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On the Asymptotic Stability of Directed Nonlinear Multiagent Network via Nonlinear Control Protocol ⋆

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Abstract: In this paper, we investigate the asymptotic stability of nonlinear agents in a directed network using a nonlinear protocol. Inspired by the V-uniformly decreasing condition, we introduce a new condition to characterize the nonlinearity of the agents, and use it to design the distributed nonlinear control protocol for the agents. Under certain conditions, we construct proper Lyapunov function to show that the agents can achieve asymptotic stability via our nonlinear control protocol. Especially, if there exist agents that are asymptotically stable, then the multiagent network must be asymptotically stable as long as the control strengths of unstable agents are large enough. Finally, an example is given to illustrate the effectiveness of our theoretical results.

Keywords: Directed networks, nonlinear protocols, nonlinear agents, consensus.

1. INTRODUCTION

In recent years, the multiagent network has drawn attention from various disciplines of science and engineering due to its broad applications: cooperative control of autonomous robots Ren et al. (2007), flight control of unmanned air vehicles Pimenta et al. (2013), distributed constrained optimization Ozdaglar et al. (2010), etc.

For the multiagent network, one key problem is to design distributed control protocol such that all the agents achieve desired dynamical behaviors, such as stability Xiang et al. (2007); Zhao et al. (2011), consensus Han et al. (2014); Huang et al. (2015), and so on. Generally, the distributed control protocol depends on two parts: the dynamics of the isolated agents and the communication graph.

At the early stage, the emphasis of the multiagent network is on the communication graph rather than on the dynamics of the isolated agents: the agents exchange information according to a communication graph, it is the exchange of information only that determines the dynamical behaviors of the variables; in the absence of communication, the agents themselves usually have no dynamics Moreau (2004); Ren et al. (2008). In this case, in order to achieve certain dynamical behavior, such as consensus, it only requires a weak form of connectivity for the communication graph.

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With the progress of its research, the dynamics of the isolated agents are more and more considered: from linear agents Qu et al. (2008); Ren et al. (Tuna); Li et al. (2010) to nonlinear agents Sarlette et al. (2009); Chung et al. (2009); Yao et al. (2009); Liang et al. (2020), from homogeneous agents Seo et al. (2009); Lin et al. (2007); Scardovi et al. (2009) to heterogeneous agents Seyboth et al. (2015); Kim et al. (2011); Wieland et al. (2011). With linear vector field being a well researched area, this paper focuses on nonlinear agents and attempt to propose proper control protocol for them.

In the existing literature, the vector field f of the nonlinear isolated agent is typically required to satisfy the Lipschitz condition Liang et al. (2017b); Lu et al. (2006); Yu et al. (2009); Abhijit et al. (2010); Maurizio et al. (2008). Under this condition, researchers proposed various of the control protocols and most of them were supposed to be a linear function of the state of the isolated agent. Moreover, quadratic Lyapunov functions were constructed to ensure the stability or consensus for general undirected networks and directed networks.

However, there exist agents that do not satisfy the Lipschitz condition. Thus, it is interesting to generalize the Lipschitz condition. To the best of our knowledge, there are very fewer relevant results about this. Some researchers relaxed it to the V-uniformly decreasing condition Xiang et al. (2007); Zhao et al. (2011); Wu et al. (2009); Xiang et al. (2009) and then investigated the stability of complex dynamical networks. The work of Liang et al. (2017a) generalized the V-uniformly decreasing condition to a more general case, and then proposed a linear control protocol such that the network achieves consensus. Unfortunately, the above results hold for undirected networks and are not applicable for general directed networks. Part of the reason for this phenomenon is that the control protocols are linear functions of the states of the isolated agents.

In this paper, the asymptotic stability of nonlinear agents in a directed network is studied. Firstly, inspired by the V-uniformly decreasing condition, a new decreasing condition is introduced to characterize the nonlinearity of the agents. Based on this decreasing condition, we propose a nonlinear distributed control protocol for the agents with directed communication graph. Then, under certain conditions, we use proper Lyapunov function to prove the asymptotic stability of the multiagent network, composed of the agents and the distributed control protocol. Especially, if there exist agents that are asymptotically stable, then the multiagent network must be asymptotically stable as long as the control strengths of unstable agents are large enough. Finally, an example is given to illustrate the validity of our theoretical results.

