Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

On the impact of edge modifications for networked control systems ⋆

Gustav Lindmark * **Claudio Altafini** *

* *Division of Automatic Control, Dept. of Electrical Engineering, Linköping University, SE-58183, Linköping, Sweden. email:* `gustav.lindmark@liu.se, claudio.altafini@liu.se`

Abstract: This paper investigates the impact of addition/removal of edges in complex networks. Growing a network by the addition of edges has for instance been suggested as a way to improve network robustness to external disturbances. Moreover, when network controllability is considered, designing edge additions is a promising alternative to add more actuation capabilities in order to improve different performance metrics. We quantify the impact of an edge modification with the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms. For networks with positive edge weights we show how the $\mathcal{H}_\infty$ norm can be computed exactly for each possible single edge modification, while for the $\mathcal{H}_2$ norm we instead obtain a lower bound. This bound is linked to the trace of the controllability Gramian, hence it can be used for instance to reduce the energy needed for control.

Keywords: Complex networks, Network topology design, Controllability, Positive systems.

1. INTRODUCTION

In recent years, complex networks have been studied from the perspectives of classical concepts like controllability, observability, robustness and more. In this context, the considered networks are influenced by external processes such as known or unknown disturbances or external control inputs, entering the networks through *input nodes*. Depending on the application, it might also be meaningful to specify *output nodes* when only the states of a subset of the nodes is observed. This could be e.g. the workload of security critical servers in a computer network, or the traffic flow along certain roads in a traffic network.

When network controllability in the binary yes/no sense is studied, the notion of structural controllability (Lin, 1974; Pichai et al., 1981; Chen et al., 2018) has proven to be very useful as it maps algebraic controllability conditions to graphical ones. Most recent contributions in the field however focus on quantitative metrics for the possibility to control a network in practice, often energy-related metrics based on the controllability Gramian (see among others Pasqualetti et al. (2014); Yan et al. (2015); Lindmark and Altafini (2018); Chen et al. (2016)). Several methods have been suggested that consider how to place a limited number of control input actuators in the network in such a way to optimize (or improve) a given metric for control energy (Pasqualetti et al., 2014; Tzoumas et al., 2016; Lindmark and Altafini, 2018).

Another way to improve the controllability of a network is by edge addition/removal/re-weighting. This approach is promising, considering the significant impact that network topology has on control performance (see for instance Parlangeli and Notarstefano (2011); Bianchin et al. (2015)). Edge modifications are often feasible in applications and correspond to changes in e.g. connectivity of computer servers or traffic routing.

When structural controllability is considered, graph-theoretic procedures can be used to identify a minimum number of edge additions that render a network controllable (Pichai et al., 1981; Chen et al., 2018). Optimizing Gramian-based controllability metrics by edge modifications is however more difficult. Even when comparing edge modifications and control input placement as two different means to improve such metrics, edge modifications are generally more difficult. This comes from the fact that the controllability Gramian as a function of control inputs (columns in the input matrix B) is simpler than as a function of the edges (elements in state update matrix A). There are however a few studies in this direction: For a given budget of edges and weights that can be added, Chanekar et al. (2019) apply differential analysis for maximization of the trace of the Gramian control energy metric. In Becker et al. (2017), re-weighting of existing edges is applied in order to reduce the worst case control energy as measured by the minimal eigenvalue of the Gramian.

Edge addition in consensus networks has received more attention, see e.g. Hagberg and Schult (2008); Summers et al. (2015); Hassan-Moghaddam et al. (2017); Zhang et al. (2017); Siami and Motee (2017). In this context, the focus is often on network robustness to external disturbances.

In this paper we do *not* approach network controllability or robustness directly, instead we investigate how edge-modifications alter the state evolution of the network. For a given network with input and output nodes (in theory all nodes can be both input and output nodes) we derive a transfer function formulation for the changes in output caused by edge-modifications. The formulation is such that

⋆ Work supported in part by a grant from the Swedish Research Council (grant n. 2015-04390 to C.A.).

it can be applied in a large scale setting where each of the $n(n-1)$ possible edges (n is the total number of nodes) are considered for modification.

Networks with exclusively positive edge weights appear in various areas. For this important class we use established theory on positive systems together with our transfer function formulation of the effects of edge modifications, to derive several new results: i) We show what is the maximal weight by which an edge can be modified without causing instability; ii) We derive an analytical expression for the $\mathcal{H}_\infty$ norm of the transfer function of the difference in output due to the edge addition, and a lower bound for the $\mathcal{H}_2$ norm, both usable in a large scale network setting. These norms can be interpreted as measures of the extent that an edge modification impacts the network. While a large impact might represent a large risk in some situations, it could represent an opportunity in other cases. For instance, the mentioned $\mathcal{H}_2$ norm has a simple relation to the trace of the controllability Gramian. Hence, adding the edges corresponding to the largest $\mathcal{H}_2$ norm is a way to reduce the energy needed for control. Also edge modifications with small impact can represent opportunities. For instance in transportation or engineering networks, removing edges with small impact can reduce maintenance costs without putting the network functionality at risk.

The rest of the paper is organized as follows. In Section 2, definitions are given and the network model is presented. In Section 3, the changes in system properties of the network due to edge modifications are modeled. The main contributions of this paper are reported in Section 4. Finally, some applications of the results are presented in Section 5.

