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# An Algebraic Approach for Discrete Dynamic Reconstruction for Switched Bilinear Systems

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**Abstract:** Estimation of the switching signal of continuous-time switched bilinear systems from input-output measurements is addressed in this paper. First, the uniqueness problem of the switching signal reconstruction from the input-output data is studied in terms of distinguishability analysis of the operating modes. In this context, a numerically verifiable condition for the operating modes distinguishability is established. This condition also provides a characterization of regularly persistent control inputs ensuring the unique determination of the switching signal. For these class of control inputs, an algorithm is then provided for the estimation problem. The proposed approach is based on a compatibility test between the input-output measurements and the dynamical behaviour of the modes.

*Keywords:* Bilinear system, switched system, regularly persistent input, mode compatibility.

## 1. INTRODUCTION

Hybrid systems (Goebel et al., 2012) constitute a mathematical tool useful for modelling processes whose behaviour involves both continuous and discrete dynamics. Switched systems represent an important subclass of these systems. The continuous dynamic of a switched system is represented by a collection of differential (or difference) equations called modes, while its discrete dynamic is described by a piecewise constant function orchestrating the switching between the operating modes. The piecewise constant function called the switching signal indicates at each time the active mode of the system. This information is crucial for instance in stability analysis of switched systems (Liberzon, 2003; Lin and Antsaklis, 2005) and for the synthesis of asynchronous switching controllers (Yuan et al., 2018; Etienne et al., 2019).

A growing amount attention has been paid in the last decades to the reconstruction of the switching signal from the input-output measurements. Studies on observability of switched systems, that is, the determination of conditions guaranteeing the uniqueness of the reconstruction of the switched signal and the initial continuous state from the input-output data are addressed in Babaali and Pappas (2005); Gómez-Gutiérrez et al. (2012); Lou and Yang (2014). In Babaali and Pappas (2005) existence problem of a control input for the identification of the switching signal, refered as observability for some input, is analysed and algebraic conditions based on the modes Markov parameters are provided for disturbance-free continuous-time LSSs (linear switched systems). In Baglietto et al. (2007), the design of control inputs allowing the estimation of the switching signal is examined for discrete-time LSSs subject to bounded measurement noises. In Lou and Yang (2014), for disturbance-free continuous-time LSSs, rank conditions are established to characterize the observability of the switching signal for all inputs. In Gómez-Gutiérrez et al. (2012), for the class of continuous-time LSSs with disturbances, a geometric characterization of observability for some or all control inputs is provided in terms of maximal invariant space is provided. All these observability studies are based on the characterization of the distinguishability property of the operating modes. This property informs on the ability of the modes to generate different output signals for some or all control inputs.

Regarding the concrete problem of reconstruction of the switching signal from the input-output measurements, various algorithms have been proposed in the literature in the last few years. For instance, for discrete-time LSSs, model-based diagnostic methods are used in Halimi et al. (2015) and Hakem et al. (2016). The approach consists in designing a residual depending on the input-output data that have particular properties for the active mode and that are sensitive to the change of mode. A similar approach is also considered in Fliess et al. (2009) for continuous-time LSSs. To avoid derivative measurements in the continuous-time case, distribution framework is used in Tian et al. (2009) and Mincarelli et al. (2011). In presence of disturbances, reduced-order observers approaches are proposed in Van Gorp et al. (2014) and Yang et al. (2015) for the reconstruction of the switching signal and the continuous state under suitable assumptions on the dimension and the detectability of the modes. Optimization-based methods have been also proposed for the identification of both the discrete and the continuous dynamics of LSSs, for more details, we suggest the readers to see the survey paper Garulli et al. (2012).

To the best of our knowledge, while several algorithms are proposed in the literature for estimating the switching signal of LSSs, development of switching signal reconstruction methods for switched systems with nonlinear continuous dynamics stills an open problem. In this context, the switching signal estimation problem is addressed in the present paper for continuous-time switched bilinear systems that are used for modelling numerous nonlinear systems (Shu et al., 2018), including thermal dynamics in buildings (Chasparis and Natschläger, 2014) for instance. Regularly persistent control input data (Bornard et al., 1989) are considered. The proposed method is based on a compatibility test between the input-output measurements collected on the system and the dynamical behaviour of the modes. For this analysis, we use the distance from the output of the system to the allowable output signals of the modes associated to the control input. Indeed, we prove that when using an output signal generated by a regularly persistent control input, in disturbance-free case this distance is zero only for modes that are compatible with the measurement data. An analysis of the distinguishability property of the operating modes is also made in the paper. This property guarantees the uniqueness of the switching signal estimation from the input-output data. On this point, we mainly focus on the distinguishability of the modes w.r.t. the control input used for the estimation of the switching signal. For regularly persistent inputs, we specify a numerically verifiable condition for modes distinguishability. This condition provides a characterization of regularly persistent inputs guaranteeing the unique determination of the switching signal. It also generalizes the controlled distinguishability condition proposed recently in Motchon et al. (2018) for linear operating modes.

Notice that a distinguishability condition for bilinear modes have been also established recently in Motchon and Pekpe (2019). Operating modes relative degrees are used to establish this condition. However as for controlled distinguishability studies of Babaali and Pappas (2005) the condition of Motchon and Pekpe (2019) only allow to verify whether there exists a control input making distinguishable the operating modes.

The paper is organized as follows. First, the switching signal estimation problem for bilinear switched systems is formulated in Section 2. The formal definition of input-output data compatibility with the dynamical behaviour of the modes as well as the concept of modes distinguishability w.r.t. an input are also presented in this section. Results related to compatibility tests are provided in Section 3.1 and the condition for distinguishability of modes w.r.t. persistent inputs is presented in Section 3.2. The proposed algorithm for estimating the switching signal which is based on the compatibility tests is provided in Section 3.3. The effectiveness of this algorithm is confirmed in Section 4 through a numerical example.

