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A spectral small-gain condition for input-to-state stability of infinite networks

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Abstract: This paper presents a tight small-gain theorem for networks composed of infinitely many finite-dimensional subsystems. Assuming that each subsystem is exponentially input-to-state stable, we show that if the gain operator, collecting all the information about the internal Lyapunov gains, has a spectral radius less than one, the overall infinite network is exponentially input-to-state stable. We illustrate the effectiveness of our result by applying it to traffic networks.

Keywords: Nonlinear systems, small-gain theorems, infinite-dimensional systems, input-to-state stability, Lyapunov methods, large-scale systems

1. INTRODUCTION

Interconnections of countably many finite-dimensional subsystems also called infinite networks, appear naturally as over-approximations of finite but very large networks with a possibly unknown number of subsystems (Jovanović and Bamieh, 2005). Applications of the theory of infinite networks are versatile. Spatially invariant systems consisting of an infinite number of components interconnected to each other in the same pattern are studied in (Bamieh et al., 2002; Curtain et al., 2009) together with applications to, e.g., vehicle platoon formation (Besselink and Johansson, 2017). Infinite networks also appear as representations of the solutions of linear and nonlinear partial differential equations over Hilbert spaces in terms of series expansions with respect to orthonormal or Riesz bases, see e.g. (Lhachemi and Shorten, 2018). A closely related approach relies on approximations of the system dynamics by partial differential and difference equations (Meurer, 2012; Kim et al., 2008), which is based on a continuum approximation in space or in time and is particularly useful for consensus or coverage type problems.

Most of the results on the stability of infinite networks are devoted either to spatially invariant or to linear systems. Recently, several attempts have been made to relax such strong restrictions (Dashkovskiy and Pavlichkov, 2020; Dashkovskiy et al., 2019; Mironchenko, 2019), by introducing max-form small-gain theorems for infinite networks, where each subsystem is individually input-to-state stable (ISS) (Sontag, 1989).

The main idea behind an ISS small-gain theory (see e.g. (Jiang et al., 1996; Dashkovskiy et al., 2010)) is to decompose a large-scale or infinite network into smaller subsystems which are ISS with respect to the neighboring subsystems, i.e., the inputs from other subsystems act as disturbances. Then, if the influence of the subsystems on each other is small enough, which is mathematically described by a small-gain condition, the stability of the overall system can be concluded.

In (Dashkovskiy and Pavlichkov, 2020) it is shown that a countably infinite network of continuous-time ISS systems is ISS, provided that the gain functions capturing the influence from the neighboring subsystems are all less than identity which is quite conservative. By means of examples, it is shown in (Dashkovskiy et al., 2019) that classic max-form small-gain conditions (SGCs) developed for finite-dimensional systems (Dashkovskiy et al., 2010) do not guarantee the stability of infinite networks of ISS systems, even if all the systems are linear. To address this issue, more restrictive robust strong SGCs are developed in (Dashkovskiy et al., 2019). While the small-gain theorems in (Dashkovskiy and Pavlichkov, 2020; Dashkovskiy et al., 2019) are formulated in terms of ISS Lyapunov functions, a trajectory-based small-gain theorem for infinite networks is provided in (Mironchenko, 2019).

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In this paper, we develop *tight* SGCs for networks consisting of a countably infinite number of finite-dimensional continuous-time systems. We assume that each subsystem is exponentially ISS with respect to internal and external inputs and equipped with an exponential ISS Lyapunov function. The associated gain functions reflecting the interaction with neighbors are assumed to be linear. Such a scenario leads to several nontrivialities. In particular, the gain operator, collecting all the information about the internal gains, acts in an infinite-dimensional space, in contrast to couplings of just $N \in \mathbb{N}$ systems of arbitrary nature (possibly infinite-dimensional). This calls for a careful choice of an infinite-dimensional state space of the overall network, and motivates the use of the theory of *positive operators* on ordered Banach spaces for the small-gain analysis. *We establish in Theorem 6.2 that if the gain operator, which is a positive operator, has the spectral radius less than one, then the whole interconnection is exponentially ISS and a so-called* coercive *exponential ISS Lyapunov function for the overall network can be constructed.*

Our main result is a *nontrivial* generalization of Proposition 3.3 in (Dashkovskiy et al., 2011) from finite to infinite networks. The result in (Dashkovskiy et al., 2011) basically relies on (Dashkovskiy et al., 2011, Lem. 3.1), which is a consequence of the Perron-Frobenius theorem. However, existing infinite-dimensional versions of the Perron-Frobenius theorem, including the Krein-Rutman theorem (Krein and Rutman, 1948), are *not* applicable to our setting as they require at least quasi-compactness of the gain operator, which is a very strong assumption. Therefore we need to develop new technical results based on the classical theory of ordered Banach spaces.