2. PRELIMINARIES

2.1 Notation

We denote by $\mathbb{R}^{n\times m}$ the set of $n \times m$ matrices with real valued entries. $\mathbb{R}^+$ is the set of non-negative real numbers, $\mathbb{N}$ the set of natural numbers and $\mathbb{N}_0$ the set of natural numbers including zero. Given a matrix $M \in \mathbb{R}^{n\times m}$, let $M_{ij},\ i \in 1,\dots,n, j \in 1,\dots,m$, denote the element on row i and column j. We use $\bar{\sigma}(M)$ for the maximal singular value of M. For M and N two matrices of the same dimension, $M \geq N$ should be interpreted element-vise, i.e. $M_{ij} \geq N_{ij}\ \forall i,j$. The spectral radius of the square matrix $M \in \mathbb{R}^{n\times n}$ is denoted by $\rho(M)$. The k-th vector of the canonical basis of $\mathbb{R}^n$ is denoted $e_k,\ k \in 1,\dots,n$. A (directed) graph $\mathcal{G}$ is indicated by the pair of its nodes and edges, $\mathcal{V} = \{1,\dots,n\}$ and $\mathcal{E} = \{(i,j),\ i,j \in 1,\dots,n\}$, or, if it is necessary to specify the edge weights, by the adjacency matrix A, i.e., $\mathcal{G} = \mathcal{G}(A)$. Then, the weight associated with the edge from node i to node j, (i,j), is A_{ji}. A path in $\mathcal{G}$ is a subgraph of the form $\mathcal{V}^* = \{i_1,\dots,i_j\}$ and with the edges $\mathcal{E}^* = \{(i_1,i_2),\dots,(i_{j-1},i_j)\}$. The path is directed from i_1 to i_j. The cardinality of the set $\mathcal{S}$ is denoted by $|\mathcal{S}|$. For $\mathcal{S} = \{i_1,\dots,i_j\} \subseteq \{1,\dots,n\}$ we define $E_{\mathcal{S}} = [e_{i_1}\ \dots\ e_{i_j}] \in \mathbb{R}^{n\times|\mathcal{S}|}$. For the vector $z \in \mathbb{R}^n$, $|z| = \sqrt{z^\top z}$ is its Euclidean norm. Given an input-output system G we use $||G||_{\mathcal{H}_2}$ and $||G||_{\mathcal{H}_\infty}$ for its $\mathcal{H}_2$ resp. $\mathcal{H}_\infty$ norms.

2.2 Network model

Consider a network represented by the directed graph $\mathcal{G}(A) = (\mathcal{V},\mathcal{E})$, where $A \in \mathbb{R}^{n\times n}$ is the weighted adjacency matrix. Each external input is assumed to act only on one node which is then called an *input node*. The set of input nodes is $\mathcal{K} \subseteq \mathcal{V}$, $|\mathcal{K}| = n_{\mathcal{K}}$. Similarly, the *output nodes* are given by the set $\mathcal{O} \subseteq \mathcal{V}$, $|\mathcal{O}| = n_{\mathcal{O}}$. Observe that $\mathcal{K} = \mathcal{V}$ and/or $\mathcal{O} = \mathcal{V}$ is possible. We consider the following discrete-time, linear, time-invariant model of the network dynamics

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the network at time $t \in \mathbb{N}_0$. $B = E_{\mathcal{K}} \in \mathbb{R}^{n\times n_{\mathcal{K}}}$ is the input matrix, $u(t) \in \mathbb{R}^{n_{\mathcal{K}}}$ is the input vector, $C = E_{\mathcal{O}}^\top \in \mathbb{R}^{n_{\mathcal{O}}\times n}$ is the output matrix, and $y(t) \in \mathbb{R}^{n_{\mathcal{O}}}$ the output vector.

In this paper we sometimes need to consider input/output relations between other sets of nodes than $\mathcal{K}$ and $\mathcal{O}$. To make the presentation clear, we introduce the following notation: For two sets of nodes, $\mathcal{S}_1 \subseteq \mathcal{V}$ and $\mathcal{S}_2 \subseteq \mathcal{V}$, the transfer function from inputs applied to $\mathcal{S}_1$ to the states of $\mathcal{S}_2$ (intended as output nodes) is denoted by

$$G^{(A)}_{\mathcal{S}_2\mathcal{S}_1} = \left[\begin{array}{c|c} A & E_{\mathcal{S}_1} \\ \hline E_{\mathcal{S}_2}^\top & 0 \end{array}\right]. \tag{2}$$

Moreover, the impulse response is

$$g^{(A)}_{\mathcal{S}_2\mathcal{S}_1}(t) = E_{\mathcal{S}_2}^\top A^t E_{\mathcal{S}_1},\ t \in \mathbb{N}_0. \tag{3}$$

With this notation we can write the system (1) as

$$y(t) = G^{(A)}_{\mathcal{O}\mathcal{K}} u(t).$$

All the networks considered in this paper have positive edge weights.

Definition 1. The linear system (A,B,C) is said to be *externally positive* if its forced output is non-negative for every non-negative input function. It is said to be *positive* if for every non-negative initial state and for every non-negative input, both its state and outputs are non-negative.

Clearly, positivity implies external positivity. A necessary and sufficient condition for (A,B,C) being externally positive is that the impulse response is non-negative. Moreover, (A,B,C) is positive if and only if $A \geq 0$, $B \geq 0$ and $C \geq 0$ (Farina and Rinaldi, 2011).