The outline of the paper is organized as follows. In Section 2, we present some basic notations and problem formulation. In Section 3, the distributed control protocol of multiagent network is proposed. In Section 4, an example is presented with numerical simulations. The paper is concluded in Section 5.

2. PRELIMINARIES

2.1 Communication Graph

For a group of interconnected agents, communication graph is always represented by $G = \{V, E, A\}$, where $V = \{v_1, \cdots, v_N\}$ denotes the set of agents, $E = \{e_{ij} \in V \times V\}$ denotes the set of connection relationships, and $A = [a_{ij}]_{N\times N}$ denotes the adjacency matrix. The adjacency matrix A of G is defined as follows: $a_{ij} > 0$ if and only if $e_{ij} \in E$; otherwise, $a_{ij} = 0$. In this paper, we only consider the simple graph, i.e., it does not contain self-loops from an agent to itself and there is at most one relationship between agents. Thus, $a_{ii} = 0$ for $i = 1, \cdots, N$. The communication graph G is said to be undirected if for any $i, j = 1, \cdots, N$, we have $a_{ij} = a_{ji}$; otherwise G is said to be directed. As the matrix accompanying the adjacency matrix A, the Laplacian matrix $L = [l_{ij}]_{N\times N}$ is defined as follows:

(1) $l_{ij} = -a_{ij}, \forall i \neq j$;

(2) and $l_{ii} = \sum\limits_{j=1, j\neq i}^{N} a_{ij}, \forall i.$

A few related definitions and results for L are given below.

Definition 1. Horn et al. (1985) A matrix $L \in \mathbb{R}^{N\times N}$ is reducible if there is a permutation matrix $P \in \mathbb{R}^{N\times N}$ such that

$$P^T L P = \begin{pmatrix} B & C \\ 0_{N-r,N} & D \end{pmatrix}, \tag{1}$$

where $1 \le r \le N-1$, B, C and D are some appropriate matrices. A matrix $L \in \mathbb{R}^{N\times N}$ is irreducible if it is not reducible.

The following assumption will be used in this paper.

Assumption 1. Suppose that Laplacian matrix L is irreducible.

Lemma 1. Wu. (2007) If L is irreducible, then there exists $\xi = (\xi_1, \cdots, \xi_N)^T$ with $\xi_i > 0$ $(i = 1, \cdots, N)$ and $\sum\limits_{i=1}^{N} \xi_i = 1$ such that $\xi^T L = 0$ and $\hat{L} := (\Xi L + L^T \Xi)/2$ is an irreducible symmetric Laplacian matrix, where $\Xi = diag(\xi_1, \cdots, \xi_N)$.

Lemma 2. Horn et al. (1985) Suppose that $L_{N\times N}$ is irreducible, diagonally dominant, and there exists i such that $l_{ii} > \sum\limits_{j=1, j\neq i}^{N} l_{ij}$. If $l_{ii} > 0, \forall i$, then $\Re(\lambda_i) > 0$ for all eigenvalues λ_i of L, where $\Re(\lambda_i)$ denotes the real part of eigenvalue λ_i.

2.2 Problem statement

Consider N nonlinear heterogeneous agents as follows:

$$\dot{x}_i = f_i(x_i) + u_i, \quad i = 1, \cdots, N, \tag{2}$$

where $x_i = (x_{i1}, \cdots, x_{in})^T \in \mathbb{R}^n$ is the state of the i-th agent, $f_i = (f_{i1}, \cdots, f_{in})^T$ is the vector field of the i-th isolated agent in $\mathbb{R}^n$, u_i is the corresponding controller for the i-th isolated agent.

Assume that all the isolated agents have a common equilibrium state $\bar{x} \in \mathbb{R}^n$, satisfying

$$f_i(\bar{x}) = 0, i = 1, \cdots, N. \tag{3}$$

Also, for the isolated agents (2), let us introduce an assumption to characterize the nonlinearity of the agents.

Assumption 2. For the vector field f_i of the isolated agent, there exist a constant θ_i, and a continuously differentiable, radially unbounded positive definite function $V_i(\cdot) : \mathbb{R}^n \to [0, +\infty)$ with $V_i(0) = 0$ such that

$$\frac{\partial V_i(x-\bar{x})}{\partial x}[f_i(x) - \theta_i \frac{\partial V_i(x-\bar{x})}{\partial x}] < 0 \tag{4}$$

holds for $\forall x \in \mathbb{R}^n$ and $x \neq \bar{x}$.

