Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Robustness to delay mismatch in consensus control under undirected graphs ⋆

Ran Huang *, **Zhengtao Ding** **

* *College of Information Science & Technology, Beijing University of Chemical Technology, Beijing, China(e-mail:huangran@mail.buct.edu.cn).*
** *Department of Electrical and Electronic Engineering, The University of Manchester, Manchester, UK(e-mail:zhengtao.ding@manchester.ac.uk)*

Abstract: This paper studies predictor-based adaptive consensus control of network-connected systems with unknown delays under undirected graphs. The approach is based on the representation of the delay state as a transport partial differential equation (PDE) and the utilization of a nonadaptive estimation delay state. Using the relative information of neighboring nodes, we propose a fully distributed adaptive consensus protocol, which is proven to achieve global consensus provided that the delay mismatch is within a small region. Simulation results performed on a group of neutrally stable systems are presented to illustrate the effectiveness of the proposed scheme.

Keywords: Consensus control, adaptive control, unknown delays, predictor-based feedback

1. INTRODUCTION

Consensus control has been an active area of research during the past two decades. A limited list of applications covers distributed optimization Li et al. (2018), distributed microgrids Zhao and Ding (2018), cooperative surveillance Ren et al. (2007), etc. Fruitful achievements have been made on consensus control particularly in the absence of time-delays in the control community; see Olfati-Saber and Murray (2004); Hong et al. (2008); Li et al. (2013); Ding (2014, 2015a,b), and the reference therein.

Time-delays are among the most common dynamic phenomena that arise in engineering practice Richard (2003). To prevent loss of stability, research effort has been devoted to consensus control problems in the presence of time-delays. The early results concentrate on consensus issues of single or high-order integrators subject to sensor delays, and can be viewed as passive design method since there is no delay compensation, see e.g., Olfati-Saber and Murray (2004); Sun and Wang (2009); Wang et al. (2013); Tian and Zhang (2012). Basically, the passive design method can be viewed as seeking the maximal allowable delay with a prescribed graph and protocol. However, the control design without delay compensation can only tolerate relatively short delays by frequency domain analysis theory. Owing to the successful application to stabilization of a single dynamic system with input delays, the predictor feedback design technique has been employed in consensus control design Wang et al. (2015); Zhou and Lin (2014); Wang et al. (2017), and the references therein. For instance, based on the reduction method Kwon and Pearson (1980), Artstein (1982), a consensus protocol was proposed for networked Lipschitz nonlinear systems with actuator delay in Wang et al. (2015), where sufficient convergence condition was derived in terms of LMIs. In Zhou and Lin (2014), a truncated predictor feedback (TPF) approach, originated from the finite spectrum assignment Manitius and Olbrot (1979), was proposed to solve the consensus problem for linear multi-agent systems with arbitrarily bounded delays. More recently, consensus of networked Lipschitz nonlinear systems based on TPF has also been studied in Wang et al. (2017). In spite of the progress, it should be noted that the aforementioned delay compensation results basically require the delay to be exactly known and suffer from being very sensitive to delay mismatch, which limits their application to engineering practice. Therefore, extra effort is needed to explore delay compensation for consensus control systems subject to large actuator and sensor delays of uncertain length.

In this paper, motivated by Krstic (2008a) and Bresch-Pietri and Krstic (2010) we study the predictor feedback consensus control of networked-connected systems with unknown time-delays via adaptive protocols. The control objective is to achieve consensus despite actuator and sensor delays of uncertain bounded length. Some features of the control scheme developed in this paper are highlighted as follows. The first feature is that the proposed protocol can be implemented in a fully distributed fashion, and the interaction of control input among nodes is avoided. The second feature is that one can seek an explicit bound on the delay mismatch that ensures asymptotic convergence in an appropriate norm.

The remainder of the paper is organized as follows. The problem is formulated in Section 2. Section 3 gives some

⋆ This work is supported in part by National Natural Science Foundation of China under Grants 91748102 and U1813220, by the Fundamental Research Funds for the Central Universities under Grants XK1802-4 and PYBZ1837, and by the CSC scholarship.

preliminary results. In Section 4, we give the predictor feedback adaptive consensus protocol design and convergence analysis. Simulation results are presented in Section 5 to validate the effectiveness of the proposed scheme, followed by the conclusion in Section 6.

Notations: $|\cdot|$ denotes the Euclidean norm, $\otimes$ denotes the Kronecker product, $\mathbb{N}$ refers to the set $\{1,\ldots,N\}$, $\mathbf{1}_N$ stands for a column vector filled with ones, I_N denotes the identity matrix of size N, $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimum and maximum eigenvalues of a square matrix respectively. For a smooth vector function $u(x,t)$, $\|u(t)\| = (\int_0^1 |u(x,t)|^2 dx)^{1/2}$ denotes the $\mathcal{L}_2[0,1]$ norm, and $\|u(t)\|_{\mathcal{H}_1} = \sqrt{\|u(t)\|^2 + \|u_x(t)\|^2}$ denotes the $\mathcal{H}_1[0,1]$ Sobolev norm, and $u_x(x,t)$, $u_t(x,t)$ respectively denote the partial differentiation with respect to x and t. For a series of vectors $X_i(t)$, $i \in \mathbb{N}$, $X(t)$ denotes the stacked vector $[X_1^T(t),\ldots,X_N^T(t)]^T$.

2. PROBLEM FORMULATION

Let the connection topology among the N nodes be specified by an undirected graph $\mathcal{G}(\mathcal{V},\mathcal{E},\mathcal{A})$, where $\mathcal{A} = [\delta_{ij}] \in \mathbb{R}^{N\times N}$ is the adjacency matrix with elements δ_{ij} denoting the connections such that $\delta_{ij} = \delta_{ji} = 1$ if there is a path between nodes i and j, and $\delta_{ij} = 0$ otherwise. Associated with $\mathcal{A}$, the Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N\times N}$ is defined in the normal way as $l_{ii} = \sum_{j=1}^N \delta_{ij}$ and $l_{ij} = -\delta_{ij}$ when $i \neq j$.