2.3 System norms and network centrality measures

Internal stability of a system on the form (1) holds true if $\rho(A) < 1$. Unless stated otherwise, in this paper we always consider internal stability.

For an arbitrary stable discrete-time LTI system $G = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ with impulse response $g(t) = CA^{t-1}B \in \mathbb{R}^{n_{\mathcal{O}}\times\ n_{\mathcal{K}}},\ t \in \mathbb{N}$ (no direct term) the $\mathcal{H}_2$ norm is given by

$$||G||^2_{\mathcal{H}_2} = \sum_{t=1}^{\infty} \sum_{\substack{i=1,\dots,n_{\mathcal{K}},\\ j=1,\dots,n_{\mathcal{O}}}} (g_{ji}(t))^2 = \sum_{t=1}^{\infty} \operatorname{Tr} g^\top(t) g(t), \tag{4}$$

which can be equivalently written as

$$||G||^2_{\mathcal{H}_2} = \mathrm{Tr}\left(CWC^\top\right), \text{ where}$$
$$W = \sum_{\tau=0}^{\infty} A^\tau BB^\top (A^\top)^\tau$$

is the (infinite time) controllability Gramian.

For network systems of the form (2)-(3), the expression (4) can be rewritten as follows. First, with the single input node $i \in \mathcal{V}$ and the single output node $j \in \mathcal{V}$,

$$||G^{(A)}_{ji}||^2_{\mathcal{H}_2} = \sum_{t=1}^{\infty} \left(e_j^\top A^{t-1} e_i\right)^2 = \sum_{\tau=0}^{\infty} ((A^\tau)_{ji})^2 := \varepsilon_{i\to j}.$$

The quantity $\varepsilon_{i\to j} \geq 0$ was introduced in Lindmark and Altafini (2019) and referred to as the *walk energy* from node i to node j. In the multiple-input-multiple-output (MIMO) case, with $\mathcal{S}_1,\ \mathcal{S}_2 \subseteq \mathcal{V}$ arbitrary, straightforward calculations give

$$||G^{(A)}_{\mathcal{S}_2\mathcal{S}_1}||^2_{\mathcal{H}_2} = \sum_{\substack{\forall i\in\mathcal{S}_1,\\ \forall j\in\mathcal{S}_2}} ||G^{(A)}_{ji}||^2_{\mathcal{H}_2}. \tag{5}$$

The following two network centrality metrics are variants of the ones proposed in Lindmark and Altafini (2019):

Definition 2. The *input-to-node* resp. *node-to-output* network centralities are given by

$$q_i = ||G^{(A)}_{i\mathcal{K}}||^2_{\mathcal{H}_2} = \sum_{\forall j\in\mathcal{K}} \varepsilon_{j\to i},\ i\in\mathcal{V},$$
$$p_i = ||G^{(A)}_{\mathcal{O}i}||^2_{\mathcal{H}_2} = \sum_{\forall j\in\mathcal{O}} \varepsilon_{i\to j},\ i\in\mathcal{V}.$$

The walk energies and the input-to-node resp. node-to-output centrality measures can be computed for all nodes even in large scale networks by solving Lyapunov equations.

The $\mathcal{H}_\infty$ norm of an LTI system is induced by the $\mathcal{L}_2$ signal norm,

$$||G||_{\mathcal{H}_\infty} = \sup_{u(t)} \frac{||Gu(t)||_{\mathcal{L}_2}}{||u(t)||_{\mathcal{L}_2}} = \sup_{\theta}\ \bar{\sigma}(G(\theta)), \tag{6}$$

where $G(\theta),\ \theta \in [-\pi,\pi]$ is the frequency function. While the norm cannot usually be computed directly for LTI systems (rather, one has to test if $||G||_\infty < \gamma$ for some $\gamma > 0$), for positive systems the following proposition can be used:

Proposition 1. (Tanaka and Langbort (2011)). Let $G(\theta)$ be the frequency function of a stable externally positive LTI system. Then

(1) $G(0) \geq 0$,
(2) $||G||_{\mathcal{H}_\infty} = \bar{\sigma}(G(0))$.

That is, the $\mathcal{H}_\infty$ norm of a stable externally positive system coincides with the spectral norm of the steady state transfer function. For the network model (2), with $A \geq 0,\ \rho(A) < 1$ and the sets $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{V}$, the steady state transfer function is

$$G^{(A)}_{\mathcal{S}_2\mathcal{S}_1}(0) = E^\top_{\mathcal{S}_2}(I-A)^{-1}E_{\mathcal{S}_1},$$

which makes exact computation of the $\mathcal{H}_\infty$ norm easy.

The following result will be used to determine internal stability of positive systems.

Proposition 2. (Farina and Rinaldi, 2011) For $A \geq 0$, $(I - A)^{-1}$ exist and is non-negative if and only if $\rho(A) < 1$.

3. THE DELTA SYSTEM

Consider a network given by A and the sets $\mathcal{K}$ and $\mathcal{O}$. Assume that the edge $(s,\mathfrak{t})$, $s,\mathfrak{t} \in \mathcal{V}$, is modified with the weight w such that the adjacency matrix of the modified network becomes

$$\bar{A} = A + e_\mathfrak{t} w e_s^\top. \tag{7}$$

We always assume that $w \geq -A_{\mathfrak{t}s}$ such that $\bar{A} \geq 0$ (i.e. the modification preserves network positivity). In the following, we use the triplet $(s,\mathfrak{t},w)$ to identify the modification.

