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Stability by the First Approximation of a Water Hammer Model

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Abstract: Usually the water hammer models in hydroelectric engineering are described by the adapted Saint-Venant Partial Differential Equations with linear and nonlinear boundary conditions. If the dynamic head and the Darcy-Weisbach losses are neglected the PDEs are linear hyperbolic and can be tackled by associating a system of Neutral Functional Differential Equations with two delays - there are two conduits (the tunnel and the penstock). The time scale analysis shows that in certain cases arising from practice the dynamics of the penstock can be considered as described by ordinary differential equations. Consequently the water hammer dynamics has now a single time delay. The stability is then discussed by analyzing the characteristic equation: frequency domain methods combined with algebraic ones are implied. In this way stability by the first approximation is obtained. From the engineering point of view the results display the stabilizing role of the surge tank.

Keywords: Distributed parameter systems, Linear systems, Time delay, Stability

1. INTRODUCTION AND PROBLEM STATEMENT

Analysis of the dynamic processes in hydroelectric power generation is concerned with two basic phenomena: water hammer and frequency-megawatt control. Both aspects have increased significance in the context of present demands for “clean” and renewable (sustainable) energy: hydroelectric power belongs to both “clean” and renewable energy and the new, also old hydroelectric plants are operated in possibly heavier conditions leading to the aforementioned dynamics.

From the mathematical and engineering points of views the first aspect, i.e. water hammer, is better described by distributed parameters (wave propagation) while, the second one is described by lumped parameters, i.e. ordinary differential equations (ODEs) Aronovich et al. (1968), Popescu et al. (2003), Popescu (2008), Kishor et al. (2007). In the recent homologated models distributed and lumped parameters can appear as mixed - see a quite recent reference in the field Munoz-Hernandez et al. (2013). For the frequency/megawatt control the models are periodically revised e.g. by IEEE task forces - see Task Force (2013).

The interest in theoretical studies for such systems also arises from the nonlinear character of the description, oscillatory phenomena and other. Last but not least, the effects of the instabilities might have huge economic and environmental consequences. The scientific answer to such challenges includes improvement of the mathematical models, but also corresponding validation of the model.

The standard validation of a mathematical model is represented by the well-posedness in the sense of J. Hadamard; it consists of establishing existence of the solution, its uniqueness and its continuous dependence on data (parameters and initial conditions). An interesting discussion on the significance for the aforementioned validation steps can be found in Courant (1956). Two other steps stages of validation have been incorporated quite recently in the validation process thus, defining the so-called *augmented validation* Răsvan (2014): existence of certain physically significant invariant sets as well as inherent (i.e. without control) stability of certain steady state – the Stability Postulate of N. G. Četaev.

We described above one of the motivations of the present paper which will consider the model of a hydroelectric power plant with the following structure: lake (water reservoir), tunnel, surge tank with throttling, penstock and hydraulic turbine (Figure 1).

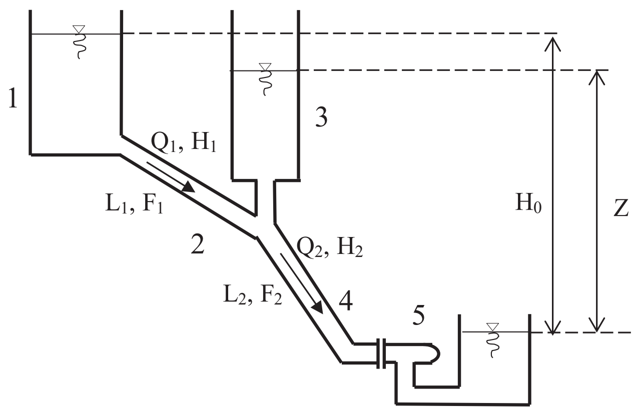

Fig. 1. Hydroelectric plant structure: 1. Lake. 2. Tunnel. 3. Surge tank. 4. Penstock. 5. Hydraulic turbine.

In the most general case, the tunnel and the penstock are considered to have distributed parameters: the description relies on the Saint-Venant equations including the Darcy-Weisbach losses. The presence of such elements with lumped parameters as the surge tank or the hydraulic turbine will give a non-standard form to the boundary conditions (BCs).