The dynamics of node i, $i \in \mathbb{N}$ is described by

$$\dot{X}_i(t) = AX_i(t) + BU_i(t - D_1), \tag{1}$$

$$Z_i(t) = \sum_{j=1}^N l_{ij} X_j(t - D_2), \tag{2}$$

where $X_i(t) \in \mathbb{R}^n$ is the state, $U_i(t) \in \mathbb{R}^m$ is the control input, A and B are system matrices with compatible dimensions, D_1 and D_2 are unknown actuator and sensor delays, respectively, and $Z_i(t)$ is the information measured by node i. Let $D = D_1 + D_2$ denote the total unknown constant delay.

Assumption 1. The Laplician matrix L of the connection graph has a single eigenvalue at 0.

Assumption 2. The pair (A,B) is stabilizable.

Assumption 3. There exists a positive constant $\bar{D}$ such that $D \in (0,\bar{D}]$

For node i, the available information only includes $Z_i(t)$, the input information $U_i(t)$ of itself, and a virtual filter signal transmitted by its neighbours. The objective is to design a fully distributed protocol $U_i(Z_i(t))$, $i \in \mathbb{N}$ with an estimate of D to achieve leaderless consensus without any global information, i.e.,

$$\lim_{t\to+\infty} X_i(t) - X_j(t) = 0, \forall i,j \in \mathbb{N}. \tag{3}$$

3. PRELIMINARY RESULTS

Motivated by Krstic (2008b) and Krstic and Smyshlyaev (2008), we use a transport PDE to model the delay state

$$\dot{X}_i(t) = AX_i(t) + Bu^i(x_0,t), \tag{4}$$

$$Du_t^i(x,t) = u_x^i(x,t), x \in (0,1) \tag{5}$$

$$u^i(1,t) = U_i(t) \tag{6}$$

with $x_0 = \frac{D_2}{D}$. Let $\hat{u}^i(x,t)$, $i \in \mathbb{N}$ be the estimate of delay state $u^i(x,t)$, which obeys the following transport PDE

$$\begin{cases} \hat{D}\hat{u}_t^i(x,t) = \hat{u}_x^i(x,t), x \in (0,1) \\ \quad \hat{u}^i(1,t) = U_i(t), \end{cases} \tag{7}$$

where $\hat{D}$ is an estimate of the total delay D. Given an estimate $\hat{D}$, denote the delay mismatch by $\tilde{D} = D - \hat{D}$, and define the delay state error $\tilde{u}^i(x,t)$ by

$$\tilde{u}^i(x,t) = u^i(x,t) - \hat{u}^i(x,t). \tag{8}$$

Remark 4. The idea of modelling the delay state by a transport PDE was first proposed in Krstic and Smyshlyaev (2008). The substantial advantage is that with such a respresentation, the backstepping method for PDEs developed in Krstic and Smyshlyaev (2008) can be used to construct a Lyapunov function for stability analysis. The corresponding results for a single dynamic system were summarized in the work Krstic (2009).

Lemma 5. (*Cauchy-Schwartz inequality*) For any two compatible vector functions $u(x,t), w(x,t) \in \mathcal{L}_2[0,1]$, it holds

$$\left(\int_0^x u^T(y,t)w(y,t)dy\right)^2 \le \int_0^x |u(y,t)|^2 dy \int_0^x |w(y,t)|^2 dy$$

for any $x \in [0,1]$.

In what follows, we give a lemma that establishes the stability result concerning a first-order hyperbolic PDE with vanishing perturbation.

Lemma 6. Let $M(x) \in \mathbb{R}^{m\times n}$ be twice differentiable with $|M(x)|$ bounded $\forall x \in [0,1]$, and $\zeta(t) \in \mathbb{R}^n$ be first-order differentiable and bounded. Consider the first-order hyperbolic PDE

$$\begin{cases} Dw_t(x,t) = w_x(x,t) + M(x)\zeta(t), x \in (0,1) \\ \quad w(1,t) = Y\zeta(t), \end{cases} \tag{9}$$

with D a positive constant and $Y \in \mathbb{R}^{m\times n}$ a constant matrix. If $\lim_{t\to+\infty} \zeta(t) = 0$, then it holds that $\lim_{t\to+\infty} \|w(t)\| = 0$ and $\lim_{t\to+\infty} w(x,t) = 0$ for any $x \in [0,1)$.

Proof. Consider a Lyapunov candidate function as

$$V(t) = D\int_0^1 (1+x)|w(x,t)|^2 dx,$$

whose derivative is given

$$\begin{aligned} \dot{V}(t) = &-|w(0,t)|^2 - \|w(t)\|^2 + 2|Y\zeta(t)|^2 \\ &+ 2\int_0^1 (1+x)w^T(x,t)M(x)\zeta(t)dx. \end{aligned}$$

Let $g(x) = (\int_0^1 |(1+x)M(x)|^2 dx)^{\frac{1}{2}}$. Using Lemma 5 and $D\|w(t)\|^2 \le V(t) \le 2D\|w(t)\|^2$, we have

$$\dot{V}(t) \le -\frac{1}{2D}V(t) + \frac{2}{\sqrt{D}}|\zeta(t)|g(x)\sqrt{V(t)} + 2|Y\zeta(t)|^2$$

Note that $\lim_{t\to+\infty} |\zeta(t)| = 0$ and $g(x)$ is bounded for any $x \in [0,1]$. According to Lemma 4.7 in Khalil (2002), the following system

$$\dot{\bar{V}}(t) = -\frac{1}{2D}\bar{V}(t) + \frac{2}{\sqrt{D}}|\zeta(t)|g(x)\sqrt{\bar{V}(t)} + 2|Y\zeta(t)|^2$$

with initial condition $\bar{V}(0) > 0$ is globally uniformly asymptotically stable. It thus follows from the comparison principle that $\lim_{t\to+\infty} V(t) = 0$ and $\lim_{t\to+\infty} \|w(t)\| = 0$. Since $w(x,t)$ is differentiable with respect to x, $x \in (0,1)$, we have $\lim_{t\to+\infty} w(x,t) = 0$, $\forall x \in [0,1)$. This completes the proof.