Denote $y(t) = G^{(A)}_{\mathcal{OK}}u(t)$ and $\bar{y}(t) = G^{(\bar{A})}_{\mathcal{OK}}u(t)$ the outputs of the networks associated to A and $\bar{A}$. For a given input sequence $u(t),\ t = 0,1,\ldots,$ the difference

$$y^\delta(t) = \bar{y}(t) - y(t) = \left(G^{(\bar{A})}_{\mathcal{OK}} - G^{(A)}_{\mathcal{OK}}\right)u(t) \tag{8}$$

is the change in the states of the output nodes due to the edge modification (7). The corresponding transfer function,

$$G^\delta = G^{(\bar{A})}_{\mathcal{OK}} - G^{(A)}_{\mathcal{OK}}, \tag{9}$$

is from now on referred to as the *delta system.*

Proposition 3. Consider a network with adjacency matrix $A \geq 0$, sets $\mathcal{K}$, $\mathcal{O}$, and a modification $(s,\mathfrak{t},w)$. If the weight $w > 0$, then G^δ is externally positive. If instead $-A_{\mathfrak{t}s} \leq w < 0$, then $-G^\delta$ is externally positive.

Proof. With a non-negative input sequence $u(t)$,

$$w > 0 \Rightarrow \bar{A} \geq A \geq 0 \Rightarrow \bar{A}^t \geq A^t \geq 0\ \forall t\in\mathbb{N}_0$$
$$\Rightarrow y^\delta(t) = G^\delta u(t) = C(\bar{A}^t - A^t)Bu(t) \geq 0\ \forall t \in \mathbb{N}_0,$$

i.e. the output of G^δ is non-negative. Hence, G^δ is externally positive by definition.

On the other hand, choosing w s.t. $-A_{\mathfrak{t}s} < w < 0$ means reducing the weight of the existing edge $(s,\mathfrak{t})$, and choosing $w = -A_{\mathfrak{t}s}$ means completely removing it. With $u(t)$ non-negative, we obtain

$$A \geq \bar{A} \geq 0 \Rightarrow -y^\delta(t) = -G^\delta u(t) \geq 0\ \forall t \in \mathbb{N}_0,$$

i.e. $-G^\delta$ is externally positive. ∎

We seek an expression for the delta system that depends explicitly on $s,\mathfrak{t}$ and w, but not on $\bar{A}$.

Proposition 4. It holds

$$G^\delta = G^{(A)}_{\mathcal{O}\mathfrak{t}} G^\delta_c G^{(A)}_{s\mathcal{K}}, \text{ where} \tag{10a}$$
$$G^\delta_c = \left(1 - wG^{(A)}_{s\mathfrak{t}}\right)^{-1} w. \tag{10b}$$

Proof. We have

$$G^\delta = \left[\begin{array}{c|c}\bar{A} & B\\ \hline C & 0\end{array}\right] - \left[\begin{array}{c|c}A & B\\ \hline C & 0\end{array}\right] = \left[\begin{array}{cc|c}\bar{A} & 0 & B\\ 0 & A & B\\ \hline C & -C & 0\end{array}\right].$$

The formulation above corresponds to the state vector $[\bar{x}^\top x^\top]^\top$, where $\bar{x}$ is the state of the modified network and x that of the original network. Define the state transformation

$$\begin{bmatrix}\tilde{x}\\ x\end{bmatrix} = \begin{bmatrix}I & -I\\ 0 & I\end{bmatrix}\begin{bmatrix}\bar{x}\\ x\end{bmatrix}, \text{ with inverse } \begin{bmatrix}\bar{x}\\ x\end{bmatrix} = \begin{bmatrix}I & I\\ 0 & I\end{bmatrix}\begin{bmatrix}\tilde{x}\\ x\end{bmatrix}.$$

Changing basis,

$$\left[\begin{array}{cc|c} I & -I & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & I \end{array}\right] \left[\begin{array}{cc|c} \bar{A} & 0 & B \\ 0 & A & B \\ \hline C & -C & 0 \end{array}\right] \left[\begin{array}{cc|c} I & I & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & I \end{array}\right]$$

$$= \left[\begin{array}{cc|c} \bar{A} & e_{\mathfrak{t}} w e_s^\top & 0 \\ 0 & A & B \\ \hline C & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} \bar{A} & e_{\mathfrak{t}} \\ \hline C & 0 \end{array}\right] w \left[\begin{array}{c|c} A & B \\ \hline e_s^\top & 0 \end{array}\right],$$

from which

$$G^\delta = G^{(\bar{A})}_{\mathcal{O}\mathfrak{t}} w G^{(A)}_{s\mathcal{K}}. \tag{11}$$

Analogous calculations give

$G^{(\bar{A})}_{\mathcal{O}\mathfrak{t}} - G^{(A)}_{\mathcal{O}\mathfrak{t}} = G^{(\bar{A})}_{\mathcal{O}\mathfrak{t}} w G^{(A)}_{s\mathfrak{t}} \Leftrightarrow G^{(\bar{A})}_{\mathcal{O}\mathfrak{t}} = G^{(A)}_{\mathcal{O}\mathfrak{t}} (1 - w G^{(A)}_{s\mathfrak{t}})^{-1}$
which together with (11) gives (10). ■

Considering (8), it is the input-output relation of G^δ that is of interest.