Remark 1. When studying the stability problem of multiagent network, we always assume that there exists $1 \le i \le N$ such that θ_i in Assumption 2 is non-negative, even positive. If not, all the θ_i are negative; from (4), all the isolated agent themselves are asymptotically stable. For this case, the additional control protocol u_i is not necessary.

In this paper, our main objective is as follows.

Definition 2. Given N heterogeneous agents described by (2) and a communicaton graph G, find a distributed control protocol u_i such that for any $\varepsilon > 0$, there exist a $T > 0$, such that $\|x_i(t) - \bar{x}\| \le \varepsilon$ for any initial conditions and all $t > T$, $i, j = 1, 2, \cdots, N$.

3. DISTRIBUTED CONTROL PROTOCOL FOR DIRECTED MULTIAGENT NETWORK

In this section, we aim to propose control protocol u_i such that the solutions $x_i(t)$ of the agents (2) converge to $\bar{x}$ for any initial conditions.

For (2), it is obvious that the control protocol should relies heavily on the dynamics of the isolated agents. Based on Assumption 2, we propose a control protocol as follows:

$$u_i = -c_i \sum_{j=1}^{N} l_{ij} [\frac{\partial V_j(x_j - \bar{x})}{\partial x_j}]^T, \quad i = 1, \cdots, N, \tag{5}$$

where $c_i > 0$ represents the control strength.

Substituting protocol (5) into the agents (2), we have the following multiagent network

$$\dot{x}_i = f_i(x_i) - c_i \sum_{j=1}^{N} l_{ij} \frac{\partial V_j(x_j - \bar{x})}{\partial x_j}, \quad i = 1, \cdots, N. \tag{6}$$

Since V_i is positive definite, then 0 is the minimum point. Also because V_i is continuously differentiable, then $\frac{\partial V_j(x_j - \bar{x})}{\partial x_j}|_{x_j = \bar{x}} = 0$. Substituting $x_i = \bar{x}, i = 1, \cdots, N$, into (6), we find that $(\bar{x}, \cdots, \bar{x}) \in \mathbb{R}^{nN}$ is a equilibrium state of multiagent network (6), i.e., $\dot{x}_i|_{x_i = \bar{x}} = 0, i = 1, \cdots, N$. In order to prove that x_i converges to $\bar{x}$, we just need to prove that $(\bar{x}, \cdots, \bar{x}) \in \mathbb{R}^{nN}$ is an asymptotically stable equilibrium state of (6).

Now, let us prove asymptotic stability of $(\bar{x}, \cdots, \bar{x}) \in \mathbb{R}^{nN}$ in (6) via Lyapunov function method, the main result is summarized as follows.

Theorem 1. Suppose that Assumptions 1 and 2 hold. If

$$\Xi C^{-1} \Theta - \hat{L} < 0, \tag{7}$$

where $\Xi = diag\{\xi_1, \cdots, \xi_N\}$, $C = diag\{c_1, \cdots, c_N\}$, $\Theta = diag\{\theta_1, \cdots, \theta_N\}$, then $(\bar{x}, \cdots, \bar{x}) \in \mathbb{R}^{nN}$ is an asymptotically stable equilibrium state of (6).

Proof.

Let $\bar{V} = \sum_{i=1}^{N} \alpha_i V(x_i - \bar{x})$, where $\alpha_i = \frac{\xi_i}{c_i}, i = 1, \cdots, N$. Now let us prove that $\bar{V}$ is the desired Lyapunov function. Clearly, $\bar{V} = 0 \Leftrightarrow x_i = \bar{x}$. Differentiating $\bar{V}$ along the trajectory of network (2), we have

$$\begin{aligned} \dot{\bar{V}} =& \sum_{i=1}^{N} \alpha_i \frac{\partial V_i(x_i - \bar{x})}{\partial x_i} (\dot{x}_i - \dot{\bar{x}}) \\ =& \sum_{i=1}^{N} \alpha_i \frac{\partial V_i(x_i - \bar{x})}{\partial x_i} \dot{x}_i \\ =& \underbrace{\sum_{i=1}^{N} \alpha_i \frac{\partial V_i(x_i - \bar{x})}{\partial x_i} f(x_i)}_{(I)} \\ & \underbrace{- \sum_{i=1}^{N} c_i \alpha_i \frac{\partial V_i(x_i - \bar{x})}{\partial x_i} \sum_{j=1}^{N} l_{ij} [\frac{\partial V_j(x_j - \bar{x})}{\partial x_j}]^T}_{(II)}. \end{aligned} \tag{8}$$