**Notation:** Throughout the rest of the paper, $\otimes$ denotes the Kronecker product symbol. The set $\mathcal{L}_2\left([\tau_1\,;\tau_2],\mathbb{R}^l\right)$ consists of all functions $\varphi\colon [\tau_1\,;\tau_2] \to \mathbb{R}^l$ such that $\|\varphi\|_{\mathcal{L}_2} := \int_{\tau_1}^{\tau_2} \|\varphi(\tau)\|_2^2\, \mathrm{d}\tau < \infty$ where $\|\cdot\|_2$ is the Euclidean norm. The $L^2$-distance from a function $\psi \in \mathcal{L}_2\left([\tau_1\,;\tau_2],\mathbb{R}^l\right)$ to a set $\mathscr{S} \subset \mathcal{L}_2\left([\tau_1\,;\tau_2],\mathbb{R}^l\right)$ is denoted $\mathrm{d}_{\mathcal{L}_2}(\psi,\mathscr{S}) := \inf_{\varphi\in\mathscr{S}} \|\psi-\varphi\|_{\mathcal{L}_2}$. For a matrix $M \in \mathbb{R}^{l\times l}$, $M^\top$ denotes its transpose matrix. If $M \succ 0$ (i.e. $M$ is symmetric positive definite), $\|\cdot\|_M$ is the weighted norm defined by $\|w\|_M = \sqrt{w^\top M\, w}$.

## 2. PROBLEM FORMULATION AND PRELIMINARIES

### 2.1 Problem formulation

Let's consider the switched bilinear system described on $[t_0\,;\infty)$ by the following equations:

$$\dot{x} = A_\sigma\, x + N_\sigma\, (u \otimes x) + B_\sigma\, u, \quad y = C_\sigma\, x, \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^\ell$ is the control input vector, $y \in \mathbb{R}^m$ is the output vector and $\sigma\colon [t_0\,;\infty) \to \mathscr{P} \subset \mathbb{N}$ is a piecewise constant function continuous from the right, modelling the switching signal of the system i.e. the signal indicating at each time the active mode of the system. For every $p \in \mathscr{P}$, $A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{n\times \ell}$, $C_p \in \mathbb{R}^{m\times n}$ and $N_p = \left[N_p^{(1)} \cdots N_p^{(\ell)}\right] \in \mathbb{R}^{n\times n\,\ell}$ (with $N_p^{(r)} \in \mathbb{R}^{n\times n}$, $r = 1, 2, \ldots, \ell$) are known matrices defining the continuous dynamic of the system. The switching times of the system are elements of the strictly increasing sequence $\{t_k\}_{k\in\mathbb{N}^\star} \subset [t_0\,;\infty)$. There are no jumps in the state vector evolution. It is also assumed that $\sigma$ admits a minimum dwell-time, that is:

*Assumption 1.* There exists $\boldsymbol{\tau}_d > 0$ such that for every $k \in \mathbb{N}$, $t_{k+1} - t_k \geq \boldsymbol{\tau}_d$.

Given a final observation time $T > t_0 + \boldsymbol{\tau}_d$, the problem discussed in this paper is concerned with the reconstruction of the switching signal on $[t_0\,;T]$ using the control input signal and the output signal collected on the system during the observation time $[t_0\,;T]$.

### 2.2 Compatibility between input-output data and modes

The operating modes of system (1) are the following bilinear systems:

$$\forall p \in \mathscr{P},\ \dot{x}_p = A_p\, x_p + N_p\, (u \otimes x) + B_p\, u,\ y_p = C_p\, x_p. \qquad (2)$$

Given $\underline{\tau} \geq t_0$, the output vector generated by mode $p \in \mathscr{P}$ at a time instant $t \geq \underline{\tau}$, when starting at $\underline{\tau}$ from an arbitrary state $x_p^o \in \mathbb{R}^n$ is denoted by $y_p\left(t, \underline{\tau}, x_p^o, u\right)$ and can be expressed explicitly as follows:

$$y_p\left(t, \underline{\tau}, x_p^o, u\right) = C_p\, \Phi_{p,\underline{\tau}}^u(t)\, x_p^o + \Psi_{p,\underline{\tau}}^u(t)\,, \qquad (3)$$

where $\Phi_{p,\underline{\tau}}^u(t) \in \mathbb{R}^{n\times n}$ defined by

$$\Phi_{p,\underline{\tau}}^u(t) = \exp\left(\int_{\underline{\tau}}^{t} \{A_p + N_p\,[u(s) \otimes I_n]\}\, \mathrm{d}s\right), \qquad (4)$$

is the transition matrix and $\Psi^u_{p,\underline{\tau}}(t) \in \mathbb{R}^m$ is given by:

$$\Psi^u_{p,\underline{\tau}}(t) = C_p \int_{\underline{\tau}}^{t} \Phi^u_{p,s}(t)\; B_p\, u(s)\, \mathrm{d}s. \qquad (5)$$

*Remark 1.* Notice that the transition matrix $\Phi^u_{p,\underline{\tau}}(t)$ verifies $\dot{\Phi}^u_{p,\underline{\tau}}(t) = (A_p + N_p[u(t) \otimes I_n])\, \Phi^u_{p,\underline{\tau}}(t)$ and $\Phi^u_{p,\underline{\tau}}(\underline{\tau}) = I_n$.

The allowable output signals that can be generated by mode $p \in \mathscr{P}$ during an observation time $[\underline{\tau}\,;\overline{\tau}] \subseteq [t_0\,;T]$ and when the input $u$ is applied to the switched bilinear system is therefore

$$\mathscr{Y}^u_p([\underline{\tau}\,;\overline{\tau}]) = \left\{ y_p\left(\cdot, \underline{\tau}, x^o_p, u\right)_{|\,[\underline{\tau};\overline{\tau}]} : x^o_p \in \mathbb{R}^n \right\}, \qquad (6)$$

where $y_p\left(\cdot, \underline{\tau}, x^o_p, u\right)_{|\,[\underline{\tau};\overline{\tau}]}$ is the restriction of $y_p\left(\cdot, \underline{\tau}, x^o_p, u\right)$ on $[\underline{\tau}\,;\overline{\tau}]$.