The work in (Dashkovskiy et al., 2019) is close in spirit to our work, since in both the stability of the network is studied on the basis of the knowledge of ISS Lyapunov functions for the subsystems and the knowledge of the gain structure. However, in (Dashkovskiy et al., 2019), the ISS Lyapunov functions for the subsystems are defined in an implication form and the gain operator is used in a max formulation, which makes it *nonlinear*, even if all the gains are linear. In contrast to (Dashkovskiy et al., 2019), in the present work, we assume the existence of exponential ISS Lyapunov functions for the subsystems in a dissipative form and assume that the gain operator is defined in a sum form. These differences make the results of this paper and the methods employed in our analysis quite different from those of (Dashkovskiy et al., 2019).

Due to the page limitation, we either omit the proofs of the results or present only the main parts of the arguments. We refer readers to the journal version of this paper (Kawan et al., 2019) for the detailed proofs and further applications to nonlinear spatially invariant systems with sector nonlinearities. In Noroozi et al. (2020) one can find further applications of our small-gain theorem for the stability of infinite *time-varying* networks, to consensus in *infinite*-agent systems, as well as to the design of *distributed observers* for infinite networks.

2. NOTATION AND PRELIMINARIES

2.1 Notation

We write $\mathbb{N} = \{1, 2, 3, \dots\}$ for the set of positive integers. $\mathbb{R}$ denotes the reals and $\mathbb{R}_+ := \{t \in \mathbb{R} : t \geq 0\}$ the nonnegative reals. For vector norms on finite- and infinite-dimensional spaces, we write $|\cdot|$. For associated operator norms, we use the notation $\|\cdot\|$. We write $A^\top$ for the transpose of a matrix A (which can be finite or infinite). We typically use Greek letters for infinite matrices and Latin ones for finite matrices. Elements of $\mathbb{R}^n$ are by default regarded as column vectors and we write $x^\top \cdot y$ for the Euclidean inner product of two vectors $x, y \in \mathbb{R}^n$. We use the same notation for dot products of vectors with infinitely many components. By ℓ^p, $p \in [1, \infty)$, we denote the Banach space of all real sequences $x = (x_i)_{i\in\mathbb{N}}$ with finite ℓ^p-norm $|x|_p < \infty$, where $|x|_p = (\sum_{i=1}^\infty |x_i|^p)^{1/p}$. We write $L^\infty(\mathbb{R}_+, \mathbb{R})$ for the Banach space of essentially bounded measurable functions from $\mathbb{R}_+$ to $\mathbb{R}$. If X is a Banach space, we write $r(T)$ for the spectral radius of a bounded linear operator $T : X \to X$. The notation $C^0(X, Y)$ stands for the set of all continuous mappings $f : X \to Y$ between metric spaces X and Y. The right upper Dini derivative of a function $\gamma : \mathbb{R} \to \mathbb{R}$ at $t \in \mathbb{R}$ is defined by

$$\mathrm{D}^+\gamma(t) := \limsup_{h\to 0+} \frac{1}{h}\big(\gamma(t+h) - \gamma(t)\big),$$

and is allowed to assume the values $\pm\infty$. The right lower Dini derivative $\mathrm{D}_+\gamma(t)$ is defined analogously, replacing lim sup with lim inf. Finally, we introduce the following classes of comparison functions frequently used in Lyapunov stability theory.

$$\mathcal{P} := \big\{\gamma \in C^0(\mathbb{R}_+, \mathbb{R}_+) : \gamma(0) = 0, \gamma(r) > 0,\ \forall r > 0\big\},$$
$$\mathcal{K} := \big\{\gamma \in \mathcal{P} : \gamma \text{ is strictly increasing}\big\},$$
$$\mathcal{K}_\infty := \big\{\gamma \in \mathcal{K} : \lim_{t\to\infty} \gamma(t) = \infty\big\}.$$

2.2 Interconnected Systems

We study the interconnection of countably many systems, each given by a finite-dimensional ordinary differential equation (ODE). Using $\mathbb{N}$ as the index set (by default), the i-th subsystem is written as

$$\Sigma_i : \quad \dot{x}_i = f_i(x_i, \bar{x}_i, u_i). \tag{1}$$

The family $(\Sigma_i)_{i\in\mathbb{N}}$ comes together with sequences $(n_i)_{i\in\mathbb{N}}$, $(m_i)_{i\in\mathbb{N}}$ of positive integers and finite index sets $I_i \subset \mathbb{N}\backslash\{i\}$, $i \in \mathbb{N}$, so that the following assumptions hold.

- The state vector x_i of Σ_i is an element of $\mathbb{R}^{n_i}$.
- The vector $\bar{x}_i$ is composed of the state vectors x_j, $j \in I_i$. The order of these vectors plays no particular role, so we do not specify it.
- The external input vector u_i is an element of $\mathbb{R}^{m_i}$.
- The right-hand side is a continuous function $f_i : \mathbb{R}^{n_i} \times \mathbb{R}^{N_i} \times \mathbb{R}^{m_i} \to \mathbb{R}^{n_i}$, where $N_i := \sum_{j\in I_i} n_j$.
- Unique local solutions of the ODE (1) (in the sense of Carathéodory) exist for all initial states $x_i^0 \in \mathbb{R}^{n_i}$ and locally essentially bounded functions $\bar{x}_i(\cdot)$ and $u_i(\cdot)$ (which are regarded as time-dependent inputs). We denote the corresponding solution by $\phi_i(t, x_i^0, \bar{x}_i, u_i)$.