Two are the main control problems of the hydroelectric power plant: (P1) quenching of the water hammer and (P2) the control of the hydraulic turbine to ensure the frequency megawatt control of the Grid areas. The first goal is achieved by a suitable choice of the surge tank: the problem is important since it results in a construction which cannot be further "adjusted". The second one is connected to the speed controller of the turbine.

The present paper focuses on the water hammer analysis in the following special case. In certain real hydroelectric plants the penstock conduit is "short" in comparison to the tunnel and it is acceptable to consider lumped parameters for the penstock while those of the tunnel are distributed. Instead of using the standard lumped parameter modeling for conduits e.g. Jaeger (1977) and other, a (yet) formal approach based on singular perturbations is applied. The newly obtained model is considered in the water hammer case. By neglecting the dynamic head and the Darcy-Weisbach losses, a system allowing association of certain functional differential equations useful in model validation is introduced. Linearization around the equilibrium in the water hammer equations is performed and stability by the first approximation follows from root location of a characteristic equation. Here the classical approach of Čebotarev and Meiman (1949) is introduced, with a specific discussion encompassing the neutral type. The qualitative problems of the global behavior are introduced, the paper ending with conclusions and perspectives of further development.

2. THE BASIC MODEL AND ITS TRANSFORMATIONS

We consider the model of the hydroelectric plant sketched in Figure 1. The basic equations describing the transients of the hydroelectric plant are the following

- the tunnel

$$\partial_{x_1}\left(H_1+\frac{1}{2g}V_1^2\right)+\frac{1}{g}\partial_t V_1+\frac{\lambda_1}{2D_1}V_1|V_1|=0$$
$$\partial_t H_1+\frac{a_1^2}{g}\partial_{x_1}V_1=0,\ H_1(0,t)=H_0,\ 0<x_1<L_1 \tag{1}$$

- the penstock

$$\partial_{x_2}\left(H_2+\frac{1}{2g}V_2^2\right)+\frac{1}{g}\partial_t V_2+\frac{\lambda_2}{2D_2}V_2|V_2|=0$$
$$\partial_t H_2+\frac{a_2^2}{g}\partial_{x_2}V_2=0,\ 0<x_2<L_2 \tag{2}$$
$$V_2(L_2,t)=\alpha_q\sqrt{H_2(L_2,t)}$$

- the surge tank and the throttling

$$F_s\frac{\mathrm{d}Z}{\mathrm{d}t}=Q_1(L_1,t)-Q_2(0,t)=F_1V_1(L_1,t)-F_2V_2(0,t)$$
$$H_1(L_1,t)=Z(t)+\lambda_s\frac{\mathrm{d}Z}{\mathrm{d}t}=H_2(0,t) \tag{3}$$

- the hydraulic turbine

$$J\Omega_0\frac{\mathrm{d}\Omega}{\mathrm{d}t}=\eta_\theta\frac{\gamma}{2g}F_\theta V_2^3(L_2,t)-N_g \tag{4}$$

The notations for the state variables are as follows

- V_i, Q_i, H_i - water velocity, water flow and piezometric head ($i=1$ accounts for the tunnel and $i=2$ for the penstock);
- H_0 - piezometric head of the lake;
- Z - water level in the surge tank;
- Ω - turbine rotating speed; Ω_0 - the synchronous speed;

Next, the notations for system's parameters are as follows

- F_i, D_i, L_i ($i=1,2$) - the cross section areas, the hydraulic diameters and the lengths of the conduits;
- F_s, λ_s - the cross section area and the coefficient of hydraulic losses for the surge tank;
- F_θ - the regulated flow area of the hydraulic turbine;
- J - the moment of inertia of the hydraulic turbine;
- γ - specific weight of the water;
- η_θ - efficiency of the hydraulic turbine;
- $N_g=\Omega_0 M_g$ - the power supplied to the hydrogenerator, where M_g is the load torque;
- λ_i, $i=1,2$ - coefficients of the Darcy-Weisbach losses through the conduits;
- a_i, $i=1,2$ - propagation speeds of the water hammer along the conduits; g - gravitation acceleration;
- α_q - a flow coefficient at the input of turbine's wicket gates

Now we shall rate the physical variables to some significant constant values of them. The introduction of the rated (scaled) variables has at least three useful outcomes: independence with respect to the measurement units, a certain reduction of the numerical ill-conditioning and a certain reduction of the number of the parameters. According to standard reference of the field Jaeger (1977), Popescu (2008), the rating values are H_0 for the piezometric heads, $F_{\theta max}$ for cross-section areas and the maximal available flow $\bar{Q}=\alpha_q F_{\theta max}\sqrt{H_0}$ for flows.