*Remark 2.* In the linear case ($N_p = 0_{n\times n}$), output spaces $\mathscr{Y}^u_p([\underline{\tau}\,;\overline{\tau}])$ are closed in $\left(\mathcal{L}_2([\underline{\tau}\,;\overline{\tau}], \mathbb{R}^m), \|\cdot\|_{\mathcal{L}_2}\right)$, see for instance Appendix A of Motchon et al. (2018). The existence of right-pseudo inverse of modes observability Gramian is a key tool to establish this result. In the bilinear case, the observability Gramian of each mode $p$ on $[\underline{\tau}\,;\overline{\tau}]$, given by

$$G^u_p(\underline{\tau}, \overline{\tau}) = \int_{\underline{\tau}}^{\overline{\tau}} \left[\Phi^u_{p,\underline{\tau}}(s)\right]^\top C_p^\top C_p \Phi^u_{p,\underline{\tau}}(s)\, \mathrm{d}s \qquad (7)$$

is nonsingular when the control input is universal (Bornard et al., 1989) for mode $p \in \mathscr{P}$ on $[\underline{\tau}\,;\overline{\tau}]$. So using a reasoning similar to that in Appendix A of Motchon et al. (2018), one can see that when the control input $u$ is universal for the modes, the output spaces $\mathscr{Y}^u_p([\underline{\tau}\,;\overline{\tau}])$ are also closed in the bilinear case.

To identify the active mode, it will be important to verify the compatibility between the system's input-output data and the dynamical behaviour of the operating modes (or simply, the M-IO Compatibility). The formal definition of the concept of M-IO Compatibility is expressed as follows (Cocquempot et al., 2004):

*Definition 1.* (M-IO Compatibility). The input-output data $u$ and $y$ collected on the switched system (1) are said to be compatible with mode $p \in \mathscr{P}$ during the observation time $[\underline{\tau}\,;\overline{\tau}]$, and we write for simplicity $p \oplus (u, y)$ on $[\underline{\tau}\,;\overline{\tau}]$, if $y_{|[\underline{\tau};\overline{\tau}]} \in \mathscr{Y}^u_p([\underline{\tau}\,;\overline{\tau}])$.

Compatibility tests allow to identify modes that can produce the output data used to infer the discrete dynamic of the switched system.

### *2.3 From distinguishability to active mode identification*

A worst case where it is difficult to detect the active mode of the switched system using the input-output data is the situation where the data are compatible with at least two operating modes, that is: $\exists\,(p, q) \in \mathscr{P} \times \mathscr{P},\;\; y_{|[\underline{\tau};\overline{\tau}]} \in \mathscr{Y}^u_p([\underline{\tau}\,;\overline{\tau}]) \cap \mathscr{Y}^u_q([\underline{\tau}\,;\overline{\tau}])$. To better understand this problem, an analysis of controlled distinguishability of modes (Babaali and Pappas, 2005; Motchon et al., 2018), recalled in Definition 2 is required.

*Definition 2.* (Controlled distinguishability). Two modes $p \in \mathscr{P}$ and $q \in \mathscr{P}$ of system (1) are said to be controlled distinguishable w.r.t. $u$ on $[\underline{\tau}\,;\overline{\tau}]$ if $\mathscr{Y}^u_p([\underline{\tau}\,;\overline{\tau}]) \cap \mathscr{Y}^u_q([\underline{\tau}\,;\overline{\tau}]) = \emptyset$. The input $u$ is called in this case a discerning input of the modes on $[\underline{\tau}\,;\overline{\tau}]$.

Characterization of controlled distinguishability is important to know *a priori* whether the active mode is uniquely identifiable or not from the input-output data. Augmented systems of modes (Gómez-Gutiérrez et al., 2012; Motchon et al., 2018) are main tools for addressing operating modes distinguishability problem [1]. In the bilinear case which is the setting of the present paper, the augmented system associated to any pair $(p, q) \in \mathscr{P} \times \mathscr{P}$ of modes is the bilinear system defined by the following equations (Motchon and Pekpe, 2019):

$$\dot{x}_{pq} = A_{pq}x_{pq} + N_{pq}(u \otimes x_{pq}) + B_{pq}u,\;\; y_{pq} = C_{pq}x_{pq} \qquad (8)$$

where $x_{pq}(t) = \left[x_p^\top(t)\; x_q^\top(t)\right]^\top \in \mathbb{R}^{2n}$ and $y_{pq}(t) \in \mathbb{R}^m$ are the state vector and the output vector, $A_{pq} = \mathrm{diag}(A_p, A_q)$, $B_{pq} = \left[B_1^\top\; B_2^\top\right]^\top$, $C_{pq} = [C_p \;\; -C_q]$ and $N_{pq} = \left[N^{(1)}_{pq} \cdots N^{(\ell)}_{pq}\right]$ with $N^{(r)}_{pq} = \mathrm{diag}\left(N^{(r)}_p, N^{(r)}_q\right)$, $r = 1, 2, \ldots, \ell$; we recall that $N^{(r)}_p$ and $N^{(r)}_q$ are the $rth$ block matrix of $N_p$ and $N_q$, respectively. The output vector at the time instant $t \geq \underline{\tau}$ generated by the augmented system when the input $u$ is applied to the augmented system, starting from the state $x^o_{pq}$ at the time $\underline{\tau}$ is denoted by $y_{pq}\left(t, \underline{\tau}, x^o_{pq}, u\right)$. An explicit expression of this vector can be obtained similarly to (3)–(5). The set of all output signals generated by the augmented system associated to modes $p$ and $q$ is then given by

$$\mathscr{Y}^u_{pq}([\underline{\tau}\,;\overline{\tau}]) = \left\{ y_{pq}\left(\cdot, \underline{\tau}, x^o_{pq}, u\right)_{|\,[\underline{\tau};\overline{\tau}]} : x^o_{pq} \in \mathbb{R}^{2n} \right\}. \qquad (9)$$

*Remark 3.* For every distinct modes $p \in \mathscr{P}$ and $q \in \mathscr{P}$, $\mathscr{Y}^u_{pq}([\underline{\tau}\,;\overline{\tau}])$ is also closed in $\left(\mathcal{L}_2([\underline{\tau}\,;\overline{\tau}], \mathbb{R}^m), \|\cdot\|_{\mathcal{L}_2}\right)$ when the observability Gramian $G^u_{pq}(\underline{\tau}, \overline{\tau})$ of their augmented system on $[\underline{\tau}\,;\overline{\tau}]$ is nonsingular. This matrix is explicitly given by $G^u_{pq}(\underline{\tau}, \overline{\tau}) = \int_{\underline{\tau}}^{\overline{\tau}} \left[\Phi^u_{pq,\underline{\tau}}(s)\right]^\top C_{pq}^\top C_{pq}\, \Phi^u_{pq,\underline{\tau}}(s)\, \mathrm{d}s$ with $\Phi^u_{pq,\underline{\tau}}(s) = \exp\left(\int_{\underline{\tau}}^{s} \{A_{pq} + N_{pq}[u(\theta) \otimes I_{2n}]\}\, \mathrm{d}\theta\right)$ the transition matrix of the augmented system.