4. ADAPTIVE CONSENSUS CONTROL WITH NONADAPTIVE PREDICTORS

In this section, we propose an adaptive consensus control design on the basis of predictor feedback.

4.1 Adaptive consensus protocol design

Introduce a virtual filter for node i as

$$\dot{\xi}_i(t) = A\xi_i(t) + h_i(t)F\check{\chi}_i(t) + B\hat{u}^i(0,t), \tag{10}$$

where $\check{\chi}_i(t) = \sum_{j=1}^N l_{ij}\chi_j(t) = \sum_{j=1}^N l_{ij}\xi_j(t) - Z_i(t)$ with

$$\chi_j(t) = \xi_j(t) - X_j(t - D_2), j \in \mathbb{N}$$

the filter gain F is a negative definite matrix satisfying

$$A^T F^{-1} + F^{-1}A + I_n > 0, \tag{11}$$

and the adaptive gain $h_i(t)$ is updated by

$$\dot{h}_i(t) = \begin{cases} d_i|\check{\chi}_i(t)|^2, & \text{if } h_i(t) < \bar{h}, \\ 0, & \text{else} \end{cases} \tag{12}$$

with d_i a positive scalar and initial condition $h_i(0) < \bar{h}$. Define an auxiliary variable $G_i(x,t)$, $x \in [0,1]$, $i \in \mathbb{N}$ by

$$G_i(x,t) = \int_0^x e^{A\hat{D}(x-y)} B\hat{u}^i(y,t)dy. \tag{13}$$

Based on the virtual filter, for an estimate $\hat{D}$ we propose the following node-based adaptive consensus protocol

$$U_i(t) = Ke^{A\hat{D}}\xi_i(t) + \hat{D}KG_i(1,t), i \in \mathbb{N} \tag{14}$$

where K is chosen to make $A_K = A+BK$ Hurwitz, $G_i(1,t)$ is defined in (13), and $\xi_i(t)$ is the filter signal. In view of (10) and (13), we introduce the following distributed spatially causal state transformation

$$\hat{w}^i(x,t) = \hat{u}^i(x,t) - Ke^{A\hat{D}x}\xi_i(t) - \hat{D}KG_i(x,t). \tag{15}$$

Let

$$\rho_i(t) = [X_i^T(t - D_2) \quad \chi_i^T(t)]^T,$$

then we have [1]

$$\dot{\rho}_i = A_s\rho_i + h_i(t)F_s\check{\rho}_i + B_s\hat{w}^i(0,t) + B_{s_1}\tilde{u}^i(0,t). \tag{16}$$

where $\check{\rho}_i(t) = \sum_{j=1}^N l_{ij}\rho_j(t)$, $B_s = [B^T \;\; 0_{m\times n}]^T$, and

$$A_s = \begin{bmatrix} A_K & BK \\ 0_n & A \end{bmatrix}, F_s = \begin{bmatrix} 0_n & 0_n \\ 0_n & F \end{bmatrix}, B_{s_1} = \begin{bmatrix} B \\ -B \end{bmatrix},$$

Remark 7. For a network of N nodes, let $\mathcal{S}$ denote the set of nonzero eigenvalues of the Laplician matrices associated with all connected undirected graphs. Set $\bar{h}$ such that $2\bar{h}\varrho > 1$ with ϱ given by

$$\varrho = \min\{\lambda|\lambda \in \mathcal{S}\}. \tag{17}$$

It is observed that ϱ only depends on the number of nodes in the network N, and N can be derived by each node via discrete-time average consensus protocol proposed in Olfati-Saber and Murray (2004). Particularly, we can directly set $\bar{h}$ sufficiently large. Therefore, the adaptive law (12) is implementable without using global information.

[1] In what follows, the time variable t will be occasionally suppressed to save space.

By noting $\mathbf{1}_N$ is the left eigenvalue of L corresponding to the eigenvalue at 0, we consider a state transformation

$$\varphi(t) = (M \otimes I_{2n})\rho(t), \tag{18}$$

where $M = I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}^T$, and $\varphi(t) = [\varphi_1^T(t), \ldots, \varphi_N^T(t)]^T$ with $\varphi_i(t) = \rho_i(t) - \frac{1}{N}\sum_{j=1}^N \rho_j(t)$. Note that the rank of M is $N-1$ with one eigenvalue at 0 and others at 1. The eigenvector corresponding to eigenvalue at 0 is $\mathbf{1}_N$, which implies that $\rho_1(t) = \cdots = \rho_N(t)$ if and only if $\varphi(t) = 0$. Hence, the leaderless consensus is achieved by showing that $\lim_{t\to+\infty}\varphi(t) = 0$.

We also consider the following transformations

$$\begin{aligned} \bar{\xi} &= (M \otimes I_n)\xi, \bar{w}(x) = (M \otimes I_m)\hat{w}(x), \\ \bar{u}(x) &= (M \otimes I_m)\hat{u}(x), \bar{\tilde{u}}(x) = (M \otimes I_m)\tilde{u}(x). \end{aligned} \tag{19}$$

Based on the above transformations and (7), (10) and (15), we obtain that for $x \in (0,1)$,

$$\begin{cases} \hat{D}\bar{w}_t(x) = \bar{w}_x(x) - MH(t) \otimes (\hat{D}Ke^{A\hat{D}x}F)\check{\chi}, \\ \bar{w}(1,t) = 0, \end{cases} \tag{20}$$

where $H(t) = \text{diag}\{h_1(t), \ldots, h_N(t)\}$. Using $\bar{w}(1,t) = 0$ for any t, we further have

$$\begin{cases} \hat{D}\bar{w}_{tx}(x) = \bar{w}(x) - MH(t) \otimes (\hat{D}^2KAe^{A\hat{D}x}F)\check{\chi}, \\ \bar{w}_x(1,t) = \hat{D}(MH(t) \otimes Ke^{A\hat{D}}F)\check{\chi}. \end{cases} \tag{21}$$