Denote by lower case letters $h_i(\xi_i,t)$, $q_i(\xi_i,t)$, $z(t)$, f_θ the rated values of H_i, Q_i, Z, F_θ, respectively; here $\xi_i=x_i/L_i$ are the rated space coordinates along the two conduits – the water tunnel and the turbine penstock while $Q_i=F_iV_i$ are the water flows along the conduits; ν_g denotes the rated load mechanical power of the hydraulic turbine and $\varphi=\Omega/\Omega_0$ is the rated rotating speed of the turbine.

We introduce further the following time constants for each of the two conduits (tunnel and penstock) respectively

- the launching (start) time constant $T_{wi}=(L_i\bar{Q})(gH_0F_i)^{-1}$;
- the fill up time constant (at maximally available flow) $T_i=(F_iL_i)/\bar{Q}$;
- the propagation time constant $T_{pi}=L_i/a_i$, i=1,2

Introduce also the surge tank time constant $T_s=(F_sH_0)/\bar{Q}$ which is of the type "fill up" since it represents the fill up time of the surge tank at maximally available flow $\bar{Q}$, up to the maximally available piezometric head H_0.

We perform finally a change of time scale by introducing the rated time $\tau=t/T_1$, rating the time variable to the fill up time constant of the tunnel. In this way equations (1)- (4) become

$$\frac{T_{wi}}{T_1}\partial_\tau q_i+\partial_{\xi_i}\left(h_i+\frac{T_{wi}}{T_i}q_i^2\right)+\frac{1}{2}(\lambda_i g)\frac{L_i}{D_i}q_i|q_i|=0$$
$$\left(\frac{T_{pi}}{T_{wi}}\right)^2\frac{T_{wi}}{T_1}\partial_\tau h_i+\partial_{\xi_i}q_i=0\ ;\ h_1(0,\tau)=0\ ;\ 0<\xi_i<1 \tag{5}$$

$$\frac{T_s}{T_1}\frac{\mathrm{d}z}{\mathrm{d}\tau}=q_1(1,\tau)-q_2(0,\tau)$$
$$h_1(1,\tau)=z(\tau)+\frac{\lambda_s}{T_1}\frac{\mathrm{d}z}{\mathrm{d}\tau}=h_2(0,\tau) \tag{6}$$

$$q_2(1,\tau) = f_\theta(\tau)\sqrt{h_2(1,\tau)}\ , \ \frac{T_a}{T_1}\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = f_\theta(\tau)q_2^3(1,\tau) - \nu_g \tag{7}$$

In what remains of the paper we shall start from this model with rated variables and parameters.

3. STABILITY AND CONTROL PROBLEMS

Consider first some comments about the aforementioned model (5)-(7): it is in fact a boundary value problem (BVP) for a nonlinear system of PDEs. Since the boundary conditions (6) and (7) are in a feedback connection with the ODEs, we call this BVP nonstandard. The entire system is highly nonlinear: the PDEs contain both the quadratic dynamic heads and the quadratic Darcy-Weisbach losses. The boundary condition at the hydraulic turbine is nonlinear as well as the expression of the turbine available mechanical power.

On the other hand, the aforementioned nonlinear terms of the PDEs are considered negligible throughout the hydraulic engineering literature Aronovich et al. (1968), Jaeger (1977), Popescu et al. (2003), Popescu (2008). For instance, the Darcy-Weisbach losses are usually neglected in waterhammer and transients computations Popescu et al. (2003), Popescu (2008). In this case equations (5) are hyperbolic equations of conservation laws. The space variation of the dynamic head i.e. $\partial_{\xi_i}(q_i^2)$ is usually negligible in comparison to the other space variation - of the piezometric head $\partial_{\xi_i}h_i$. This assertion is documented by registered data from some hundreds of hydroelectric plants of former USSR Aronovich et al. (1968). With this assumption equations (5) are just linear hyperbolic PDEs - more precisely the lossless and distortionless wave propagation equations. The boundary conditions remain nevertheless nonlinear at $\xi_2 = 1$ ($x_2 = L_2$), where the hydraulic turbine is located. The turbine model is linear in the control variable f_θ but nonlinear with respect to the processed water flow $q_2(1,t)$.