In the ODE (1), we consider $\bar{x}_i(\cdot)$ as an *internal input* and $u_i(\cdot)$ as an *external input* (which may be a disturbance or a control input). The interpretation is that the subsystem Σ_i is affected by finitely many neighbors, indexed by I_i, and its external input.

To define the interconnection of the subsystems Σ_i, we consider the state vector $x = (x_i)_{i\in\mathbb{N}}$, the input vector $u = (u_i)_{i\in\mathbb{N}}$ and the right-hand side $f(x,u) := (f_1(x_1,\bar{x}_1,u_1), f_2(x_2,\bar{x}_2,u_2),\ldots)$. The interconnection is then formally written as

$$\Sigma: \quad \dot{x} = f(x,u). \tag{2}$$

To handle this infinite-dimensional ODE properly, we choose appropriate Banach spaces $X \subset \prod_{i\in\mathbb{N}} \mathbb{R}^{n_i}$ and $U \subset \prod_{i\in\mathbb{N}} \mathbb{R}^{m_i}$ and restrict f to $X\times U$. As a natural choice, we use ℓ^p-type spaces for both X and U, and impose conditions on f to guarantee existence and uniqueness of solutions. *Our goal is to show that Σ is exponentially input-to-state stable (eISS) if each Σ_i admits an eISS Lyapunov function and a small-gain condition is satisfied.*

3. WELL-POSEDNESS

We want to model the state space X of Σ as a Banach space of sequences $x = (x_i)_{i\in\mathbb{N}}$ with $x_i \in \mathbb{R}^{n_i}$. The most natural choice is an ℓ^p-space. To define such a space, we first fix a norm on each $\mathbb{R}^{n_i}$ (that might not only depend on the dimension n_i but also on the index i). For brevity, we omit the index in our notation and simply write $|\cdot|$ for each of these norms. Then, for every $p \in [1,\infty)$, we put

$$\ell^p(\mathbb{N},(n_i)) := \Big\{ x = (x_i)_{i\in\mathbb{N}} : x_i \in \mathbb{R}^{n_i},\ \sum_{i\in\mathbb{N}} |x_i|^p < \infty \Big\}.$$

Equipped with the norm $|x|_p := (\sum_{i\in\mathbb{N}} |x_i|^p)^{1/p}$, the space $\ell^p(\mathbb{N},(n_i))$ becomes a separable Banach space, which can be shown using standard arguments, see e.g. (Dunford and Schwartz, 1957).

As the state space of the system Σ, we consider $X := \ell^p(\mathbb{N},(n_i))$ for a fixed $p \in [1,\infty)$. Similarly, for a fixed $q \in [1,\infty)$, we consider the *external input space* $U := \ell^q(\mathbb{N},(m_i))$, where we fix norms on $\mathbb{R}^{m_i}$ that we simply denote by $|\cdot|$ again. The space of admissible *external input functions* is defined by

$$\mathcal{U} := \big\{ u : \mathbb{R}_+ \to U : u \text{ is measurable and essentially bounded} \big\}. \tag{3}$$

A continuous mapping $\xi : I \to X$, defined on an interval $I = [0,T_*)$ with $T_* \in (0,\infty]$, is called a *solution* of the infinite-dimensional ODE (2) with initial value $x^0 \in X$ for the external input $u \in \mathcal{U}$ provided that the two conditions

$$f(\xi(t),u(t)) \in X \text{ and } \xi(t) = x^0 + \int_0^t f(\xi(s),u(s))\mathrm{d}s,$$

hold for all $t \in I$, where the integral is the Bochner integral for Banach space valued functions.

If for each $x^0 \in X$ and $u \in \mathcal{U}$, a unique local solution exists, we say that the system is *well-posed* and write $\phi(\cdot,x^0,u)$ for any such solution. As usual, we consider the maximal extension of $\phi(\cdot,x^0,u)$ and write $J_{\max}(x^0,u)$ for its interval of existence. We say that the system is *forward complete* if $J_{\max}(x^0,u) = \mathbb{R}_+$ for all $(x^0,u) \in X \times \mathcal{U}$.

Denoting by $\pi_i : X \to \mathbb{R}^{n_i}$ the canonical projection onto the i-th component (which is a bounded linear operator) and writing $u(t) = (u_1(t),u_2(t),\ldots)$, we obtain

$$\begin{aligned}\pi_i\phi(t,x^0,u) &= x_i^0 + \int_0^t \pi_i f(\phi(s,x^0,u),u(s))\mathrm{d}s \\ &= x_i^0 + \int_0^t f_i(\pi_i\phi(s,x^0,u),(\pi_j\phi(s,x^0,u))_{j\in I_i},u_i(s))\mathrm{d}s,\end{aligned}$$

which implies that $t \mapsto \pi_i\phi(t,x^0,u)$ solves the ODE $\dot{x}_i = f_i(x_i,\bar{x}_i,u_i)$ for the internal input $\bar{x}_i(\cdot) := (\pi_j\phi(\cdot,x^0,u))_{j\in I_i}$ and the external input $u_i(\cdot)$. Hence,

$$\pi_i\phi(t,x^0,u) = \phi_i(t,x_i^0,\bar{x}_i,u_i) \quad \text{for all } t \in J_{\max}(x^0,u).$$

Sufficient conditions for the existence and uniqueness of solutions (and forward completeness) can be obtained from the general theory of Carathéodory differential equations on Banach spaces, see (Aulbach and Wanner, 1996) as a general reference for systems with bounded generators.