It follows from (5), (7) and (8) that

$$\begin{cases} \hat{D}D\bar{\tilde{u}}_t(x) = \hat{D}\bar{\tilde{u}}_x(x) - \tilde{D}\bar{u}_x(x,t), x \in (0,1), \\ \bar{\tilde{u}}(1,t) = 0. \end{cases} \tag{22}$$

Note $\check{\rho}_i(t) = \sum_{j=1}^N l_{ij}\varphi_j(t)$ and let $\check{\varphi}_i(t) = \sum_{j=1}^N l_{ij}\varphi_j(t)$. In view of (4), (15)-(19), the dynamics of $\varphi_i(t)$ is given by

$$\begin{aligned} \dot{\varphi}_i =& A_s\varphi_i + h_i(t)F_s\check{\varphi}_i + B_s\bar{w}^i(0,t) \\ &- \frac{1}{N}\sum_{j=1}^N h_j(t)F_s\check{\varphi}_j + B_{s_1}\bar{\tilde{u}}^i(0,t), \end{aligned}$$

which can be written in the following compact form

$$\begin{aligned} \dot{\varphi}(t) =& \big(I_N \otimes A_s + (MH(t)L) \otimes F_s\big)\varphi(t) \\ &+ I_N \otimes B_s\bar{w}(0,t) + I_N \otimes B_{s_1}\bar{\tilde{u}}(0,t). \end{aligned} \tag{23}$$

Besides, it enables us to write the dynamics of $h_i(t)$ as

$$\dot{h}_i(t) = \begin{cases} d_i\check{\varphi}_i^T(t)\Gamma_s\check{\varphi}_i(t), & \text{if } h_i(t) < \bar{h}, \\ 0, & \text{else} \end{cases} \tag{24}$$

where

$$\Gamma_s = \begin{bmatrix} 0_n & 0_n \\ 0_n & I_N \end{bmatrix}. \tag{25}$$

4.2 Convergence Analysis

This subsection presents the convergence analysis of the proposed adaptive consensus protocol. For $x \in [0,1]$, define

$$g_1(x) = I_N \otimes Ke^{A_K\hat{D}x}, g_2(x) = I_N \otimes Ke^{A_K\hat{D}x}B,$$
$$g_3(x) = I_N \otimes KA_Ke^{A_K\hat{D}x}, g_4(x) = I_N \otimes KA_Ke^{A_K\hat{D}x}B,$$

and then we have

$$\bar{u}(x) = \bar{w}(x) + g_1(x)\bar{\xi} + \hat{D}\int_0^x g_2(x-y)\bar{w}(y)dy, \tag{26}$$

$$\begin{aligned} \bar{u}_x(x) =& \bar{w}_x(x) + \hat{D}\Big[(I_N \otimes KB)\bar{w}(x) \\ &+ g_3(x)\bar{\xi} + \hat{D}\int_0^x g_4(x-y)\bar{w}(y)dy\Big]. \end{aligned} \tag{27}$$

Lemma 8. Let $\bar{\Lambda} = I_N \otimes [I_n \;\; I_n]$. For the state transformations (26) and (27), there exist positive scalars $c_1, \ldots, c_4$ such that the following holds

$$\|\bar{u}(t)\|^2 \leq c_1\|\bar{w}(t)\|^2 + c_2|\bar{\Lambda}\varphi(t)|^2, \quad (28)$$

$$\|\bar{u}_x(t)\|^2 \leq 4\|\bar{w}_x(t)\|^2 + c_3\|\bar{w}(t)\|^2 + c_4|\bar{\Lambda}\varphi(t)|^2. \quad (29)$$

Proof. Using (26), the definition of $\mathcal{L}_2[0,1]$ norm, the Young's inequality and Lemma 5, we obtain

$$\|\bar{u}(t)\|^2 \leq 3\|\bar{w}(t)\|^2 + 3\int_0^1 |g_1(x)|^2 dx |\bar{\xi}(t)|^2 + 3\hat{D}^2 \int_0^1 \int_0^x |g_2(x-y)|^2 dy dx \|\bar{w}(t)\|^2.$$

By noting $\bar{\xi}(t) = \bar{\Lambda}\varphi(t)$, the inequality (28) is derived with the choice of $c_1 = 3(1 + \hat{D}^2 \int_0^1 \int_0^x |g_2(x-y)|^2 dy dx)$ and $c_2 = 3\int_0^1 |g_1(x)|^2 dx$. The proof of the other inequality can be carried out in a similar fashion, and thus is omitted.

Theorem 9. For the network-connected system with node dynamics (1), the filter (10), and the adaptive gain law (12) under Assumptions 1-3, there exists a ϵ^* such that for any $|\tilde{D}| < \epsilon^*$ the adaptive protocol (14) solves the consensus problem, the variables $|\xi_i(t)|$, $|U_i(t)|$, $\|\hat{u}^i(t)\|$, $\|\tilde{u}^i(t)\|$, $\|\hat{w}^i(t)\|_{\mathcal{H}_1}$, $i \in \mathbb{N}$, converge to zero, and the adaptive gain $h_i(t)$ converges to some finite value.