For (I), based on Assumption 2, one has

$$\begin{aligned} & \sum_{i=1}^{N} \alpha_i \frac{\partial V_i(x_i - \bar{x})}{\partial x_i} f(x_i) \\ \leq& \alpha_i \frac{\partial V_i(x_i - \bar{x})}{\partial x_i} \theta_i [\frac{\partial V_i(x_i - \bar{x})}{\partial x_i}]^T \\ =& \frac{\partial V_i(x_i - \bar{x})}{\partial x_i} \frac{\xi_i}{c_i} \theta_i [\frac{\partial V_i(x_i - \bar{x})}{\partial x_i}]^T \\ =& \frac{\partial V}{\partial x} [(\Xi C^{-1} \Theta) \otimes I_n] [\frac{\partial V}{\partial x}]^T, \end{aligned} \tag{9}$$

where $\frac{\partial V}{\partial x} = (\frac{\partial V_1(x_1 - \bar{x})}{\partial x_1}, \cdots, \frac{\partial V_N(x_N - \bar{x})}{\partial x_N})$.

For (II), one has

$$\begin{aligned} & \sum_{i=1}^{N} c_i \alpha_i \frac{\partial V_i(x_i - \bar{x})}{\partial x_i} \sum_{j=1}^{N} l_{ij} [\frac{\partial V_j(x_j - \bar{x})}{\partial x_j}]^T \\ =& \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial V_i(x_i - \bar{x})}{\partial x_i} \xi_i l_{ij} [\frac{\partial V_j(x_j - \bar{x})}{\partial x_j}]^T \\ =& \frac{\partial V}{\partial x} [\hat{L} \otimes I_n] [\frac{\partial V}{\partial x}]^T. \end{aligned} \tag{10}$$

From (I) and (II), one has

$$\begin{aligned} \dot{\bar{V}} =& \frac{\partial V}{\partial x} [(\Xi C^{-1} \Theta) \otimes I_n] [\frac{\partial V}{\partial x}]^T - \frac{\partial V}{\partial x} [\hat{L} \otimes I_n] [\frac{\partial V}{\partial x}]^T \\ =& \frac{\partial V}{\partial x} [(\Xi C^{-1} \Theta - \hat{L}) \otimes I_n] [\frac{\partial V}{\partial x}]^T \\ \leq& 0. \end{aligned}$$

Based on the above, $\bar{V}$ is a Lyapunov function, and then the proof is completed.

Remark 2. It should be pointed out that Theorem 1 can also be used to check the asymptotic stability of multiagent systems with non-autonomous agents $f(t, x)$. In fact, if we modify $f(x)$ in Assumption 2 to $f(t, x)$, Theorem 1 also holds.

Remark 3. There are two significant differences between our results and the results in Xiang et al. (2007): on one hand, the assumptions about the nonlinearity of the agents are different, the nonlinear agents in Xiang et al. (2007) are stabilizable via linear feedback but the nonlinear agents in this paper are stabilizable via nonllinear feedback; on the other hand, the result in Xiang et al. (2007) is valid only for undirected network while Theorem 1 is valid for both undirected network and directed network.

Theorem 1 show that if expression (7) holds, then multi-agent network (6) is asymptotically stable. Note that expression (7) is composed of Θ (determined by the isolated agents), C (determined by the control strengths c_i), Ξ (determined by communication graph) and $\hat{L}$ (determined by communication graph). Thus, the asymptotic stability multiagent network (6) is influenced by three factors: the isolated agents, the control strengths and the communication graph.

For multiagent network (6), a further question is as follows: given the communication graph, what kind of agents can be asymptotically stable by choosing proper control strengths $c_i, i = 1, \cdots, N$? The following result offers an answer for this question.