4. LARGE SCALE ANALYSIS OVER ALL EDGES

With the delta system it is possible to characterize exactly the implications of a specific edge modification $(s, \mathfrak{t}, w)$. In this section we present results for the case when each of the $n(n-1)$ possible edges $(s, \mathfrak{t})$ in a positive large scale network have to be considered for modification. The results are made possible by the the fact that the formulation (10) does not depend explicitly on $\bar{A}$. This allows computationally heavy operations to be performed only once, and to reuse the results for the analysis of each single edge.

4.1 Stability bounds and the $\mathcal{H}_\infty$ norm of G^δ

We begin with a result that concern the stability of the modified network. Theorem 1 below follows from Son and Hinrichsen (1996, Theorem 5), a more general result on internal stability of positive systems subject to parameter perturbations. We provide a proof here since Son and Hinrichsen (1996) consider continuous time systems and a more involved problem formulation than we need.

Theorem 1. Consider a network with adjacency matrix $A \geq 0$, $\rho(A) < 1$, and the edge modification $(s, \mathfrak{t}, w)$, $w > 0$. If the original network has no path $\mathfrak{t} \to s$, then the modified network is internally stable for any $w > 0$. On the other hand, if there is a path $\mathfrak{t} \to s$, then the modified network is internally stable if and only if $0 \leq w < 1/\left((I - A)^{-1}\right)_{s\mathfrak{t}}$.

Proof. First note that for $A \geq 0$, $\rho(A) < 1$,

$$e_s^\top (I - A)^{-1} e_{\mathfrak{t}} = e_s^\top \left(I + A + A^2 + \dots\right) e_{\mathfrak{t}}$$
$$\begin{cases} = 0 \text{ if } \nexists \text{ a path } \mathfrak{t} \to s, \\ > 0 \text{ if } \exists \text{ a path } \mathfrak{t} \to s. \end{cases}$$

By Proposition 2, $\rho(\bar{A}) < 1$ holds true if and only if $(I - \bar{A})^{-1} \geq 0$. We will show that this is the case for the weights specified by the theorem.

Sufficiency: Using the matrix inversion lemma (Horn and Johnson, 2012), we obtain

$$(I - \bar{A})^{-1} = (I - A - e_{\mathfrak{t}} w e_s^\top)^{-1}$$
$$= (I - A)^{-1} + \frac{(I - A)^{-1} e_{\mathfrak{t}} w e_s^\top (I - A)^{-1}}{1 - w e_s^\top (I - A)^{-1} e_{\mathfrak{t}}}. \tag{12}$$

Given that the matrix $(I - A)^{-1} \geq 0$, the expression (12) is non-negative if the denominator is s.t.

$$1 - w e_s^\top (I - A)^{-1} e_{\mathfrak{t}} > 0.$$

This is true for any $w > 0$ if $\nexists$ a path $\mathfrak{t} \to s$, and for $0 < w < 1/\left((I - A)^{-1}\right)_{s\mathfrak{t}}$ if $\exists$ a path $\mathfrak{t} \to s$.

Necessity: Assume that $\exists$ a path $\mathfrak{t} \to s$ and

$$w > 1/\left((I - A)^{-1}\right)_{s\mathfrak{t}}$$
$$\Rightarrow 0 > 1 - w\left((I - A)^{-1}\right)_{s\mathfrak{t}} > -w\left((I - A)^{-1}\right)_{s\mathfrak{t}}$$
$$\Rightarrow \frac{1}{1 - w\left((I - A)^{-1}\right)_{s\mathfrak{t}}} < \frac{1}{-w\left((I - A)^{-1}\right)_{s\mathfrak{t}}}.$$

The element on row s and column $\mathfrak{t}$ of (12) is

$$((I - \bar{A})^{-1})_{s\mathfrak{t}}$$
$$= \left((I - A)^{-1}\right)_{s\mathfrak{t}} + \frac{\left((I - A)^{-1}\right)_{s\mathfrak{t}} w \left((I - A)^{-1}\right)_{s\mathfrak{t}}}{1 - w\left((I - A)^{-1}\right)_{s\mathfrak{t}}}$$
$$< \left((I - A)^{-1}\right)_{s\mathfrak{t}} + \frac{\left((I - A)^{-1}\right)_{s\mathfrak{t}} w \left((I - A)^{-1}\right)_{s\mathfrak{t}}}{-w\left((I - A)^{-1}\right)_{s\mathfrak{t}}}$$
$$= 0,$$

i.e. it does *not* hold $(I - \bar{A})^{-1} \geq 0$, hence from Proposition 2 the modified network is *not* stable. Finally, with $w = 1/\left((I - A)^{-1}\right)_{s\mathfrak{t}}$, the inverse $(I - \bar{A})$ does not exist. This case corresponds to $\bar{A}$ marginally stable. ■

When the conditions on the weight w of Theorem 1 are met, then both $G^{(A)}_{\mathcal{O}\mathcal{K}}$ and $G^{(\bar{A})}_{\mathcal{O}\mathcal{K}}$ are stable. Hence, also $G^\delta = G^{(\bar{A})}_{\mathcal{O}\mathcal{K}} - G^{(A)}_{\mathcal{O}\mathcal{K}}$ is stable and $||G^\delta||_{\mathcal{H}_2}$, $||G^\delta||_{\mathcal{H}_\infty}$ are bounded.