Proof. Since A_K is Hurwitz, there exist positive definite matrices P and $Q \in \mathbb{R}^{n\times n}$ such that $A_K^T P + PA_K = -Q$. Consider a Lyapunov functional candidate as

$$V(t) = V_1(t) + V_2(t), \quad (30)$$

with

$$V_1 = \varphi^T(t)(L\otimes\Pi)\varphi(t) + \epsilon_1 \sum_{i=1}^N \frac{\big(h_i(t)-\alpha\big)^2}{d_i}, \quad (31)$$

$$V_2 = \epsilon_2\hat{D}\int_0^1 (1+x)\Big(|\bar{w}(x,t)|^2 + |\bar{w}_x(x,t)|^2\Big)dx, + \epsilon_3 D\int_0^1 (1+x)|\bar{\tilde{u}}(x,t)|^2 dx \quad (32)$$

where $\Pi = \mathrm{diag}\{\epsilon_0 P, -\epsilon_1 F^{-1}\}$ with F determined by (11), α, ϵ_0, ϵ_1 ϵ_2 and ϵ_3 are positive constants to be designed later. Let $\mathbb{N}_1$ be defined as $\mathbb{N}_1 = \{i | h_i(t) < \bar{h}\}$ and $\mathbb{N}_2 = \mathbb{N}\setminus\mathbb{N}_1$. The derivative of $V_1(t)$ along the solution to (23) and (24) is obtained as

$$\begin{aligned}\dot{V}_1 =& \varphi^T(t)\Big[\big(LH(t)L\big)\otimes\big(\Pi F_s + F_s^T\Pi\big) + L\otimes\Pi A_s \\ &+ L\otimes A_s^T\Pi\Big]\varphi(t) + 2\varphi^T(t)\big(L\otimes\Pi B_s\big)\bar{w}(0,t) \\ &+ 2\varphi^T(t)\big(L\otimes\Pi B_{s_1}\big)\bar{\tilde{u}}(0,t) \\ &+ 2\epsilon_1\sum_{i\in\mathbb{N}_1}\big(h_i(t)-\alpha\big)\check{\varphi}_i^T(t)\Gamma_s\check{\varphi}_i^T(t). \end{aligned} \quad (33)$$

Let L_i be the ith column of L, thus L can be written as $L = [L_1, \ldots L_N]$. By using $\Pi F_s + F_s^T\Pi = -2\epsilon_1\Gamma_s$, $(L_i \otimes I_{2n})\Gamma_s(L_i^T\otimes I_{2n}) = (L_iL_i^T)\otimes\Gamma_s$, $\check{\varphi}_i(t) = (L_i^T\otimes I_{2n})\varphi(t)$, and the Young's inequality, we obtain

$$\begin{aligned}\dot{V}_1 \leq& \varphi^T\Big[\frac{2}{\epsilon_2}L^2\otimes\Psi + \frac{2}{\epsilon_2}I_{2Nn} - 2\epsilon_1\bar{h}\sum_{i\in\mathbb{N}_2}\big(L_iL_i^T\otimes\Gamma_s\big) \\ &- 2\alpha\epsilon_1\sum_{i\in\mathbb{N}_1}\big(L_iL_i^T\otimes\Gamma_s\big) + L\otimes\big(\Pi A_s + A_s^T\Pi\big)\Big]\varphi \\ &+ \frac{\epsilon_2}{2}|\bar{w}(0,t)|^2 + \frac{\epsilon_2}{2}|L\otimes\Pi B_{s_1}|^2|\bar{\tilde{u}}(0,t)|^2. \end{aligned} \quad (34)$$

with $\Psi = \Pi B_s B_s^T\Pi$. Using (25), integration by parts and $(M\otimes I_n)\check{\chi}(t) = L\otimes[0_n \;\; I_n]\varphi(t)$, we have

$$\begin{aligned}&\int_0^1 (1+x)\big(\bar{w}^T(x,t)\bar{w}_x(x,t) + \bar{w}_x^T(x,t)\bar{w}_{xx}(x,t)\big)dx \\ &- \hat{D}\int_0^1 (1+x)\bar{w}^T(x,t)(MH(t)\otimes Ke^{A\hat{D}x}F)\check{\chi}dx \\ &- \hat{D}^2\int_0^1 (1+x)\bar{w}_x^T(x,t)(MH(t)\otimes KAe^{A\hat{D}x}F)\check{\chi}dx \\ \leq& -\frac{|\bar{w}(0,t)|^2 + |\bar{w}_x(0,t)|^2}{2} - \frac{\|\bar{w}\|^2_{\mathcal{H}_1}(t)}{4} \\ &+ \bar{h}^2\gamma\varphi^T(t)(L^2\otimes\Gamma_s)\varphi(t), \end{aligned} \quad (35)$$

where γ is given by

$$\gamma = \gamma_1 + \hat{D}^2\int_0^1 (1+x)^2(|Ke^{A\hat{D}x}F|^2 + \hat{D}^2|KAe^{A\hat{D}x}F|^2)dx$$

Let $\Phi = [I_n \;\; I_n]^T[I_n \;\; I_n]$. In view of (35), Lemma 5 and (29) in Lemma 8, we have that the derivative of $V_2(t)$ along the solution to (20) and (21) satisfies

$$\begin{aligned}\dot{V}_2 \leq& 2\epsilon_2\gamma\bar{h}^2\varphi^T(L^2\otimes\Gamma_s)\varphi + \frac{2\epsilon_3 c_4|\tilde{D}|}{\hat{D}}\varphi^T(I_N\otimes\Phi)\varphi \\ &- \epsilon_2|\bar{w}(0,t)|^2 - \epsilon_2\frac{\|\bar{w}\|^2_{\mathcal{H}_1}}{2} - \epsilon_2|\bar{w}_x(0,t)|^2 \\ &- \epsilon_3\Big(|\bar{\tilde{u}}(0,t)|^2 + (1 - \frac{2|\tilde{D}|}{\hat{D}})\|\bar{\tilde{u}}(t)\|^2 \\ &- \frac{2|\tilde{D}|}{\hat{D}}\big(4\|\bar{w}_x(t)\|^2 + c_3\|\bar{w}(t)\|^2\big)\Big). \end{aligned} \quad (36)$$