Corollary 1. Suppose that the conditions in Theorem 1 hold. Additionally, if Θ is not positive semi-definite, then for any Laplacian matrix L, there exists matrix C such that $(\bar{x}, \cdots, \bar{x}) \in \mathbb{R}^{nN}$ is an asymptotically stable equilibrium state of (6).

Proof. According to Theorem 1, we just to need to prove that $\Xi C^{-1}\Theta - \hat{L} < 0$.

Since Θ is not positive semi-definite, then there exists i such that $\theta < 0$. Without loss of generality, suppose that $\theta_i < 0, i = 1, \cdots, k$; $\theta_i \geq 0, i = k+1, \cdots, N$. Let $\Theta_1 = diag\{\theta_1, \cdots, \theta_k, 0 \cdots, 0\}$ be an $N \times N$ matrix and $\Theta_2 = \Theta - \Theta_1$, then Θ_1 is negative definite, Θ_2 is non-negative definite.

Since L is irreducible, then Ξ is a positive definite diagonal matrix. Note that Θ_1 is a negative semi-definite diagonal matrix, C^{-1} is a positive definite diagonal matrix, then $\Xi C^{-1}\Theta_1$ is a negative semi-definite diagonal matrix. Since L is an irreducible Laplacian matrix, then $\hat{L}$ is an irreducible, diagonally dominant matrix. Moreover, $-(\Theta_1 - \hat{L})$ is irreducible, diagonally dominant, and there exists a row such that this row is strictly diagonally dominant. By Lemma 2, $-(\Xi C^{-1}\Theta_1 - \hat{L})$ is positive definite and then $\Xi C^{-1}\Theta_1 - \hat{L}$ is negative definite.

Clearly, $\Xi C^{-1}\Theta - \hat{L} = \Xi C^{-1}\Theta_1 - \hat{L} + \Xi C^{-1}\Theta_2$. For given $c_i, i = 1, \cdots, k$, let $c_i, i = k+1, \cdots, N$, be large enough, then $\frac{1}{c_i}, i = k+1, \cdots, N$, are small enough. For this case, $\Xi C^{-1}\Theta_2$ also is small enough and then $\Xi C^{-1}\Theta - \hat{L}$ is negative definite.

As a result, for given L, suppose that there exists negative definite matrix Θ_1, if we choose large enough $c_i (i = k+1, \cdots, N)$, then $\Xi C^{-1}\Theta - \hat{L}$ must be negative definite.

This completes the proof.

Remark 4. In the proof of Corollary 1, we not only prove the existence of c_i (or matrix C), but also provide a way to find it. Suppose that $\theta_i < 0, i = 1, \cdots, k$; $\theta_i \geq 0, i = k+1, \cdots, N$. Under the conditions of Corollary 1, for given control strengths $c_i, i = 1, \cdots, k$, $(\bar{x}, \cdots, \bar{x}) \in \mathbb{R}^{nN}$ is an asymptotically stable equilibrium state of (6) if the rest of the control strength $c_i, i = k+1, \cdots, N$ are large enough.

4. AN EXAMPLE WITH NUMERICAL SIMULATIONS

In this section, an example with numerical simulations is presented to illustrate the applicability of our theoretical results obtained in Section 3.

Example 1. Consider system (2) with 3 agents, where the agents are described by: $f_1(a_1, a_2) = f_2(a_1, a_2) = \begin{cases} a_1 - a_2^3 \\ a_1 + (1 - \dfrac{a_2^2}{3})a_2 \end{cases}$, $f_3(a_1, a_2) = \begin{cases} -a_1 + a_2 \\ 2a_1 - 3a_2 \end{cases}$, $a_1, a_2 \in \mathbb{R}$, Laplacian matrix $L = \begin{pmatrix} 4 & -2 & -2 \\ -1 & 2 & -1 \\ -3 & 0 & 3 \end{pmatrix}$. Clearly, $\bar{x} = (0, 0)$, L is irreducible and then Assumption 1 holds.