A. The model (5)-(7) (and its simplified versions) will be considered in the study of what are called *normal and abnormal exploitation regimes*. As already mentioned in Section 1, the normal regimes are defined by the loaded turbine - part of the overall frequency megawatt control of the Grid. These regimes are considered for turbine controller design.

The abnormal regimes are determined by the so called *turbine shut down* - "ignited" by some sudden load fall. In these regimes the most important phenomenon is the water hammer along the two conduits. Due to its possible huge damaging effects, it has to be quenched; the quenching of water hammer oscillations can be viewed as a stability problem and the stabilizing "device" (subsystem) is the surge tank.

The time scale of the water hammer phenomena requires taking into account of the distributed parameters of the two conduits. This is the case of the analysis in Halanay and Popescu (1987) or more recently in Danciu et al. (2019b,a). We recall briefly the results in these papers. Neglecting the space variations of the dynamic heads and the Darcy-Weisbach losses, the two sets of PDEs become linear. Moreover, since the turbine shut down is assumed complete ($f_\theta(\tau) \equiv 0$) the only nonlinear boundary condition becomes linear. Consequently the stability problem concerns the following linear non-standard BVP for linear hyperbolic PDEs in one space dimension

$$\begin{aligned}
&\frac{T_{wi}}{T_1}\partial_\tau q_i + \partial_{\xi_i}h_i = 0\\
&\left(\frac{T_{pi}}{T_{wi}}\right)^2\frac{T_{wi}}{T_1}\partial_\tau h_i + \partial_{\xi_i}q_i = 0\ ;\ 0<\xi_i<1\\
&h_1(0,\tau)\equiv 1\ ,\ \frac{T_s}{T_1}\frac{\mathrm{d}z}{\mathrm{d}\tau} = q_1(1,\tau) - q_2(0,\tau)\\
&h_1(1,\tau) = z(\tau) + \frac{\lambda_s}{T_1}\frac{\mathrm{d}z}{\mathrm{d}\tau} = h_2(0,\tau)\ ;\ q_2(1,\tau)\equiv 0
\end{aligned} \tag{8}$$

It can be shown that its characteristic equation is described by a *first degree quasi-polynomial with two rationally independent time delays*. The algebraic difficulties of the stability criteria have been avoided by applying the Lyapunov method under the features of the *energy identity*.

B. Here we shall consider a simplified case resulting from the fact that system (8) is *a system with several time scales*. For instance, in the real case of the "Bicaz" Romanian hydroelectric plant Popescu (2008) one has $T_{w1}/T_1 \approx 1.46\times 10^{-2}$ while $T_{w2}/T_1 \approx 3.8\times 10^{-4}$. At the same time $T_s/T_1 \approx 0.5$. Consequently we may take $T_{w2}/T_1 = 0$. However $T_{p2}/T_{w2} \approx 9$ hence $(T_{p2}/T_{w2})^2(T_{w2}/T_1)$ cannot be considered 0. But we can change the time scales by introducing the auxiliary variable $\hat{q}_2$ *via* $q_2 = (T_{p2}/T_{w2})^2\hat{q}_2$. Equations (8) will be now for $i=2$

$$\begin{aligned}
&\left(\frac{T_{p2}}{T_{w2}}\right)^2\frac{T_{w2}}{T_1}\partial_\tau \hat{q}_2 + \partial_{\xi_2}h_2 = 0\\
&\frac{T_{w2}}{T_1}\partial_\tau h_2 + \partial_{\xi_2}\hat{q}_2 = 0\ ;\ 0<\xi_2<1
\end{aligned} \tag{9}$$

Considering the second equation of (9) as singularly perturbed we let $(T_{w2}/T_1) = 0$ hence $\hat{q}_2(\cdot,\tau)$ is constant on $[0,1]$. Integrating the first equation of (9) with respect to ξ_2 from 0 to 1 we obtain

$$\left(\frac{T_{p2}}{T_{w2}}\right)^2\frac{T_{w2}}{T_1}\frac{\mathrm{d}\hat{q}_2}{\mathrm{d}\tau} + h_2(1,\tau) - h_2(0,\tau) = 0 \tag{10}$$

hence

$$\frac{T_{w2}}{T_1}\frac{\mathrm{d}q_2}{\mathrm{d}\tau} + h_2(1,\tau) - h_2(0,\tau) = 0 \tag{11}$$