Let $\Xi = [\frac{\mathbf{1}_N}{\sqrt{N}} \;\; \bar{\Xi}^T]$ with $\bar{\Xi} \in \mathbb{R}^{(N-1)\times N}$ be such an unitary matrix that $\Xi^T L \Xi = J = \mathrm{diag}\{0, \lambda_2, \ldots, \lambda_N\}$, where $\lambda_2 \leq \cdots \leq \lambda_N$ are the nonzero eigenvalues of L. We also consider the following state transformation

$$\eta(t) = (\Xi^T\otimes I_{2n})\varphi(t), \quad (37)$$

where $\eta(t) = [\eta_1^T(t), \ldots, \eta_N^T(t)]^T$. Note that the first row of Ξ^T is $\frac{\mathbf{1}_N^T}{\sqrt{N}}$, and thus we have $\eta_1(t) \equiv 0$. Since $2\bar{h}\lambda_2 > 1$, it enables us to choose

$$\alpha = \frac{\epsilon_2\gamma\bar{h}^2}{\epsilon_1} + \frac{1}{2\lambda_2}, i \in \mathbb{N}, \quad (38)$$

$$\epsilon_1 = \frac{2\bar{h}^2\lambda_2\gamma}{2\bar{h}\lambda_2 - 1}\epsilon_2. \quad (39)$$

Summing up (34) and (36), combining the similar terms, and using (37)-(39) and $\lambda_2 \leq \lambda_i$, we obtain

$$\begin{aligned}\dot{V} \leq& \sum_{i=2}^N \lambda_i\eta_i^T\Big[\Upsilon + \frac{2\epsilon_3 c_4|\tilde{D}|}{\hat{D}\lambda_2}\Phi + \frac{2\lambda_N}{\epsilon_2}\Psi + \frac{2}{\lambda_2\epsilon_2}I_{2n}\Big]\eta_i \\ &- (\frac{\epsilon_2}{2} - \frac{8\epsilon_3|\tilde{D}|}{\hat{D}})\|\bar{w}_x(t)\|^2 - \epsilon_3(1 - \frac{2|\tilde{D}|}{\hat{D}})\|\bar{\tilde{u}}(t)\|^2 \\ &- \epsilon_2|\bar{w}_x(0,t)|^2 - (\frac{\epsilon_2}{2} - \frac{2c_3\epsilon_3|\tilde{D}|}{\hat{D}})\|\bar{w}(t)\|^2) \\ &- (\epsilon_3 - \frac{\epsilon_2}{2}|L\otimes\Pi B_{s_1}|^2)|\bar{\tilde{u}}(0,t)|^2 - \frac{\epsilon_2}{2}|\bar{w}(0,t)|^2. \end{aligned} \quad (40)$$

where

$$\Upsilon = \begin{bmatrix} -\epsilon_0 Q & \epsilon_0 PBK \\ \epsilon_0(PBK)^T & \epsilon_1\big(-A^TF^{-1} - F^{-1}A - I_n\big)\end{bmatrix}, \quad (41)$$

Let $-\bar{\Omega} = \Upsilon + \frac{2\lambda_N}{\epsilon_2}\Psi + \frac{2}{\lambda_2\epsilon_2}I_{2n}$, and choose

$$\epsilon_2 = \frac{1}{\lambda_2\epsilon_0^2}. \tag{42}$$

By substituting (39) and (42) into $\bar{\Omega}$, we have by Schur compliment lemma that $\bar{\Omega} > 0$ if and only if

$$\frac{2\bar{h}^2\gamma}{(2\bar{h}\lambda_2 - 1)\epsilon_0^2}\Omega - 2\epsilon_0^2 I_n > 0, \tag{43}$$

$$\begin{aligned}&\epsilon_0(Q - 2\lambda_2\lambda_N\epsilon_0^3 PBB^T P - 2\epsilon_0 I_n)\\ &- \epsilon_0^4 PBK\Big(\frac{2\bar{h}^2\gamma}{2\bar{h}\lambda_2 - 1}\Omega - 2\epsilon_0^4 I_n\Big)^{-1}(PBK)^T > 0,\end{aligned} \tag{44}$$

where $\Omega = A^T F^{-1} + F^{-1}A + I_n > 0$ by (11). There exists a small $\epsilon_0 > 0$ such that (43) and (44) hold simultaneously due to $Q > 0$ and $\Omega > 0$, and thus $\bar{\Omega}_1 > 0$. Choose

$$\begin{aligned}\epsilon_3 >& \frac{\epsilon_2}{2}|L \otimes \Pi B_{s_1}|^2,\\ \epsilon^* =& \min\Big\{\frac{\lambda_2\lambda_{\min}(\bar{\Omega})D}{\lambda_2\lambda_{\min}(\bar{\Omega}) + 2\epsilon_3 c_4\lambda_{\max}(\Phi)}, \frac{D}{3},\\ &\frac{\epsilon_2 D}{\epsilon_2 + 4c_3\epsilon_3}, \frac{\epsilon_2 D}{\epsilon_2 + 16\epsilon_3}\Big\}\\ \mu =& \min\Big\{\lambda_{\min}(\bar{\Omega}) - \frac{2\epsilon_3 c_4\lambda_{\max}(\Phi)}{\hat{D}\lambda_2}|\tilde{D}|, \frac{\epsilon_2}{2} - \frac{8\epsilon_3}{\hat{D}}|\tilde{D}|,\\ &\frac{\epsilon_2}{2} - \frac{2c_3\epsilon_3}{\hat{D}}|\tilde{D}|, \epsilon_3(1 - \frac{2|\tilde{D}|}{\hat{D}})\Big\}.\end{aligned}$$