4. EXPONENTIAL INPUT-TO-STATE STABILITY

Having a well-posed interconnection (2) with state space $X = \ell^p(\mathbb{N},(n_i))$ and external input space $U = \ell^q(\mathbb{N},(m_i))$, it is natural to study its stability with respect to both initial conditions and external inputs. The concept of input-to-state stability is suitable for both of these purposes.

We equip the (linear) space $\mathcal{U}$ of external inputs with the sup-norm $|u|_{q,\infty} := \text{ess sup}_{t\geq 0} |u(t)|_q$ and work with the following definition of exponential input-to-state stability.

Definition 4.1. System Σ in (2) is called *exponentially input-to-state stable (eISS)* if it is forward complete and there exist $a, M > 0$ and $\gamma \in \mathcal{K}$ such that for any initial state $x^0 \in X$ and any $u \in \mathcal{U}$ the corresponding solution satisfies

$$|\phi(t,x^0,u)|_p \leq M\mathrm{e}^{-at}|x^0|_p + \gamma(|u|_{q,\infty}) \quad \text{for all } t \geq 0.$$

Next, we provide an alternative definition of input-to-state stability by means of eISS Lyapunov functions. First, for any continuous $V : X \to \mathbb{R}$, let us define the *orbital derivative* at $x \in X$ for the external input $u \in \mathcal{U}$ by

$$\mathrm{D}^+V_u(x) := \mathrm{D}^+V(\phi(t,x,u))_{|t=0},$$

where the right-hand side is the right upper Dini derivative of $t \mapsto V(\phi(t,x,u))$ at $t = 0$. Then, the definition of exponential ISS Lyapunov functions is as follows.

Definition 4.2. A continuous function $V : X \to \mathbb{R}_+$ is called an *eISS Lyapunov function* for the system Σ if there exist constants $\underline{\omega}, \overline{\omega}, b, \kappa > 0$ and $\gamma \in \mathcal{K}_\infty$ such that

$$\underline{\omega}|x|_p^b \leq V(x) \leq \overline{\omega}|x|_p^b, \tag{4a}$$

$$\mathrm{D}^+V_u(x) \leq -\kappa V(x) + \gamma(|u|_{q,\infty}), \tag{4b}$$

hold for all $x \in X$ and $u \in \mathcal{U}$.

The importance of eISS Lyapunov functions is due to the following result, which is a variation of (Dashkovskiy and Mironchenko, 2013, Thm. 1), and thus we omit the proof.

Proposition 4.3. If there exists an eISS Lyapunov function for Σ, then Σ is eISS.

5. ASSUMPTIONS ON THE SUBSYSTEMS AND THE GAIN OPERATOR

Our main objective is to develop conditions for exponential input-to-state stability of the interconnection of countably many subsystems (1), depending on certain stability properties of subsystems.

We assume that each subsystem Σ_i, given by (1), is eISS and there exist continuous eISS Lyapunov functions with linear gains for all Σ_i. Restating the concept of an eISS Lyapunov function (Definition 4.2) for the subsystem Σ_i, we see that the gain γ in this definition indicates the influence of the aggregated input onto the system. For our purposes, this information is not sufficient as we would like to know how each j-th subsystem influences each i-th subsystem as in the next assumption.

Assumption 5.1. For each $i \in \mathbb{N}$ there is a continuous function $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$, such that for certain $p, q \in [1, \infty)$

- There are constants $\underline{\alpha}_i, \overline{\alpha}_i > 0$ so that for all $x_i \in \mathbb{R}^{n_i}$
$$\underline{\alpha}_i |x_i|^p \leq V_i(x_i) \leq \overline{\alpha}_i |x_i|^p. \tag{5}$$
- There are constants $\lambda_i, \gamma_{ij}, \gamma_{iu} > 0$ so that the following holds: for each $x_i \in \mathbb{R}^{n_i}$, $u_i \in L^\infty(\mathbb{R}_+, \mathbb{R}^{m_i})$ and each internal input $\bar{x}_i \in C^0(\mathbb{R}_+, \mathbb{R}^{N_i})$ and for almost all t in the maximal interval of existence of $\phi_i(t) := \phi_i(t, x_i, \bar{x}_i, u_i)$ one has
$$\begin{aligned}\mathrm{D}^+(V_i \circ \phi_i)(t) \leq -\lambda_i V_i(\phi_i(t)) + \sum_{j \in I_i} \gamma_{ij} V_j(x_j(t)) \\ + \gamma_{iu} |u_i(t)|^q,\end{aligned} \tag{6}$$
where we denote the components of $\bar{x}_i$ by $x_j(\cdot)$.
- For all t in the maximal interval of existence of ϕ_i
$$\mathrm{D}_+(V_i \circ \phi_i)(t) < \infty.$$

Furthermore, we assume that the following uniformity conditions hold for the constants introduced above.