We would like now to substitute $h_2(1,\tau)$ and $h_2(0,\tau)$ from the boundary conditions of (8). However this is not possible since the condition $q_2(1,\tau)\equiv 0$ which results from (7) by letting $f_\theta \equiv 0$ leaves $h_2(1,\tau)$ undetermined.

Consequently we have to reconsider the turbine shutdown up to some minimal $\underline{f}_\theta$ corresponding to a minimal (residual) load of the hydraulic turbine. With this dependence at $\xi_2 = 1$ we obtain the following nonstandard BVP for the hyperbolic PDEs in one space dimension

$$\begin{aligned}
&\frac{T_{w1}}{T_1}\partial_\tau q_1 + \partial_{\xi_1}h_1 = 0\\
&\left(\frac{T_{p1}}{T_{w1}}\right)^2\frac{T_{w1}}{T_1}\partial_\tau h_1 + \partial_{\xi_1}q_1 = 0\ ;\ 0<\xi_1<1
\end{aligned} \tag{12}$$

$$
\begin{aligned}
&h_1(0,\tau)\equiv 1\ ;\ h_1(1,\tau)=z(\tau)+\frac{\lambda_s}{T_1}\frac{\mathrm{d}z}{\mathrm{d}\tau}\\
&\frac{T_s}{T_1}\frac{\mathrm{d}z}{\mathrm{d}\tau}=q_1(1,\tau)-q_2(\tau)\\
&\frac{T_{w2}}{T_1}\frac{\mathrm{d}q_2}{\mathrm{d}\tau}+\frac{1}{\sqrt{\underline{f}_\theta}}q_2^2-z(\tau)-\frac{\lambda_s}{T_1}\frac{\mathrm{d}z}{\mathrm{d}\tau}=0
\end{aligned}
\tag{13}
$$

It is now quite clear that we have two hyperbolic PDEs with their boundary conditions in feedback connection with a system of ODEs describing the dynamics of the surge tank level and of the flow through the lumped parameter penstock. Denote, in order to simplify the writing of further computations

$$
\gamma_i:=\frac{T_{pi}}{T_{wi}}\ ,\ \theta_i:=\frac{T_{wi}}{T_1}\ ,\ i=1,2\ ;\ \theta_s:=\frac{T_s}{T_1}\ ,\ \lambda_s':=\frac{\lambda_s}{T_s} \tag{14}
$$

Further, we eliminate $\mathrm{d}z/\mathrm{d}\tau$ in the boundary conditions and rewrite system (12) as follows

$$
\begin{aligned}
&\theta_1\partial_\tau q_1+\partial_{\xi_1}h_1=0\ ;\ \gamma_1^2\theta_1\partial_\tau h_1+\partial_{\xi_1}q_1=0\\
&h_1(0,\tau)\equiv 1\ ;\ h_1(1,\tau)-\lambda_s'q_1(1,\tau)=z(\tau)-\lambda_s'q_2(\tau)\\
&\theta_s\frac{\mathrm{d}z}{\mathrm{d}\tau}=q_1(1,\tau)-q_2(\tau)\\
&\theta_2\frac{\mathrm{d}q_2}{\mathrm{d}\tau}=\lambda_s'q_1(1,\tau)+z-\lambda_s'q_2-\frac{1}{\sqrt{\underline{f}_\theta}}q_2^2
\end{aligned}
\tag{15}
$$

As shows the last equation above, by its quadratic term arising from the nonlinear boundary condition $q_2(1,\tau)=f_\theta(\tau)\sqrt{h_2(1,\tau)}$, system (15) is a nonlinear system.