Considering the augmented system allows to recast the distinguishability analysis into output-zeroing problems (Gómez-Gutiérrez et al., 2012). For controlled distinguishability, a formulation of the output-zeroing problem based on the admissible output space of the augmented system is provided by the following result (Motchon and Pekpe, 2019):

*Proposition 1.* Two modes $p \in \mathscr{P}$ and $q \in \mathscr{P}$ of the switched system (1) are controlled distinguishable w.r.t. $u$ on $[\underline{\tau}\,;\overline{\tau}]$ iff $\mathbf{0}^{[m]} \notin \mathscr{Y}^u_{pq}([\underline{\tau}\,;\overline{\tau}])$, where $\mathbf{0}^{[m]}$ denotes the zero output signal.

## 3. MAIN RESULTS

### *3.1 A criteria for testing M-IO Compatibility*

For testing M-IO Compatibility, we propose the following geometric characterization:

[1] The augmented system associated to two modes is a dynamical system whose state vector is the concatenation of the modes state vectors and its output is the difference of their output signals.

*Proposition 2.* (M-IO Compatibility). Let $u$ be an universal input of mode $p \in \mathscr{P}$ on $[\underline{\tau}\,;\overline{\tau}]$. Then

$$p \oplus (u, y) \text{ on } [\underline{\tau}\,;\overline{\tau}] \iff \mathrm{d}_{\mathcal{L}_2}\left(y, \mathscr{Y}_p^u\left([\underline{\tau}\,;\overline{\tau}]\right)\right) = 0. \quad (10)$$

**Proof.** The proof follows from the fact that a point is an element of a close set iff the distance from the point to the set is zero. The closure property of $\mathscr{Y}_p^u\left([\underline{\tau}\,;\overline{\tau}]\right)$ is discussed in Remark 2.

*Remark 4.* Notice that when $\mathscr{Y}_p^u\left([\underline{\tau}\,;\overline{\tau}]\right)$ is not closed, $\mathrm{d}_{\mathcal{L}_2}\left(y, \mathscr{Y}_p^u\left([\underline{\tau}\,;\overline{\tau}]\right)\right) = 0$ does not necessary implies that data $u$ and $y$ are compatible with mode $p$.

For simplicity in the presentation, we define

$$\Delta_p^{u,y}\left(\underline{\tau}, \overline{\tau}\right) := \mathrm{d}_{\mathcal{L}_2}\left(y, \mathscr{Y}_p^u\left([\underline{\tau}\,;\overline{\tau}]\right)\right). \quad (11)$$

Proposition 3 gives a numerical method for the computation of $\Delta_p^{u,y}\left(\underline{\tau}, \overline{\tau}\right)$, needed for applying the M-IO Compatibility result established in Proposition 2.

*Proposition 3.* Assume that $u$ is universal for mode $p \in \mathscr{P}$ on $[\underline{\tau}\,;\overline{\tau}]$. Then

$$\Delta_p^{u,y}\left(\underline{\tau}, \overline{\tau}\right) = \sqrt{\left\|\Psi_{p,\underline{\tau}}^u - y\right\|_{\mathcal{L}_2}^2 - \left\|\Upsilon_{p,\underline{\tau}}^{u,y}\left(\overline{\tau}\right)\right\|_{H_p^u\left(\underline{\tau},\overline{\tau}\right)}^2}, \quad (12)$$

where $H_p^u\left(\underline{\tau}, \overline{\tau}\right) = \left[G_p^u\left(\underline{\tau}, \overline{\tau}\right)\right]^{-1}$, $\Psi_{p,\underline{\tau}}^u$ is given by (5) and $\Upsilon_{p,\underline{\tau}}^{u,y}\left(\overline{\tau}\right) \in \mathbb{R}^n$ is defined by

$$\Upsilon_{p,\underline{\tau}}^{u,y}\left(\overline{\tau}\right) = \int_{\underline{\tau}}^{\overline{\tau}} \left[\Phi_{p,\underline{\tau}}^u\left(s\right)\right]^\top C_p^\top \left[\Psi_{p,\underline{\tau}}^u\left(s\right) - y\left(s\right)\right] \mathrm{d}s,$$

with $\Phi_{p,\underline{\tau}}^u$ the transition matrix function. Furthermore, $\Psi_{p,\underline{\tau}}^u$ and $\Upsilon_{p,\underline{\tau}}^{u,y}$ verify:

$$\begin{bmatrix}\Psi_{p,\underline{\tau}}^u\left(t\right)\\ \Upsilon_{p,\underline{\tau}}^{u,y}\left(t\right)\end{bmatrix} = \begin{bmatrix}C_p & 0_{m\times n}\\ 0_{n\times n} & \left[\Phi_{p,\underline{\tau}}^u\left(t\right)\right]^\top\end{bmatrix}\begin{bmatrix}\eta_p\left(t\right)\\ \nu_p\left(t\right)\end{bmatrix},\ \forall t \geq \underline{\tau} \quad (13)$$

with $\eta_p$ and $\nu_p$, the solutions of the differential equations:

$$\begin{bmatrix}\dot{\eta}_p\\ \dot{\nu}_p\end{bmatrix} = \Lambda_p \begin{bmatrix}\eta_p\\ \nu_p\end{bmatrix} + \Gamma_p\left(\begin{bmatrix}u\\ y\end{bmatrix} \otimes \begin{bmatrix}\eta_p\\ \nu_p\end{bmatrix}\right) + \Pi_p \begin{bmatrix}u\\ y\end{bmatrix}, \quad (14a)$$