Assumption 5.2. (a) There are constants $\underline{\alpha}, \overline{\alpha} > 0$ so that for all $i \in \mathbb{N}$
$$\underline{\alpha} \leq \underline{\alpha}_i \leq \overline{\alpha}_i \leq \overline{\alpha}. \tag{7}$$
(b) There is a constant $\underline{\lambda} > 0$ so that for all $i \in \mathbb{N}$
$$\underline{\lambda} \leq \lambda_i. \tag{8}$$
(c) There is a constant $\overline{\gamma}_u > 0$ so that for all $i \in \mathbb{N}$
$$\gamma_{iu} \leq \overline{\gamma}_u. \tag{9}$$

The above assumptions enforce stability properties of the subsystems Σ_i. In order to speak about the interconnection of all subsystems in (1), we should define the state space for the interconnection as well as the space of input values. The inequalities (5) and (6) suggest the following choice: $X = \ell^p(\mathbb{N}, (n_i))$ and $U = \ell^q(\mathbb{N}, (m_i))$. Thus, we suppose:

Assumption 5.3. The system Σ with state space $X = \ell^p(\mathbb{N}, (n_i))$ and external input space $U = \ell^q(\mathbb{N}, (m_i))$ is well-posed.

In order to formulate a small-gain condition, we further introduce the following infinite nonnegative matrices by collecting the coefficients from (6):
$$\Lambda := \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \ldots), \quad \Gamma := (\gamma_{ij})_{i,j \in \mathbb{N}},$$
where we put $\gamma_{ij} := 0$ whenever $j \notin I_i$. We also introduce the infinite matrix
$$\Psi := \Lambda^{-1}\Gamma = (\psi_{ij})_{i,j \in \mathbb{N}}, \quad \psi_{ij} = \frac{\gamma_{ij}}{\lambda_i}. \tag{10}$$

Under an appropriate boundedness assumption, the matrix Ψ acts as a linear operator on ℓ^1 by
$$(\Psi x)_i = \sum_{j=1}^\infty \psi_{ij} x_j \quad \text{for all } i \in \mathbb{N}.$$

We call $\Psi : \ell^1 \to \ell^1$ the *gain operator* associated with the decay rates λ_i and coefficients γ_{ij}.

We assume that Γ is a bounded operator from ℓ_1 to ℓ_1.

Assumption 5.4. The matrix $\Gamma = (\gamma_{ij})$ satisfies
$$\|\Gamma\|_{1,1} = \sup_{j \in \mathbb{N}} \sum_{i=1}^\infty \gamma_{ij} < \infty, \tag{11}$$
where the double index on the left-hand side indicates that we consider the operator norm induced by the ℓ^1-norm both on the domain and codomain of the operator Γ.

Remark 5.5. Assumption 5.4 implies that there is a constant $\overline{\gamma} > 0$ s.t. $0 < \gamma_{ij} \leq \overline{\gamma}$ for all $i \in \mathbb{N}$, $j \in I_i$. ◇

Lemma 5.6. Suppose that Assumptions 5.4 and 5.2(b) hold. Then $\Psi : \ell^1 \to \ell^1$ is a bounded operator.

6. SMALL-GAIN THEOREM

In this section, we prove that the interconnected system Σ is eISS under the given assumptions, provided that the spectral radius of the gain operator satisfies $r(\Psi) < 1$.

First, we state a central technical lemma:

Lemma 6.1. Assume that the spectral radius of Ψ satisfies $r(\Psi) < 1$ and that there exists a constant $\overline{\lambda} > 0$ such that $\lambda_i \leq \overline{\lambda}$ for all $i \in \mathbb{N}$. Then

(i) there exist a vector $\mu = (\mu_i)_{i \in \mathbb{N}}$ satisfying $\underline{\mu} \leq \mu_i \leq \overline{\mu}$ for all $i \in \mathbb{N}$ with constants $\underline{\mu}, \overline{\mu} > 0$, as well as a constant $\lambda_\infty > 0$ so that
$$\frac{[\mu^\top(-\Lambda + \Gamma)]_i}{\mu_i} \leq -\lambda_\infty \quad \text{for all } i \in \mathbb{N};$$
(ii) for every $\rho > 0$ we can choose the vector μ and the constant λ_∞ so that $\lambda_\infty \geq (1 - r(\Psi))\underline{\lambda} - \rho$.

By Proposition 4.3, our objective can be accomplished by finding an eISS Lyapunov function for the interconnection Σ under the small-gain condition $r(\Psi) < 1$. This is accomplished by the following *small-gain theorem*, which is the main result of the paper and a direct generalization of (Dashkovskiy et al., 2011, Prop. 3.3) where this result has been shown for finite networks.

Theorem 6.2. Consider the infinite interconnection Σ, composed of subsystems Σ_i, $i \in \mathbb{N}$, with fixed $p, q \in [1, \infty)$, and let the following assumptions hold.