With the above choice of α, ϵ_0, ϵ_1, ϵ_2, ϵ_3, ϵ^*, and μ, if $|\tilde{D}| < \epsilon^*$, it holds that $\mu > 0$ and

$$\dot{V} \le -\mu\Big[\sum_{i=2}^{N}\lambda_i|\eta_i(t)|^2 + \|\bar{\tilde{u}}(t)\|^2 + \|\bar{w}(t)\|_{\mathcal{H}_1}^2\Big], \tag{45}$$

which leads to the stability result, i.e., $V(t) \le V(0)$, $\forall t \ge 0$. By noting $\eta_1(t) \equiv 0$, integrating (45) from 0 to $+\infty$, one can obtain that $|\eta(t)|$, $\|\bar{\tilde{u}}(t)\|$, and $\|\bar{w}(t)\|_{\mathcal{H}_1}$ are square integrable. The boundedness of $\|\bar{\tilde{u}}(t)\|$, and $\|\bar{w}(t)\|_{\mathcal{H}_1}$ implies that $|\bar{\tilde{u}}(x,t)|$, $|\bar{w}(x,t)|$, and $|\bar{w}_x(x,t)|$ are uniformly bounded for any $(x,t) \in [0,1] \times [0,+\infty)$. By Barbalat's Lemma, we arrive at $|\eta(t)|$, $\|\bar{w}(t)\|_{\mathcal{H}_1}$, $\|\bar{\tilde{u}}(t)\|$ asymptotically converge to zero as t goes to $+\infty$. By noting (37), we have $\lim_{t\to+\infty}\varphi(t) = 0$, which indicates that the consensus is achieved.

Using (10) and (15), we have that for $x \in (0,1)$,

$$\begin{cases}\hat{D}\hat{w}_t^i(x,t) = \hat{w}_x^i(x,t) - h_i(t)\hat{D}Ke^{A\hat{D}x}F\check{\chi}_i(t),\\ \hat{w}^i(1,t) = 0.\end{cases} \tag{46}$$

By noting $\lim_{t\to+\infty}\check{\chi}_i(t) = 0$ and $h_i(t)$ is bounded, system (46) can be seen as a special case of system (9), and thus it follows from Lemma 6 that $\lim_{t\to+\infty}\|\hat{w}^i(t)\| = 0$ and $\lim_{t\to+\infty}\hat{w}^i(x,t) = 0$, $\forall x \in [0,1]$. Moreover, we can rephrase (10) as $\dot{\xi}_i(t) = A_K\xi_i(t) + B\hat{w}^i(0,t) + h_i(t)F\check{\chi}_i(t)$, thus it holds $\lim_{t\to+\infty}\xi_i(t) = 0$. From (46) we obtain

$$\begin{cases}\hat{D}\hat{w}_{tx}^i(x,t) = \hat{w}_{xx}^i(x,t) - h_i(t)\hat{D}^2KAe^{A\hat{D}x}F\check{\chi}_i(t)\\ \hat{w}_x^i(1,t) = h_i(t)\hat{D}Ke^{A\hat{D}}F\check{\chi}_i(t).\end{cases} \tag{47}$$

Using Lemma 6 again we have $\lim_{t\to+\infty}\|\hat{w}_x^i(t)\| = 0$ and $\lim_{t\to+\infty}\hat{w}_x^i(x,t) = 0$, $\forall x \in [0,1]$. Similar as the discussion in Lemma 8, there exist positive scalars b_1 and b_2 such that $\|\hat{u}^i(t)\|^2 \le b_1\|\hat{w}^i(t)\|^2 + b_2|\xi_i(t)|^2$, and thus $\|\hat{u}^i(t)\|$ converges to zero. By noting $U_i(t) = \hat{u}^i(1,t) = \hat{u}^i(1,t)$, we finally obtain that the variables $|U_i(t)|$, $\|\tilde{u}^i(t)\|$, $i \in \mathbb{N}$, converge to zero. This completes the proof.

Remark 10. It is observed that (43) and (44) are only related to the parameter ϵ_0, which implies the choice of the parameters is compatible. Namely, once ϵ_0 is solved by (43) and (44), the parameters ϵ_2, ϵ_1, α, ϵ_3, ϵ^* will be successively determined.

5. SIMULATION STUDY

In this section, simulation examples are presented to illustrate the effectiveness of the theoretical result. Consider a network of 4 nodes described by (1) and (2) with

$$A = \begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}, B = \begin{bmatrix}0\\ 1\end{bmatrix},$$

$D_1 = 4s$, $D_2 = 2s$, and the Laplacian matrix

$$L = \begin{bmatrix} 2 & -1 & 0 & -1\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ -1 & 0 & -1 & 2\end{bmatrix}.$$

Choose $K = -[1\ \ 3]$ such that A_K is Hurwitz. Solving (11) via LMI toolbox gives the filter gain matrix F in (10) as $F = -0.5011I_2$. Set $\bar{h} = 1000$ and $d_i = 0.01$, $i = 1, \dots, 4$, and one can check that $2\bar{h}\varrho > 1$. The initial conditions are set to be zero except $h_i(0) = 0.1$, $i = 1, \dots, 4$, and $X(0) = [0.5\ -0.5\ 0.75\ 0.75\ 1\ 2.5\ 5\ -1]^T$. With the above settings, simulation results in two scenarios are presented to illustrate the effectiveness of the proposed control schemes. The first one is with the total delay D underestimated, i.e., $\hat{D} = 5.4s$, while the other is with the total delay overestimated $\hat{D} = 6.6s$. The trajectories of consensus errors, adaptive gains and control input for both scenarios are shown in Fig. 1, which indicates the consensus is achieved.

6. CONCLUSION

In this paper, we have presented an adaptive consensus control design for network-connected systems with unknown delays under undirected graphs. By introducing a transformation of the estimate delay state and its inverse, an explicit Lyapunov function has been constructed to analyze system stability. One interesting feature of the result is that the proposed adaptive protocols can be implemented by each node in a fully distributed fashion, and the interaction of control input among nodes is avoided.