$$\left[\eta_p^\top\left(\underline{\tau}\right)\ \nu_p^\top\left(\underline{\tau}\right)\right] = 0_{2n}, \quad (14b)$$

where

$$\Lambda_p = \begin{bmatrix}A_p & 0_{n\times n}\\ C_p^\top C_p & -A_p^\top\end{bmatrix}, \quad \Pi_p = \mathrm{diag}\left(B_p, -C_p^\top\right), \quad (15)$$

and

$$\Gamma_p = \left[\Gamma_p^{(1)}\ \cdots\ \Gamma_p^{(\ell)}\ 0_{2n\times 2nm}\right], \quad (16)$$

with

$$\Gamma_p^k = \mathrm{diag}\left(N_p^{(r)}, -\left(N_p^{(r)}\right)^\top\right), \quad r = 1, 2, \ldots, \ell. \quad (17)$$

**Proof.** It follows from the definitions (6) and (11) that $\left[\Delta_p^{u,y}\left(\underline{\tau}, \overline{\tau}\right)\right]^2 = \inf_{x_p^o\in\mathbb{R}^n} F_p^{u,y}\left(x_p^o\right)$ where $F_p^{u,y}\left(x_p^o\right) = \left\|y_p\left(\cdot, \underline{\tau}, x_p^o, u\right) - y\right\|_{\mathcal{L}_2}^2$. It is a simple exercise to verify that $\Delta_p^{u,y}\left(\underline{\tau}, \overline{\tau}\right) = \sqrt{F_p^{u,y}\left(\mathbf{x}_p^o\right)}$ with $\mathbf{x}_p^o = -H_p^u\left(\underline{\tau}, \overline{\tau}\right)\Upsilon_{p,\underline{\tau}}^{u,y}\left(\overline{\tau}\right)$, which is equivalent to (12). Now, we will prove that $\Psi_p^u$ and $\Upsilon_p^{u,y}$ verify (13). For every $t \geq \underline{\tau}$, $\Psi_{p,\underline{\tau}}^u\left(t\right)$ defined by (5) can be rewritten as follows:

$$\Psi_{p,\underline{\tau}}^u\left(t\right) = C_p\,\overline{\Psi}_p^u\left(t\right), \quad (18)$$

where $\overline{\Psi}_p^u\left(t\right) = \int_{\underline{\tau}}^t \Phi_{p,s}\left(t\right) B_p\, u\left(s\right) \mathrm{d}s \in \mathbb{R}^n$. By observing the explicit expression (3)-(5) of the output vector of the modes, one can see that $\overline{\Psi}_p^u$ is the solution of the differential equation

$$\dot{\overline{\Psi}}_p^u\left(t\right) = A_p\,\overline{\Psi}_p^u\left(t\right) + N_p\left[u\left(t\right) \otimes \overline{\Psi}_p^u\left(t\right)\right] + B_p u\left(t\right), \quad (19a)$$

$$\overline{\Psi}_p^u\left(\underline{\tau}\right) = 0_n. \quad (19b)$$

Furthermore, it is straightforward to verify that

$$\Upsilon_{p,\underline{\tau}}^{u,y}\left(t\right) = \left[\Phi_{p,\underline{\tau}}^u\left(t\right)\right]^\top \overline{\Upsilon}_p^{u,y}\left(t\right), \quad \forall t \geq \underline{\tau}, \quad (20)$$

where $\overline{\Upsilon}_p^{u,y}\left(t\right) = \int_{\underline{\tau}}^t \overline{\Phi}_{p,s}^u\left(t\right) C_p^\top\left[C_p\,\overline{\Psi}_p^u\left(s\right) - y\left(s\right)\right] \mathrm{d}s$ with $\overline{\Phi}_{p,s}^u\left(t\right) = \exp\left(-\int_s^t \left\{A_p^\top + \left[u\left(\theta\right) \otimes I_n\right]^\top N_p^\top\right\} \mathrm{d}\theta\right)$. Observing the expression of $\overline{\Upsilon}_p^{u,y}\left(t\right)$ and $\overline{\Phi}_{p,s}^u\left(t\right)$ as well as the explicit expression (3)-(5) of the output vector of the modes, it is also a simple exercise to check that $\overline{\Upsilon}_p^{u,y}$ is the solution of the differential equation

$$\dot{\overline{\Upsilon}}_p^{u,y}\left(t\right) = -A_p^\top \overline{\Upsilon}_p^{u,y}\left(t\right) + M_p\left[u\left(t\right) \otimes \overline{\Upsilon}_p^{u,y}\left(t\right)\right] + C_p^\top C_p \overline{\Psi}_p^u\left(t\right) - C_p^\top y\left(t\right), \quad (21a)$$

$$\overline{\Upsilon}_p^{u,y}\left(\underline{\tau}\right) = 0_n, \quad (21b)$$

where $M_p = -\left[\left(N_p^{(1)}\right)^\top\ \left(N_p^{(2)}\right)^\top\ \cdots\ \left(N_p^{(\ell)}\right)^\top\right]$. Now, stacking up the differential equations (19a) and (21a) with initial conditions (19b) and (21b), and using the relation

$$\begin{pmatrix}N_p\left[u\left(t\right) \otimes \overline{\Psi}_p^u\left(t\right)\right]\\ M_p\left[u\left(t\right) \otimes \overline{\Upsilon}_p^{u,y}\left(t\right)\right]\end{pmatrix} = \Gamma_p\left(\begin{bmatrix}u\left(t\right)\\ y\left(t\right)\end{bmatrix} \otimes \begin{bmatrix}\overline{\Psi}_p^u\left(t\right)\\ \overline{\Upsilon}_p^{u,y}\left(t\right)\end{bmatrix}\right),$$

one gets that $\left[\left(\overline{\Psi}_p^u\left(t\right)\right)^\top\ \left(\overline{\Upsilon}_p^{u,y}\left(t\right)\right)^\top\right]^\top$ is solution of (14a)-(14b) and by uniqueness of the solution of this differential equation, we have $\overline{\Psi}_p^u\left(t\right) = \eta_p\left(t\right)$ and $\overline{\Upsilon}_p^{u,y}\left(t\right) = \nu_p\left(t\right)$. Therefore we deduce from (18) and (20) that (13) holds and this concludes the proof.