(i) Σ is well-posed as a system with state space $X = \ell^p(\mathbb{N}, (n_i))$, space of input values $U = \ell^q(\mathbb{N}, (m_i))$, and the external input space $\mathcal{U}$, as defined in (3).
(ii) Each Σ_i admits a continuous eISS Lyapunov function V_i so that Assumptions 5.1 and 5.2 are satisfied.
(iii) The operator $\Gamma : \ell^1 \to \ell^1$ is bounded, i.e., Assumption 5.4 holds.
(iv) The spectral radius of Ψ satisfies $r(\Psi) < 1$.

Then Σ admits an eISS Lyapunov function of the form
$$V(x) = \sum_{i=1}^\infty \mu_i V_i(x_i), \quad V : X \to \mathbb{R}_+ \tag{12}$$
for some $\mu = (\mu_i)_{i \in \mathbb{N}}$ so that $\underline{\mu} \leq \mu_i \leq \overline{\mu}$ for certain constants $\underline{\mu}, \overline{\mu} > 0$. In particular, the function V satisfies:

(a) V is continuous.
(b) There is a $\lambda_\infty > 0$ so that for all $x^0 \in X$ and $u \in \mathcal{U}$:
$$\mathrm{D}^+ V_u(x^0) \leq -\lambda_\infty V(x^0) + \overline{\mu}\,\overline{\gamma}_u |u|^q_{q,\infty}.$$
(c) For all $x \in X$ the following inequalities hold:
$$\underline{\mu}\underline{\alpha}|x|_p^p \leq V(x) \leq \overline{\mu}\,\overline{\alpha}|x|_p^p. \tag{13}$$

In particular, Σ is eISS.

Proof. First, we prove the result for the case that there is a constant $\overline{\lambda} > 0$ with

$$\lambda_i \leq \overline{\lambda} \quad \text{for all } i \in \mathbb{N}. \tag{14}$$

Inequality (14) means that the decay rates of the eISS Lyapunov functions for all subsystems are uniformly bounded. Afterwards, we treat the general case.

Step 1 (Definition of V): Lemma 6.1 yields a positive vector $\mu = (\mu_i)_{i\in\mathbb{N}} \in \ell^\infty$ whose entries are uniformly bounded away from zero, and a constant $\lambda_\infty > 0$ so that

$$\frac{[\mu^\top(-\Lambda+\Gamma)]_i}{\mu_i} \leq -\lambda_\infty \quad \forall\, i \in \mathbb{N}. \tag{15}$$

With this μ, the function V in (12) is well-defined, continuous and satisfies (13) (we skip the arguments).

Step 2 (Estimate of the orbital derivative): Fix an initial state $x^0 \in X$ and an external input $u \in \mathcal{U}$. We write $\phi(t) = \phi(t,x^0,u)$, $\phi_i(t) = \pi_i\phi(t)$, $\bar{x}_i(t) = (\pi_j\phi(t))_{j\in I_i}$, where π_i denotes the projection to the i-th component. Then for any $t > 0$ (where $\phi(t)$ is defined), we obtain

$$\frac{1}{t}\big(V(\phi(t)) - V(x^0)\big) = \frac{1}{t}\sum_{i=1}^{\infty}\mu_i\big[V_i(\phi_i(t)) - V_i(\phi_i(0))\big].$$

Since the inequalities (6) are valid for almost all positive times, the function on the right-hand side of (6) is Lebesgue integrable, and since we assume that $\mathrm{D}_+(V_i\circ\phi_i)(t) < \infty$ for all t, we can proceed using the generalized fundamental theorem of calculus (see (Hagood and Thomson, 2006, Thm. 9 and p. 42, Rmk. 5.c)) to

$$\frac{1}{t}\big(V(\phi(t)) - V(x^0)\big) \leq \frac{1}{t}\sum_{i=1}^{\infty}\int_0^t \mu_i\Big[-\lambda_i V_i(\phi_i(s)) + \sum_{j\in I_i}\gamma_{ij}V_j(\phi_j(s)) + \gamma_{iu}|u_i(s)|^q\Big]\mathrm{d}s.$$

Using the assumption (14), the notation

$$V_{\mathrm{vec}}(\phi(s)) := (V_1(\phi_1(s)), V_2(\phi_2(s)), \dots)^\top$$

and applying the Fubini-Tonelli theorem to interchange the infinite sum and the integral (we skip the verification of the assumptions of the Fubini-Tonelli theorem), one can conclude that

$$\begin{aligned}
&\frac{1}{t}\big(V(\phi(t)) - V(x^0)\big)\\
&\leq \frac{1}{t}\int_0^t \sum_{i=1}^{\infty}\mu_i\Big[-\lambda_i V_i(\phi_i(s)) + \sum_{j\in I_i}\gamma_{ij}V_j(\phi_j(s)) + \gamma_{iu}|u_i(s)|^q\Big]\mathrm{d}s\\
&= \frac{1}{t}\int_0^t \Big[\mu^\top(-\Lambda+\Gamma)V_{\mathrm{vec}}(\phi(s)) + \sum_{i=1}^{\infty}\mu_i\gamma_{iu}|u_i(s)|^q\Big]\mathrm{d}s\\
&\leq \frac{1}{t}\int_0^t \Big[-\lambda_\infty V(\phi(s)) + \overline{\mu}\,\overline{\gamma}_u|u|^q_{q,\infty}\Big]\mathrm{d}s\\
&= \frac{1}{t}\int_0^t -\lambda_\infty V(\phi(s))\,\mathrm{d}s + \overline{\mu}\,\overline{\gamma}_u|u|^q_{q,\infty},
\end{aligned}$$