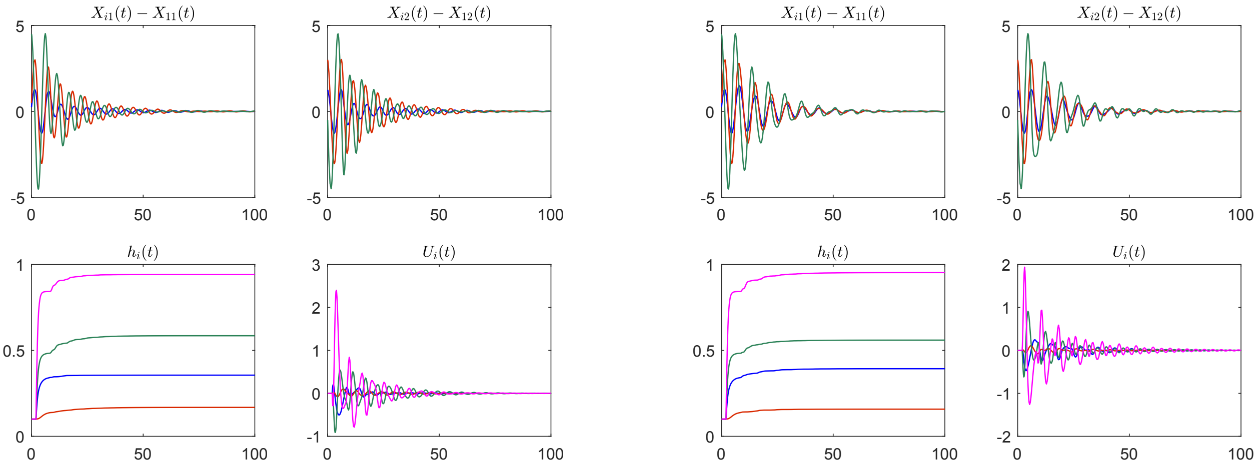
REFERENCES

Artstein, Z. (1982). Linear systems with delayed controls: A reduction. *IEEE Transactions on Automatic Control*, 27(4), 869–879.

Bresch-Pietri, D. and Krstic, M. (2010). Delay-adaptive predictor feedback for systems with unknown long actuator delay. *IEEE Transactions on Automatic Control*, 55(9), 2106–2112.

Ding, Z. (2014). Consensus control of a class of lipschitz nonlinear systems. *International Journal of Control*, 87(11), 2372–2382.

Ding, Z. (2015a). Adaptive consensus output regulation of a class of nonlinear systems with unknown high-frequency gain. *Automatica*, 51(1), 348–355.



(a) Scenario 1: $D_1 = 4s$, $D_2 = 2s$, $\hat{D} = 5.4s$

(b) Scenario 2: $D_1 = 4s$, $D_2 = 2s$, $\hat{D} = 6.6s$

Fig. 1. The response of the networked system under the adaptive protocol with non-adaptive predictors.

Ding, Z. (2015b). Consensus disturbance rejection with disturbance observers. *IEEE Transactions on Industrial Electronics*, 62(9), 5829–5837.

Hong, Y.G., Chen, G.R., and Bushnell, L. (2008). Distributed observers design for leader-following control of multi-agent networks. *Automatica*, 44(3), 846–850.

Khalil, H. (2002). *Nonlinear Systems, Third Edition*. Prentice Hall.

Krstic, M. (2008a). Lyapunov tools for predictor feedbacks for delay systems: Inverse optimality and robustness to delay mismatch. *Automatica*, 44(11), 2930–2935.

Krstic, M. (2008b). On compensating long actuator delays in nonlinear control. *IEEE Transactions on Automatic Control*, 53(7), 1684–1688.

Krstic, M. (2009). *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*. Birkhauser Boston.

Krstic, M. and Smyshlyaev, A. (2008). Backstepping boundary control for first-order hyperbolic pdes and application to systems with actuator and sensor delays. *Systems & Control Letters*, 57(9), 750–758.

Kwon, W. and Pearson, A. (1980). Feedback stabilization of linear systems with delayed control. *IEEE Transactions on Automatic Control*, 25(2), 266–269.

Li, Z., Ding, Z., Sun, J., and Li, Z. (2018). Distributed adaptive convex optimization on directed graphs via continuous-time algorithms. *IEEE Transactions on Automatic Control*, 63(5), 1434–1441.

Li, Z.K., Ren, W., Liu, X.D., and Xie, L.H. (2013). Distributed consensus of linear multi-agent systems with adaptive dynamic protocols. *Automatica*, 49(7), 1986–1995.

Manitius, A. and Olbrot, A. (1979). Finite spectrum assignment problem for systems with delays. *IEEE Transactions on Automatic Control*, 24(4), 541–552.

Olfati-Saber, R. and Murray, R.M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9), 1520–1533.

Ren, W., Beard, R.W., and Atkins, E.M. (2007). Information consensus in multivehicle cooperative control. *IEEE Control Systems Maganize*, 27(2), 71–82.

Richard, J.P. (2003). Time-delay systems: an overview of some recent advances and open problems. *Automatica*, 39(10), 1667–1694.

Sun, Y.G. and Wang, L. (2009). Consensus of multi-agent systems in directed networks with nonuniform time-varying delays. *IEEE Transactions on Automatic Control*, 54(7), 1607–1613.

Tian, Y.P. and Zhang, Y. (2012). High-order consensus of heterogeneous multi-agent systems with unknown communication delays. *Automatica*, 48(6), 1205–1212.

Wang, C., Zuo, Z., Lin, Z., and Ding, Z. (2015). Consensus control of a class of lipschitz nonlinear systems with input delay. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 62(11), 2730–2738.

Wang, C., Zuo, Z., Lin, Z., and Ding, Z. (2017). A truncated prediction approach to consensus control of lipschitz nonlinear multiagent systems with input delay. *IEEE Transactions on Control of Network Systems*, 4(4), 716–724.

Wang, X., Saberi, A., Stoorvogel, A., Grip, H., and Yang, T. (2013). Consensus in the network with uniform constant communication delay. *Automatica*, 49(8), 2461–2467.

Zhao, T. and Ding, Z. (2018). Distributed finite-time optimal resource management for microgrids based on multi-agent framework. *IEEE Transactions on Industrial Electronics*, 65(8), 6571–6580.

Zhou, B. and Lin, Z. (2014). Consensus of high-order multi-agent systems with large input and communication delays. *Automatica*, 50(2), 452–464.