### *3.2 Characterization of modes distinguishability*

To characterize controlled-distinguishability of modes, the following necessary and sufficient condition is proposed:

*Proposition 4.* (Controlled distinguishability). Let $u$ be an universal input of the augmented system associated to a pair $(p, q) \in \mathscr{P} \times \mathscr{P}$ of distinct modes i.e. an input such that $G_{pq}^u\left(\underline{\tau}, \overline{\tau}\right)$ is nonsingular. Then modes $p$ and $q$ are controlled distinguishable w.r.t. $u$ on $[\underline{\tau}\,;\overline{\tau}]$ iff

$$\Delta_{pq}^{u,\mathbf{0}^{[m]}}\left(\underline{\tau}, \overline{\tau}\right) := \mathrm{d}_{\mathcal{L}_2}\left(\mathbf{0}^{[m]}, \mathscr{Y}_{pq}^u\left([\underline{\tau}\,;\overline{\tau}]\right)\right) > 0. \quad (22)$$

**Proof.** Since $u$ is universal on $[\underline{\tau}\,;\overline{\tau}]$ for the augmented system associated to modes $p$ and $q$, then $\mathscr{Y}_{pq}^u\left([\underline{\tau}\,;\overline{\tau}]\right)$ is also closed in $\left(\mathcal{L}_2\left([\underline{\tau}\,;\overline{\tau}], \mathbb{R}^m\right), \|\cdot\|_{\mathcal{L}_2}\right)$; see Remark 3. This implies that $\mathbf{0}^{[m]} \notin \mathscr{Y}_{pq}^u\left([\underline{\tau}\,;\overline{\tau}]\right)$ iff condition (22) holds. Consequently, we deduce from Proposition 1 that condition (22) is necessary and sufficient for the controlled distinguishability of modes $p$ and $q$ w.r.t. the input $u$ on $[\underline{\tau}\,;\overline{\tau}]$.

*Remark 5.* An explicit expression of $\Delta_{pq}^{u,\mathbf{0}^{[m]}}\left(\underline{\tau}, \overline{\tau}\right)$ can be obtained similarly to (12). With this explicit expression, Proposition 4 then generalizes to bilinear operating modes, Lemma 6 of Motchon et al. (2018) that characterizes controlled distinguishability for linear operating modes. Furthermore, the value of $\Delta_{pq}^{u,\mathbf{0}^{[m]}}\left(\underline{\tau}, \overline{\tau}\right)$ can be obtained using a numerical method as in Proposition 3.

### *3.3 Reconstruction of the switching signal*

The notation $\widehat{t}_k$, $k \in \mathbb{N}$ will be used for the estimate of $t_k$. By convention $\widehat{t}_0 = t_0$. The estimated switching times $\{\widehat{t}_k\}_{k\in\mathbb{N}}$ will be determined successively. For the tests of controlled distinguishability and M-IO Compatibility, we suppose that:

*Assumption 2.* $\forall p \in \mathscr{P}$, $G_p^u\left(\widehat{t}_k, \widehat{t}_k + \boldsymbol{\tau}_d\right) \succ 0$ and $\forall\,(p,q) \in \mathscr{P} \times \mathscr{P}$, $p \neq q$, $G_{pq}^u\left(\widehat{t}_k, \widehat{t}_k + \boldsymbol{\tau}_d\right) \succ 0$.

This hypothesis expresses a regularly persistence condition of the excitation $u$ for the operating modes and the augmented systems (Bornard et al., 1989). Furthermore, for simplicity, we assume that $u$ is a discerning control for the modes:

*Assumption 3.* $\forall\,(p,q) \in \mathscr{P} \times \mathscr{P}$, $p \neq q$, modes $p$ and $q$ are controlled distingushable w.r.t $u$ on $[\widehat{t}_k\,;\widehat{t}_k + \boldsymbol{\tau}_d]$.

A procedure for designing regularly persistent discerning inputs may be proposed based on the result of Proposition 4.

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**Offline algorithm for the estimation of $\sigma$**

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**Inputs:** $t_0$, $T$, dwell-time $\boldsymbol{\tau}_d$, data $u_{|[t_0;T]}$ and $y_{|[t_0;T]}$, and small scalars $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$.

**Step 1.** **Initialisation**: Fix $k = 0$ and $\widehat{t}_k = t_0$.

**Step 2.** **Detection of the active modes** $\widehat{p}_k$: $\forall p \in \mathscr{P}$, compute $\Delta_p^{u,y}\left(\widehat{t}_k, \widehat{t}_k + \boldsymbol{\tau}_d\right)$ using (12)–(17). Identify the mode $\widehat{p}_k$ satisfying the relation $\Delta_{\widehat{p}_k}^{u,y}\left(\widehat{t}_k, \widehat{t}_k + \boldsymbol{\tau}_d\right) \leq \varepsilon_1$ and go to **Step 3.**

**Step 3.** **Estimation of the switching times** $\widehat{t}_{k+1}$:
Fix $a_k = \widehat{t}_k$ and $b_k = T$.
**While** $|a_k - b_k| > \varepsilon_2$, compute $c_k = (a_k + b_k)/2$.
  **If** $\Delta_{\widehat{p}_k}^{u,y}\left(\widehat{t}_k, c_k\right) > \varepsilon_1$ then $b_k \longleftarrow c_k$
  **Else**, $a_k \longleftarrow c_k$. **End if.**
  $\widehat{\tau}_{k+1} \longleftarrow b_k$
**End while**

**Step 4.** **Reinitialisation:** until all active modes are detected i.e. until $\Delta_{\widehat{p}_k}^{u,y}\left(\widehat{t}_k, T\right) \leq \varepsilon_1$, $k \longleftarrow k+1$ and $\widehat{t}_k \longleftarrow \widehat{t}_{k+1}$, and go to **Step 2.**