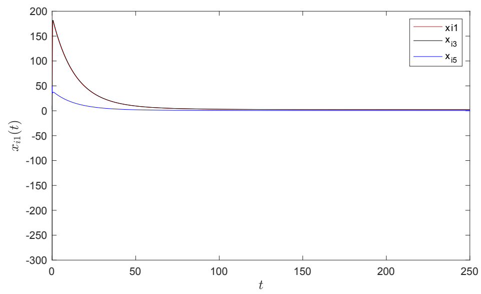
Firstly, let us show that $f_i, i = 1, 2, 3$, satisfy Assumption 2. For f_1, f_2, let $V_1(a_1, a_2) = V_2(a_1, a_2) = 0.5 * a_1^2 + 0.5 * a_2^2 + 0.25 * a_2^4$, $\theta_1 = \theta_2 = 1.5$, then it is easy to verify that Assumption 2 holds. For f_3, let $V_3(a_1, a_2) = \frac{7}{4} * a_1^2 + \frac{5}{4} * a_1 * a_2 + \frac{3}{8} * a_2^2$, $\theta_3 = -\frac{1}{4}$, then Assumption 2 also holds.

Next, based on the above discussion, the control protocol is given as follows: $u_i = -c_i \sum_{j=1}^{N} l_{ij} [\frac{\partial V_j(x_j)}{\partial x_j}]^T, \quad i = 1, 2, 3,$ where $\frac{\partial V_1(x_1)}{\partial x_1}|_{x_1=(a_1,a_2)} = \frac{\partial V_2(x_2)}{\partial x_2}|_{x_2=(a_1,a_2)} = (a_1, a_2 + a_2^3)$, $\frac{\partial V_3(x_3)}{\partial x_3}|_{x_3=(a_1,a_2)} = (\frac{7}{2} * a_1 + \frac{5}{4} * a_2, \frac{5}{4} * a_1 + \frac{3}{4} * a_2)$, and $c_i, i = 1, 2, 3$, are undetermined control strengths.

Finally, let us show that $\Xi C^{-1}\Theta - \hat{L} < 0$. From L, we have $\Xi = \frac{1}{3} diag\{1, 1, 1\}$, $\hat{L} = \frac{1}{2} \begin{pmatrix} 8 & -3 & -5 \\ -3 & 4 & -1 \\ -5 & -1 & 6 \end{pmatrix}$. Moreover,

$$\Xi C^{-1}\Theta - \hat{L} = \frac{1}{2} \left[\frac{2}{3} \begin{pmatrix} \dfrac{3}{2c_1} & 0 & 0 \\ 0 & \dfrac{3}{2c_2} & 0 \\ 0 & 0 & -\dfrac{1}{4c_3} \end{pmatrix} - \begin{pmatrix} 8 & -3 & -5 \\ -3 & 4 & -1 \\ -5 & -1 & 6 \end{pmatrix} \right].$$

Without loss of generality, let $c_3 = \frac{1}{4}$. If $c_1 = c_2 > 9$, then $\Xi C^{-1}\Theta - \hat{L} < 0$, which means that $(\bar{x}, \cdots, \bar{x}) \in \mathbb{R}^{nN}$ is an asymptotically stable equilibrium state of (6). Figure 1 also show this fact via the numerical simulation.



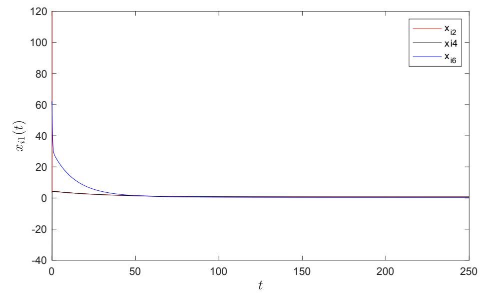


Fig. 1. State evolutions of jth-component of $\mathbf{x}_i(t)$ in Example (1), where $i = 1, 2, 3$, $j = 1, 2$, initial condition: $\mathbf{x}_1(0) = (10, 120), \mathbf{x}_2(0) = (-300, -40), \mathbf{x}_3(0) = (50, 60)$.

5. CONCLUSION

In this brief, we have investigated the asymptotic stability problem of nonlinear agents in a directed networks. Based on the Lipschitz condition, we introduce an new assumption on the nonlinearity of the agents. With the aid of this assumption, a distributed nonlinear control protocol is proposed for asymptotic stability. We found that if there exists agent that is asymptotically stable, then the multiagent network must be asymptotically stable as long as the control strengths of unstable agents are large enough.

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