We can interpret Theorem 1 in terms of cycles in the network: $G^{(A)}_{s\mathfrak{t}} > 0$ if there is a path $\mathfrak{t} \to s$. This path forms a cycle with the modified edge $(s, \mathfrak{t}, w)$. As a consequence, only edge additions that create new cycles, or edge modifications that increase the weight of an existing edge that is part of a cycle, may cause instability. Edge removal or reduction of the weight of an existing edge in a positive network will on the other hand never cause instability (Farina and Rinaldi, 2011, p. 43). To see this, consider (7) with $-A_{\mathfrak{t}s} \leq w < 0$ implying that $A \geq \bar{A} \geq 0$. Let $\bar{x}_e(t)$ resp. $x_e(t)$ denote the free motion of the modified resp. original network, then with $\bar{x}_e(0) = x_e(0)$ it follows that $x_e(t) \geq \bar{x}_e(t) \geq 0$ $\forall t$.

The $\mathcal{H}_\infty$ norm of G^δ can be computed exactly.

Theorem 2. Consider a network with adjacency matrix $A \geq 0$, $\rho(A) < 1$ and the sets $\mathcal{K}$, $\mathcal{O}$. For the edge modification $(s, \mathfrak{t}, w)$, $-A_{\mathfrak{t}s} \leq w < 1/((I - A)^{-1})_{s\mathfrak{t}}$, it holds

$$||G^\delta||_{\mathcal{H}_\infty} = \frac{\sqrt{\sum_{o \in \mathcal{O}} \left((I - A)^{-1}\right)^2_{o\mathfrak{t}}}\, |w| \sqrt{\sum_{k \in \mathcal{K}} \left((I - A)^{-1}\right)^2_{sk}}}{1 - \left((I - A)^{-1}\right)_{s\mathfrak{t}} w} \tag{13}$$

Proof. The condition $-A_{\mathfrak{t}s} \leq w < 1/((I - A)^{-1})_{s\mathfrak{t}}$ implies that $\bar{A} \geq 0$, $\rho(\bar{A}) < 1$ and the norm $||G^\delta||_{\mathcal{H}_\infty}$ is bounded.

For positive weights, $0 < w < 1/((I - A)^{-1})_{s\mathfrak{t}}$, G^δ is externally positive, hence $||G^\delta||_{\mathcal{H}_2} = \bar{\sigma}(G^\delta(0))$. Notice that

$G^\delta(0)$ is a rank one matrix. We can write

$$G^\delta(0) = G^{(A)}_{\mathcal{O}\mathrm{t}}(0) G^\delta_c(0) G^{(A)}_{s\mathcal{K}}(0)$$
$$= \frac{G^{(A)}_{\mathcal{O}\mathrm{t}}(0)\mathrm{sign}(G^\delta_c(0))}{|G^{(A)}_{\mathcal{O}\mathrm{t}}(0)|} |G^{(A)}_{\mathcal{O}\mathrm{t}}(0)|\ |G^\delta_c(0)|\ |G^{(A)}_{s\mathcal{K}}(0)|\ \frac{G^{(A)}_{s\mathcal{K}}(0)}{|G^{(A)}_{s\mathcal{K}}(0)|}, \quad (14)$$

where $\mathrm{sign}(\cdot)$ is the signum function. The equation (14) is a singular value decomposition since $G^{(A)}_{s\mathcal{K}}(0)/|G^{(A)}_{s\mathcal{K}}(0)|$ and $G^{(A)}_{\mathcal{O}\mathrm{t}}(0)\mathrm{sign}(G^\delta_c(0))/|G^{(A)}_{\mathcal{O}\mathrm{t}}(0)|$ are unit row resp. column vectors. By identification, the positive scalar $|G^{(A)}_{\mathcal{O}\mathrm{t}}(0)|\ |G^\delta_c(0)|\ |G^{(A)}_{s\mathcal{K}}(0)|$ is the only (hence the maximal) singular value. Evaluating it gives (13).

For $-A_{\mathrm{t}s} \le w < 0$, instead $-G^\delta$ is externally positive. In this case, replace $G^\delta_c(0)$ with $-G^\delta_c(0)$ in equation (14) to obtain a singular value decomposition of $-G^\delta(0)$. Then use $||G^\delta||_{\mathcal{H}_\infty} = ||-G^\delta||_{\mathcal{H}_\infty} = \bar{\sigma}(-G^\delta(0))$. ■

The computational complexity in evaluating the stability bounds of Theorem 1 and $||G^\delta||_{\mathcal{H}_\infty}$ lies in the matrix inversion. This however has to be done only once for all possible edge modifications (s, t, w).

4.2 The $\mathcal{H}_2$ norm of G^δ

The next lemma establishes two properties of positive systems that will be used to bound $||G^\delta||_{\mathcal{H}_2}$.