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Notice that since for all $p \in \mathscr{P}$, $\Delta_p^{u,y}\left(\widehat{t}_k, \widehat{t}_k + \boldsymbol{\tau}_d\right)$ are obtained with numerical errors using (12)–(17), in **Step 2**, for the detection of the active mode, inequality $\Delta_{\widehat{p}_k}^{u,y}\left(\widehat{t}_k, \widehat{t}_k + \boldsymbol{\tau}_d\right) \leq \varepsilon_1$ is used instead of the relation $\Delta_{\widehat{p}_k}^{u,y}\left(\widehat{t}_k, \widehat{t}_k + \boldsymbol{\tau}_d\right) = 0$. In **Step 3**, $\widehat{t}_{k+1}$ is computed as $\widehat{t}_{k+1} = \inf\{t\colon t > \widehat{t}_k,\ \Delta_{\widehat{p}_k}^{u,y}\left(\widehat{t}_k, t\right) > \varepsilon_1\}$ using a procedure similar to the bisection method. Condition $\Delta_{\widehat{p}_k}^{u,y}\left(\widehat{t}_k, c_k\right) > \varepsilon_1$ means that $\widehat{p}_k$ is not active on the whole interval $[\widehat{t}_k\,; c_k]$. The system has switched on the time interval $[\widehat{t}_k\,; c_k]$ and the switching time $t_{k+1}$ is then located in this interval.

## 4. AN ILLUSTRATIVE EXAMPLE

Let's consider the switched bilinear system (1) with the three operating modes ($\mathscr{P} = \{1, 2, 3\}$) described by the following matrices: $A_1 = \begin{bmatrix} -0.19 & 0.4 \\ -0.1763 & -1.19 \end{bmatrix}$, $A_2 = \begin{bmatrix} -0.25 & 0.5 \\ -0.75 & -1.5 \end{bmatrix}$, $A_3 = \begin{bmatrix} -0.146 & 0.5 \\ -0.17 & -1.25 \end{bmatrix}$, $B_1 = B_2 = B_3 = \begin{bmatrix} 1.7 & -1.9 \\ 1.8 & -1 \end{bmatrix}$, $C_1 = [1\ 0]$, $C_2 = [1\ 1]$, $C_3 = [0\ 1]$, $N_1 = \begin{bmatrix} -0.97 & 0.09 & 1 & 3 \\ 0.08 & 0.05 & - & 2 \end{bmatrix}$, $N_2 = \begin{bmatrix} -0.87 & 0.1 & 1 & 3 \\ 0.05 & 0.07 & -1 & 2 \end{bmatrix}$ and $N_3 = \begin{bmatrix} -0.097 & 0.09 & 1 & 3 \\ 0.06 & 0.04 & -1 & 2 \end{bmatrix}$. Algorithm proposed in Subsection 3.3 is applied in this part for the reconstruction of the switching signal

$$\sigma(t) = \begin{cases} 1 & \text{if} \quad t \in [3.52\,;8.45) \cup [17.5\,;20], \\ 2 & \text{if} \quad t \in [0\,;3.52) \cup [12.25\,;15.2), \\ 3 & \text{if} \quad t \in [8.45\,;12.25) \cup [15.2\,;17.5), \end{cases}$$

from the sinusoidal input signal $u(t) = [u_1(t)\ u_2(t)]^\top = [1.5\cos(2.5\,t)\ 2.5\sin(5\,t)]^\top$ and the output signal of Figure 1 generated by $u$ and $\sigma$.

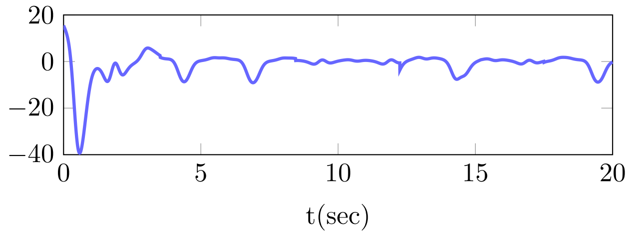


Fig. 1. Output signal $y$ generated by $u$ and $\sigma$

We choose $\boldsymbol{\tau}_d = 2s$ and $\varepsilon_1 = \varepsilon_2 = 10^{-3}$. We have verified successively on intervals $[\widehat{t}_k\,;\widehat{t}_k + \boldsymbol{\tau}_d)$ that Assumption 2 of regularly persistence of the input as well as Assumption 3 of distinguishability are satisfied. For page limitation, these numerical results are not presented. Estimated switching times and actives modes obtained with the proposed method are given in Table 1 (see columns 3 and 6).

Table 1. Numerical results for the estimation of switching times and active modes

| Iteration $k$ | $t_k$ | $\widehat{t}_k$ | $\lvert t_k - \widehat{t}_k \rvert$ | $\sigma(t_k)$ | $\widehat{\sigma}\left(\widehat{t}_k\right)$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 2 | 2 |
| 1 | 3.52 | 3.5213 | 0.0012708 | 1 | 1 |
| 2 | 8.45 | 8.4528 | 0.0028076 | 3 | 3 |
| 3 | 12.25 | 12.25 | 0.0002756 | 2 | 2 |
| 4 | 15.2 | 15.256 | 0.055753 | 3 | 3 |
| 5 | 17.5 | 17.504 | 0.0040873 | 1 | 1 |

These results prove the efficiency of the proposed method. All the active modes are well estimated with a minimum delay of 0.00028 second and a maximum delay of 0.056 second (see column 4 of Table 1). The different compatibility tests that provide the active mode detection and switching times estimation results of Table 1 are presented in Table 2.