where we use (15) to show the second inequality above. Since $s \mapsto V(\phi(s))$ is continuous, one obtains

$$\begin{aligned}
\mathrm{D}^+V_u(x^0) &= \limsup_{t\to0+}\frac{1}{t}\big(V(\phi(t)) - V(x^0)\big)\\
&\leq -\lambda_\infty V(x^0) + \overline{\mu}\,\overline{\gamma}_u|u|^q_{q,\infty}.
\end{aligned}$$

Hence, (4b) holds for V with $\kappa = \lambda_\infty$ and $\gamma(r) = \overline{\mu}\,\overline{\gamma}_u r^q$.

Step 3 (Proof of eISS): We showed that properties (a)–(c) are satisfied for V. Thus, V is an eISS Lyapunov function for Σ and Σ is eISS by Proposition 4.3. Hence, for the case of uniformly upper-bounded λ_i the theorem is proved.

Step 4: If the decay rates λ_i are unbounded, one can use the previous arguments and invoke the upper-semicontinuity of the spectral radius to show the claim. □

7. EXAMPLE: A ROAD TRAFFIC MODEL

In this example, we apply our approach to a variant of the road traffic model from (de Wit et al., 2012). We consider a traffic network divided into infinitely many cells, indexed by $i \in \mathbb{N}$. Each cell i represents a subsystem Σ_i described by a differential equation of the following form

$$\Sigma_i : \dot{x}_i = -\Big(\frac{v_i}{l_i} + e_i\Big)x_i + D_i\bar{x}_i + B_iu_i, \quad x_i, u_i \in \mathbb{R}, \tag{16}$$

with the following structure

- $e_i = 0, D_i = c\frac{v_{i+1}}{l_{i+1}}, \bar{x}_i = x_{i+1}, B_i = 0$ if $i \in S_1 := \{1,3\}$;
- $e_i = 0, D_i = c\frac{v_{i+4}}{l_{i+4}}, \bar{x}_i = x_{i+4}, B_i = r > 0$ if $i \in S_2 := \{4+8c : c \in \mathbb{N}\cup\{0\}\}$;
- $e_i = 0, D_i = c\frac{v_{i-4}}{l_{i-4}}, \bar{x}_i = x_{i-4}, B_i = \frac{r}{2}$ if $i \in S_3 := \{5+8c : c \in \mathbb{N}\cup\{0\}\}$;
- $e_i = 0, D_i = c(\frac{v_{i-1}}{l_{i-1}}, \frac{v_{i+4}}{l_{i+4}})^\top, \bar{x}_i = (x_{i-1}, x_{i+4}), B_i = 0$ if $i \in S_4 := \{6+8c : c \in \mathbb{N}\cup\{0\}\}$;
- $e_i = e \in (0,1)$, $D_i = c(\frac{v_{i-4}}{l_{i-4}}, \frac{v_{i+1}}{l_{i+1}})^\top, \bar{x}_i = (x_{i-4}, x_{i+1}), B_i = 0$ if $i \in S_5 := \{9+8c : c \in \mathbb{N}\cup\{0\}\}$;
- $e_i = 0, D_i = c(\frac{v_{i+1}}{l_{i+1}}, \frac{v_{i+4}}{l_{i+4}})^\top, \bar{x}_i = (x_{i+1}, x_{i+4}), B_i = 0$ if $i \in S_6 := \{2+8c : c \in \mathbb{N}\cup\{0\}\}$;
- $e_i = 0, D_i = c(\frac{v_{i-4}}{l_{i-4}}, \frac{v_{i-1}}{l_{i-1}})^\top, \bar{x}_i = (x_{i-4}, x_{i-1}), B_i = 0$ if $i \in S_7 := \{7+8c : c \in \mathbb{N}\cup\{0\}\}$;
- $e_i = 2e, D_i = c(\frac{v_{i-1}}{l_{i-1}}, \frac{v_{i+4}}{l_{i+4}})^\top, \bar{x}_i = (x_{i-1}, x_{i+4}), B_i = 0$ if $i \in S_8 := \{8+8c : c \in \mathbb{N}\cup\{0\}\}$;
- $e_i = 0, D_i = c(\frac{v_{i-4}}{l_{i-4}}, \frac{v_{i+1}}{l_{i+1}})^\top, \bar{x}_i = (x_{i-4}, x_{i+1}), B_i = 0$ if $i \in S_9 := \{11+8c : c \in \mathbb{N}\cup\{0\}\}$;

where, for all $i \in \mathbb{N}$, $0 \leq v_i \leq \overline{v}$, $0 < \underline{l} \leq l_i \leq \overline{l}$, and $c \in (0, 0.5)$. In (16), l_i is the length of a cell in kilometers (km), and v_i is the flow speed of the vehicles in kilometers per hour (km/h). The state of each subsystem Σ_i, i.e. x_i, is the density of traffic, given in vehicles per cell, for each cell i of the road. The scalars B_i represent the number of vehicles that can enter the cells through entries that are controlled by u_i. In particular, $u_i = 1$ means green light and $u_i = 0$ means red light. Moreover, the constants e_i represent the percentage of vehicles that leave the cells using available exits.