Lemma 1. Let G and H be two externally positive systems with impulse responses $g(t)$ resp. $h(t)$. Assuming matching input/output dimensions, the following hold,

P1. $||G+H||^2_{\mathcal{H}_2} \ge ||G||^2_{\mathcal{H}_2} + ||H||^2_{\mathcal{H}_2}$.
P2. For G a multiple input single output (MISO) system and H a single input multiple output (SIMO) system, $||HG||_{\mathcal{H}_2} \ge ||H||_{\mathcal{H}_2}||G||_{\mathcal{H}_2}$.

Proof. Because of space constraints we only outline how the lemma can be proved. For P1, use (4) and the fact that both $g(t),\ h(t) \ge 0\ \forall t$. External positivity is also required for P2, which can be proved when both G and H are SISO systems by substituting the convolution formula for the impulse response of HG into (4) and applying appropriate changes of variables. For $G = [G_1\ \dots\ G_{n_\mathcal{K}}]$ and $H = [H_1\ \dots\ H_{n_\mathcal{O}}]^\top$, the result is then obtained from the SISO result and using (5), noting that the matrix element $(HG)_{ji} = H_j G_i$. ■

Remark 1. Notice that P2 does not hold in general for G and H positive multiple-input-multiple-output (MIMO) systems. For instance, with $G = H$ both given by $[y_1(t)\ y_2(t)]^\top = [u_1(t-1)\ u_2(t-1)]^\top$, it is $||HG||^2_{\mathcal{H}_2} = 2 < ||H||^2_{\mathcal{H}_2}||G||^2_{\mathcal{H}_2} = 4$.

Theorem 3. Consider a network with adjacency matrix $A \ge 0$, $\rho(A) < 1$ and the sets $\mathcal{K}$, $\mathcal{O}$. For the edge modification (s, t, w), $-A_{\mathrm{t}s} \le w < 1/((I-A)^{-1})_{s\mathrm{t}}$, the $\mathcal{H}_2$-norm of the delta system is bounded by

$$||G^\delta||^2_{\mathcal{H}_2} \ge p_\mathrm{t}\ \frac{w^2}{1-\varepsilon_{\mathrm{t}\to s} w^2}\ q_s. \quad (15)$$

Proof. The conditions

$$\begin{cases} 1 > w((I-A)^{-1})_{s\mathrm{t}} = w\left(I + A + A^2 + \dots\right)_{s\mathrm{t}}, \\ w > 0, \\ A \ge 0 \end{cases}$$

imply

$$\begin{aligned} 1 &> w^2\left(\left(I + A + A^2 + \dots\right)_{s\mathrm{t}}\right)^2 \\ &\ge w^2\left((I_{s\mathrm{t}})^2 + (A_{s\mathrm{t}})^2 + ((A^2)_{s\mathrm{t}})^2 + \dots\right) \\ &= w^2 \varepsilon_{\mathrm{t}\to s}. \end{aligned}$$

It follows from Definition 1 that externally positive systems in series or in parallel constitute an externally positive system. Hence we conclude that the feedback loop

$$G^\delta_c = w\left(1 + G^{(A)}_{s\mathrm{t}} w + (G^{(A)}_{s\mathrm{t}} w)(G^{(A)}_{s\mathrm{t}} w) + \dots\right)$$

is externally positive, and

$$\begin{aligned} ||G^\delta_c||^2_{\mathcal{H}_2} &= \left|\left| w\left(1 + G^{(A)}_{s\mathrm{t}} w + (G^{(A)}_{s\mathrm{t}} w)(G^{(A)}_{s\mathrm{t}} w) + \dots\right)\right|\right|^2_{\mathcal{H}_2} \\ &\ge w^2\left(1 + ||G^{(A)}_{s\mathrm{t}} w||^2_{\mathcal{H}_2} + ||G^{(A)}_{s\mathrm{t}} w G^{(A)}_{s\mathrm{t}} w||^2_{\mathcal{H}_2} + \dots\right) \\ &\ge w^2\left(1 + ||G^{(A)}_{s\mathrm{t}} w||^2_{\mathcal{H}_2} + ||G^{(A)}_{s\mathrm{t}} w||^2_{\mathcal{H}_2}||G^{(A)}_{s\mathrm{t}} w||^2_{\mathcal{H}_2} + \dots\right) \\ &= \frac{w^2}{1-\varepsilon_{\mathrm{t}\to s} w^2}, \end{aligned}$$

where the properties P1 and P2 are used in the first resp. second inequality. We also use $||G^{(A)}_{s\mathrm{t}} w||^2_{\mathcal{H}_2} = \varepsilon_{\mathrm{t}\to s} w^2 < 1$ and geometric series. Finally, since $G^\delta = G^{(A)}_{\mathcal{O}\mathrm{t}} G^\delta_c G^{(A)}_{s\mathcal{K}}$, i.e. three positive systems in series, we can apply Property P2 to obtain

$$||G^\delta||^2_{\mathcal{H}_2} \ge \left|\left|G^{(A)}_{\mathcal{O}\mathrm{t}}\right|\right|^2_{\mathcal{H}_2} ||G^\delta_c||^2_{\mathcal{H}_2} \left|\left|G^{(A)}_{s\mathcal{K}}\right|\right|^2_{\mathcal{H}_2} = \frac{p_\mathrm{t} w^2 q_s}{1-\varepsilon_{\mathrm{t}\to s} w^2}.$$

