity and by orthogonality between $(u_i - \bar{u}_i)$ and $\bar{u}_i$ the following system holds:

$$
\begin{aligned}
&i \in \mathcal{I},\\
&0 = r_i(u_i - \bar{u}_i)^{2k_i - 1} + c_i(x - \bar{x})^{2k_i - 1}
\end{aligned}
$$

$$+\sum_{j\neq i}\epsilon_{ij}(x-\bar{x})^{2(k_i-1)}(u_j-\bar{u}_j)$$
$$+\left[\hat{V}_{i,xm}(t,m)-\int \hat{V}_{i,xm}(t,m)(x)m(dx)\right]b_{2i}, \quad \text{(A.2a)}$$
$$0=\bar{r}_i\bar{u}_i^{2\bar{k}_i-1}+\bar{c}_i\bar{x}^{2\bar{k}_i-1}+\sum_{j\neq i}\bar{\epsilon}_{ij}\bar{x}^{2(\bar{k}_i-1)}\bar{u}_j$$
$$+\left[\int \hat{V}_{i,xm}(t,m)(x)m(dx)\right]\bar{b}_{2i}. \quad \text{(A.2b)}$$

The linear state-and-mean-field-type feedback strategy $u_i=-\eta_i(x-\int ym(dy))-\bar{\eta}_i\int ym(dy),\ i\in\mathcal{I}$ solves the system if the coefficients satisfy

$$i\in\mathcal{I},$$
$$0=-r_i\eta_i^{2k_i-1}-\sum_{j\neq i}\epsilon_{ij}\eta_j+b_{2i}\alpha_i+c_i, \quad \text{(A.3a)}$$
$$0=-\bar{r}_i\bar{\eta}_i^{2\bar{k}_i-1}-\sum_{j\neq i}\bar{\epsilon}_{ij}\bar{\eta}_j+\bar{b}_{2i}\bar{\alpha}_i+\bar{c}_i. \quad \text{(A.3b)}$$

The integrand Hamiltonian of i becomes

$$\hat{H}_i=\Bigg[q_i+r_i\eta_i^{2k_i}-2k_ic_i\eta_i$$
$$+2k_i\sum_{j\neq i}\epsilon_{ij}\eta_i\eta_j\Bigg]\frac{(x-\int ym(dy))^{2k_i}}{2k_i}$$
$$+2k_i\alpha_i\left[b_1-\sum_{j\in\mathcal{I}}b_{2j}\eta_j\right]\frac{(x-\int ym(dy))^{2k_i}}{2k_i}$$
$$+2k_i(2k_i-1)\alpha_i\frac{1}{2}\tilde{\sigma}^2\frac{(x-\int ym(dy))^{2k_i}}{2k_i}$$
$$+\alpha_i\int_{\Theta}[(1+\tilde{\mu})^{2k_i}-1-2k_i\tilde{\mu}]\nu(d\theta)\frac{(x-\int ym(dy))^{2k_i}}{2k_i}$$
$$+\left[\bar{q}_i+\bar{r}_i\bar{\eta}_i^{2\bar{k}_i}-2\bar{k}_i\bar{c}_i\bar{\eta}_i+2\bar{k}_i\sum_{j\neq i}\bar{\epsilon}_{ij}\bar{\eta}_i\bar{\eta}_j\right]\frac{(\int ym(dy))^{2\bar{k}_i}}{2\bar{k}_i}$$
$$+2\bar{k}_i\bar{\alpha}_i\left[\bar{b}_1-\sum_j\bar{b}_{2j}\bar{\eta}_j\right]\frac{(\int ym(dy))^{2\bar{k}_i-1}}{2\bar{k}_i}+\tilde{\epsilon}_2. \quad \text{(A.4)}$$

By identification the coefficients α_i solve the following ordinary differential equation:

$$0=\dot{\alpha}_i+q_i+r_i\eta_i^{2k_i}-2k_ic_i\eta_i+2k_i\sum_{j\neq i}\epsilon_{ij}\eta_i\eta_j$$
$$+2k_i\alpha_i[b_1-\sum_{j\in\mathcal{I}}b_{2j}\eta_j]+2k_i(2k_i-1)\alpha_i\frac{1}{2}\tilde{\sigma}^2$$
$$+\alpha_i\int_{\Theta}[(1+\tilde{\mu})^{2k_i}-1-2k_i\tilde{\mu}]\nu(d\theta), \quad \text{(A.5a)}$$
$$\alpha_i(T)\ =\text{to be determined}, \quad \text{(A.5b)}$$
$$0=\dot{\bar{\alpha}}_i+\bar{q}_i+\bar{r}_i\bar{\eta}_i^{2\bar{k}_i}-2\bar{k}_i\bar{c}_i\bar{\eta}_i+2\bar{k}_i\sum_{j\neq i}\bar{\epsilon}_{ij}\bar{\eta}_i\bar{\eta}_j$$
$$+2\bar{k}_i\bar{\alpha}_i[\bar{b}_1-\sum_j\bar{b}_{2j}\bar{\eta}_j], \quad \text{(A.5c)}$$
$$\bar{\alpha}_i(T)\ =\text{to be determined}. \quad \text{(A.5d)}$$

The aggregate mean-field term $\int ym(t,dy)$ can be derived in a semi-explicit way by taking the expected value of the state dynamics. It follows that

$$\int ym(t,dy)=\left[\int ym(0,dy)\right]e^{\int_0^t\left[\bar{b}_1-\sum_j\bar{b}_{2j}\bar{\eta}_j\right]dt'}.$$

This completes the proof. ■