It can be shown easily that the interconnected system Σ with state space $X := \ell^2(\mathbb{N},(n_i))$ and input space $U := \ell^2(\mathbb{N},(m_i))$ is well-posed.

Furthermore, each subsystem Σ_i admits an eISS Lyapunov function of the form $V_i(x_i) = \frac{1}{2}x_i^2$. The function V_i satisfies (5) and (6) for all $i \in \mathbb{N}$ with $\underline{\alpha}_i = \overline{\alpha}_i = \frac{1}{2}$, $\lambda_i = 2(\frac{v_i}{l_i} + e_i - 2\varepsilon_i)$, $\gamma_{ij} = \frac{\|cD_i\|^2}{2\varepsilon_i}$ for all $j \in I_i$, $\gamma_{iu} = \frac{B_i^2}{2\varepsilon_i}$, for an appropriate choice of $0 < \underline{\varepsilon} \leq \varepsilon_i \leq \overline{\varepsilon}$ such that $0 < \underline{\lambda} := 2\big(\underline{v}/\overline{l} - 2\overline{\varepsilon}\big) \leq \lambda_i$. In that way, one can readily observe that

$$0 < \gamma_{ij} \leq \frac{\overline{v}^2}{\underline{\varepsilon}\,\underline{l}^2} =: \overline{\gamma} < \infty, \quad 0 < \gamma_{iu} \leq \frac{r^2}{2\underline{\varepsilon}} =: \overline{\gamma}_u < \infty.$$

Additionally, the infinite matrix $\Psi := \Lambda^{-1}\Gamma = (\psi_{ij})_{i,j\in\mathbb{N}} = (\gamma_{ij}/\lambda_i)_{i,j\in\mathbb{N}}$, for Λ and Γ defined in (10), has the following structure.

- $i \in S_1 \Rightarrow (\gamma_{ij} \neq 0 \Leftrightarrow j = i+1)$;
- $i \in S_2 \Rightarrow (\gamma_{ij} \neq 0 \Leftrightarrow j = i+4)$;
- $i \in S_3 \Rightarrow (\gamma_{ij} \neq 0 \Leftrightarrow j = i-4)$;
- $i \in S_4 \Rightarrow (\gamma_{ij} \neq 0 \Leftrightarrow j \in \{i-1, i+4\})$;
- $i \in S_5 \Rightarrow (\gamma_{ij} \neq 0 \Leftrightarrow j \in \{i-4, i+1\})$;
- $i \in S_6 \Rightarrow (\gamma_{ij} \neq 0 \Leftrightarrow j \in \{i+1, i+4\})$;
- $i \in S_7 \Rightarrow (\gamma_{ij} \neq 0 \Leftrightarrow j \in \{i-4, i-1\})$;
- $i \in S_8 \Rightarrow (\gamma_{ij} \neq 0 \Leftrightarrow j \in \{i-1, i+4\})$;
- $i \in S_9 \Rightarrow (\gamma_{ij} \neq 0 \Leftrightarrow j \in \{i-4, i+1\})$.

The spectral radius $r(\Psi)$ can be estimated by

$$r(\Psi) \leq \|\Psi\| = \sup_{j\in\mathbb{N}} \sum_{i=1}^{\infty} \psi_{ij} \leq 2\frac{\overline{\gamma}}{\underline{\lambda}}.$$

Hence, any choice of the constants ε_i such that

$$(2(c\overline{v})^2/\underline{\varepsilon}\,\underline{l}^2)/((\underline{v}/\overline{l}) - 2\overline{\varepsilon}) < 1,$$

for all $i \in \mathbb{N}$, leads to $r(\Psi) < 1$.

Hence, by Theorem 6.2 there exists $\mu = (\mu_i)_{i\in\mathbb{N}} \in \ell^\infty$ satisfying $\underline{\mu} \leq \mu_i \leq \overline{\mu}$ with constants $\underline{\mu}, \overline{\mu} > 0$ such that the function $V(x) = \frac{1}{2}\sum_{i=1}^{\infty} \mu_i x_i^2$ is an eISS Lyapunov function for the interconnected system Σ.

8. CONCLUSIONS

In this paper, we developed sufficient small-gain type conditions for showing exponential ISS of networks consisting of countably infinite numbers of exponentially ISS subsystems, which are finite-dimensional themselves. The proposed small-gain conditions, expressed in terms of the spectral radius of the resulting gain operator, are tight (see Kawan et al. (2019) for a detailed tightness analysis) and can be effectively checked for large classes of systems. In the spirit of (Noroozi et al., 2018), we plan to investigate the necessity of the proposed small-gain conditions.

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