■

5. APPLICATIONS

5.1 Edge modifications and the degree of controllability

Edge modifications can be used as a mean to improve the degree of controllability of a network (Becker et al., 2017; Chanekar et al., 2019). One metric for the energy needed for control is $\mathrm{Tr}(W)$, or $\mathrm{Tr}(CWC^\top)$ when only the states of certain output nodes are considered. In this context, equation (15) provides a lower bound on the increment that the edge addition (s, t, w), $0 < w < 1/((I-A)^{-1})_{s\mathrm{t}}$, gives to the trace of the Gramian of a positive network. With W the controllability Gramian for (A, B) and $\bar{W}$ the controllability Gramian for $(\bar{A}, B)$, it is

$$\begin{aligned} \mathrm{Tr}\left(C\bar{W}C^\top\right) &= ||G^{(\bar{A})}_{\mathcal{O}\mathcal{K}}||^2_{\mathcal{H}_2} = ||G^{(A)}_{\mathcal{O}\mathcal{K}} + G^\delta||^2_{\mathcal{H}_2} \\ &\ge ||G^{(A)}_{\mathcal{O}\mathcal{K}}||^2_{\mathcal{H}_2} + ||G^\delta||^2_{\mathcal{H}_2} \\ &\ge \mathrm{Tr}\left(CWC^\top\right) + \frac{p_\mathrm{t} w^2 q_s}{1-\varepsilon_{\mathrm{t}\to s} w^2}. \end{aligned}$$

One way to improve $\mathrm{Tr}\left(CWC^\top\right)$ is therefore to make edge modifications in a greedy manner, choosing the edges that correspond to the largest bounds (15).

5.2 Numerical results

To illustrate the metrics for the impact of edge modifications, we compute them here for a random Erdős–Rényi network with 500 nodes. Edges are generated with probability 0.02 and with weights that are first sampled from the uniform distribution over]0 1], and then rescaled such that the prespecified spectral radius $\rho(A) = 0.9$ is obtained.

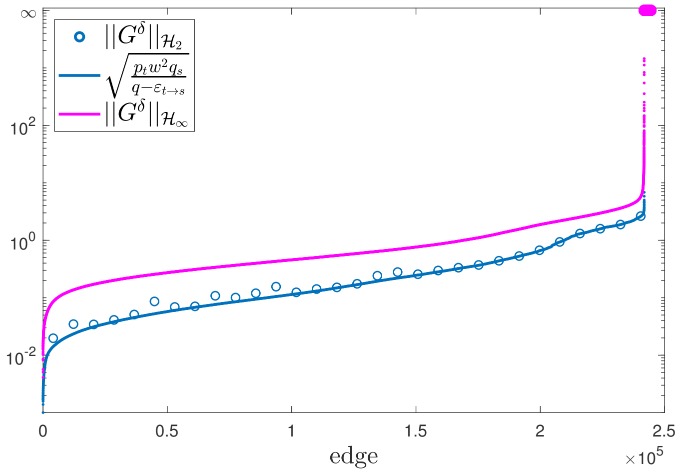

Fig. 1. Numerical computations on a random Erdős–Rényi network with 500 nodes. For all $s, \mathfrak{t} \in \mathcal{V}$, $s \neq \mathfrak{t}$ and weight $w = 10$, the plots show the norm $||G^\delta||_{\mathcal{H}_\infty}$ and the lower bound of $||G^\delta||_{\mathcal{H}_2}$. Edges are ordered along the x-axis in ascending order in both cases. For a subselection of 30 edges, also the exact value of $||G^\delta||_{\mathcal{H}_2}$ has been computed.

(Hence, the network is positive and stable.) 50 input nodes and 100 output nodes are randomly selected.

The metric $||G^\delta||_{\mathcal{H}_\infty}$ is computed and plotted in Figure 1 for each single edge modification $(s, \mathfrak{t}, w)$, $s, \mathfrak{t} \in \mathcal{V}$, $s \neq \mathfrak{t}$ and $w = 10$. Some modifications result in instability and $||G^\delta||_{\mathcal{H}_\infty}$ unbounded in accordance with Theorem 1, these correspond to the marks at the top right corner of the figure. The lower bound (15) on $||G^\delta||_{\mathcal{H}_2}$ is also plotted for each possible single edge modification. For comparison, the exact values are computed for a few edges using standard Matlab routines. (This is however infeasible to do for all edges due to the computational cost.) It appears that the bound is consistently close to the exact value in the cases where it has been computed.

6. CONCLUSIONS

The particular structure of the transfer function G^δ that we derive for the changes in network output due to an edge modification enables us to quantify the impact of each possible edge modification in a large scale network. Whether we use the $\mathcal{H}_2$ norm or the $\mathcal{H}_\infty$ norm as metric, the impact from modifying the edge $(s, \mathfrak{t})$ depends on three network properties: the strength of the connections from (i) the input nodes to s, (ii) from $\mathfrak{t}$ to the output nodes, and (iii) feedback connections from $\mathfrak{t}$ to s. As a possible application of our results, we show how the proposed $\mathcal{H}_2$ metric can be used to design edge modifications that improve the trace of the controllability Gramian control energy metric. We intend to further explore other ways in which our results can be used to improve different metrics for the degree of controllability. Moreover, the stability margins we present for networks subject to edge modifications has possible applications to network robustness/fragility. In this case, an interesting extension of our results would be for instance to consider the stability margins when $k \in 2, 3, \ldots$ arbitrary edges are allowed to be modified.

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