## 5. CONCLUSION

A method for detecting the active modes and estimating the switching times for continuous-time switched bilinear

Table 2. Compatibility tests for the modes

| Iteration $k$ | $\Delta_p^{u,y}\left(\widehat{t}_k, \widehat{t}_k + \boldsymbol{\tau}_d\right)$ | | |
|---|---|---|---|
| | $p = 1$ | $p = 2$ | $p = 3$ |
| 0 | 137.51 | $6.0657 \times 10^{-8}$ | 80.949 |
| 1 | $1.6514 \times 10^{-6}$ | 1.3401 | 3.5772 |
| 3 | 1.9893 | 2.4465 | $1.8625 \times 10^{-7}$ |
| 3 | 0.70282 | 0.00027665 | 4.821 |
| 4 | 25.764 | 6.3962 | $3.8743 \times 10^{-11}$ |
| 5 | $2.2373 \times 10^{-8}$ | 1.0863 | 11.585 |

systems is proposed in disturbance-free situation. It is based on a compatibility test between the input-output data and the dynamical behaviour of the operating modes. The method applies only for data generated by persistent inputs that distinguish the input-output behaviour of the modes. A characterization of these inputs is also provided in terms of a necessary and sufficient condition for modes controlled distinguishability. Based on the established distinguishability condition, future works will focus on the experiment design problem for the proposed switching signal estimation algorithm i.e. the synthesis of persistent discerning control inputs of the modes. An extension of the results to bilinear systems with disturbances or measurement noises is also envisaged.

## REFERENCES

Babaali, M. and Pappas, G. (2005). Observability of switched linear systems in continuous time. In *Hybrid Systems: Computation and Control*, volume 3414 of the series Lecture Notes in Computer Science, 103–117. Springer Berlin Heidelberg.

Baglietto, M., Battistelli, G., and Scardovi, L. (2007). Active mode observability of switching linear systems. *Automatica*, 43(8), 1442–1449.

Bornard, G., Couenne, N., and Celle, F. (1989). Regularly persistent observers for bilinear systems. In J. Descusse and M. Fliess (eds.), *New Trends in Nonlinear Control Theory*, 130–140. Springer, Berlin, Heidelberg.

Chasparis, G.C. and Natschläger, T. (2014). Supervisory system identification for bilinear systems with application to thermal dynamics in buildings. In *2014 IEEE International Symposium on Intelligent Control (ISIC)*, 832–837. Juan Les Pin.

Cocquempot, V., El Mezyani, T., and Staroswiecki, M. (2004). Fault detection and isolation for hybrid systems using structured parity residuals. In *5th Asian Control Conference (IEEE Cat. No.04EX904)*, volume 2, 1204–1212. Melbourne, Victoria, Australia.

Etienne, L., Motchon, K.M.D., and Lecoeuche, S. (2019). Stability analysis for switched uncertain nonlinear systems with dwell time and delay in the active mode detection. *IEEE Control Systems Letters*, 3(2), 362–367.

Fliess, M., Join, C., and Perruquetti, W. (2009). Real-time estimation of the switching signal for perturbed switched linear systems. In *3rd IFAC Conference on Analysis and Design of Hybrid Systems, ADHS0́9*, CDROM. Zaragoza, Spain.

Garulli, A., Paoletti, S., and Vicino, A. (2012). A survey on switched and piecewise affine system identification. *IFAC Proceedings Volumes*, 45(16), 344–355. 16th IFAC Symposium on System Identification.

Goebel, R., Sanfelice, R.G., and Teel, A.R. (2012). *Hybrid dynamical systems: modeling, stability, and robustness.* Princeton University Press.

Gómez-Gutiérrez, D., Ramírez-Treviño, A., Ruiz-León, J., and Di Gennaro, S. (2012). On the observability of continuous-time switched linear systems under partially unknown inputs. *IEEE Transactions On Automatic Control*, 57(3), 732–738.

Hakem, A., Cocquempot, V., and Pekpe, K.M. (2016). Switching time estimation and active mode recognition using a data projection method. *International Journal of Applied Mathematics and Computer Science*, 26(4), 827–840.

Halimi, M., Millerioux, G., and Daafouz, J. (2015). A tutorial on model-based modes detection and discernibility for switched affine discrete-time systems. *IEEE Transactions on Automatic Control*, 60(6), 1501–1514.

Liberzon, D. (2003). *Switching in systems and control.* Birkhäuser Basel.

Lin, H. and Antsaklis, J. (2005). Stability and stabilizability of switched linear systems: a short survey of recent results. *IEEE Transactions on Automatic Control*, 54(2), 308–322.

Lou, H. and Yang, R. (2014). Necessary and sufficient conditions for distinguishability of linear control systems. *Acta Mathematicae Applicatae Sinica, English Series*, 30(2), 473–482.

Mincarelli, D., Floquet, T., and Belkoura, L. (2011). Active mode and switching time estimation for switched linear systems. In *50th IEEE Conference on Decision and Control and European Control Conference*, 1854–1859. Orlando, FL, USA.

Motchon, K.M.D. and Pekpe, K.M. (2019). Necessary and sufficient condition for controlled distinguishability of continuous-time bilinear systems. *IEEE Transactions On Automatic Control*, 64(7), 3013–3018.

Motchon, K.M.D., Pekpe, K.M., and Cassar, J.P. (2018). Robust discerning controls for the operating modes of linear switched systems subject to bounded unknown inputs. *Automatica*, 96, 159–165.

Shu, L., De Schutter, B., and Li, D. (2018). Stabilizing controller design for state-based switching bilinear systems. In *Proceedings of the 2018 IEEE Conference on Decision and Control (CDC)*, 6452–6457. Miami Beach, FL, USA.

Tian, Y., Floquet, T., Belkoura, L., and Perruquetti, W. (2009). Switching time estimation for linear switched systems: an algebraic approach. In *48h IEEE Conference on Decision and Control held jointly with 28th Chinese Control Conference*, 3909–3913. Shanghai, China.

Van Gorp, J., Defoort, M., Veluvolu, K.C., and Djemai, M. (2014). Hybrid sliding mode observer for switched linear systems with unknown inputs. *Journal of the Franklin Institute*, 351(7), 3987–4008.

Yang, J., Chen, Y., and Wang, X. (2015). Active mode identification and continuous state estimation for switched linear systems with unknown inputs and slow switching signal. *Circuits, Systems, and Signal Processing*, 34, 2193–2211.

Yuan, S., Zhang, L., De Schutter, B., and Baldi, S. (2018). A novel lyapunov function for a non-weighted $\mathcal{L}_2$ gain of asynchronously switched linear systems. *Automatica*, 87, 310–317.