C. From now on we shall discuss the stability by the first approximation of the equilibrium of (15). At this point we compute this equilibrium from its equations deduced from (15) by letting the time derivatives to 0.

$$
\begin{aligned}
&\overline{h}'(\xi_1)\equiv 0\ ,\ \overline{q}'(\xi_1)\equiv 0\ ;\ \overline{h}_1(0)=1\\
&\overline{h}_1(1)-\lambda_s'\overline{q}_1(1)=\overline{z}-\lambda_s'\overline{q}_2\ ;\ \overline{q}_1(1)=\overline{q}_2\\
&\lambda_s'\overline{q}_1(1)+\overline{z}-\lambda_s'\overline{q}_2-\frac{1}{\sqrt{\underline{f}_\theta}}\overline{q}_2^2=0
\end{aligned}
\tag{16}
$$

The solution is easily obtained from (16) as follows

$$
\begin{aligned}
&\overline{h}_1(\xi_1)\equiv\overline{h}_1(0)=\overline{h}_1(1)=\overline{z}=1\\
&\overline{q}_1(\xi_1)\equiv\overline{q}_1(1)=\overline{q}_2\ ;\ \overline{q}_2^{\pm}=\pm\sqrt[4]{\underline{f}_\theta}
\end{aligned}
\tag{17}
$$

The two values of the flow $\overline{q}_2$ speak about two (mathematically) possible steady state regimes among which one - with negative flow $\overline{q}_2^-$ - is probably unstable since water cannot flow upstream indefinitely. Therefore it is worth studying stability by the first approximation of both steady states.

4. THE ASSOCIATED SYSTEM OF FUNCTIONAL DIFFERENTIAL EQUATIONS

Introduce first the following deviations with respect to the steady state $(\overline{q}_2,1,1,\overline{q}_2)$:

$$
\begin{aligned}
&\Phi_1(\xi_1,\tau)=q_1(\xi_1,\tau)-\overline{q}_2,\ \chi(\xi_1,\tau)=h_1(\xi_1,\tau)-1\\
&\zeta(\tau)=z(\tau)-1,\ \Phi_2(\tau)=q_2(\tau)-\overline{q}_2
\end{aligned}
\tag{18}
$$

where $\overline{q}_2=\overline{q}_2^+>0$ or $\overline{q}_2=\overline{q}_2^-<0$, i.e. $\overline{q}_2=\pm\sqrt[4]{\underline{f}_\theta}$. The BVP in deviations becomes

$$
\begin{aligned}
&\theta_1\partial_\tau\Phi_1+\partial_{\xi_1}\chi_1=0,\ \gamma_1^2\theta_1\partial_\tau\chi_1+\partial_{\xi_1}\Phi_1=0\\
&\chi_1(0,\tau)\equiv 0,\ \chi_1(1,\tau)-\lambda_s'\Phi_1(1,\tau)=\zeta(\tau)-\lambda_s'\Phi_2(\tau)\\
&\theta_s\frac{\mathrm{d}\zeta}{\mathrm{d}\tau}=\Phi_1(1,\tau)-\Phi_2(\tau)\\
&\theta_2\frac{\mathrm{d}\Phi_2}{\mathrm{d}\tau}=\lambda_s'\Phi_1(1,\tau)+\zeta-\left(\lambda_s'\pm\frac{2}{\sqrt[4]{\underline{f}_\theta}}\right)\Phi_2-\frac{1}{\sqrt[4]{\underline{f}_\theta}}\Phi_2^2
\end{aligned}
\tag{19}
$$

where the sign $\pm$ under the brackets points to the positive (up-down) flow through the penstock and to the negative (down-up), respectively. Observe that (19) is a nonstandard BVP like (15).

For this case, with linear hyperbolic PDEs, the analysis of the BVP can be done *via* the method developed in the papers of Myshkis and his co-workers Myshkis and Shlopak (1957), Abolinia and Myshkis (1960), Myshkis and Filimonov (1981, 2008), later in the papers of K.L. Cooke (and D. W. Krumme) Cooke (1970), Cooke and Krumme (1968) and also in Răsvan (2014). We describe briefly the approach.

Introduce first the Riemann invariants of (19), together with the converse relations

$$
\begin{aligned}
&r_1^{\pm}(\xi_1,\tau)=\frac{1}{2}[\pm\Phi_1(\xi_1,\tau)+\gamma_1\chi_1(\xi_1,\tau)]\\
&\Phi_1(\xi_1,\tau)=r_1^+(\xi_1,\tau)-r_1^-(\xi_1,\tau)\\
&\chi_1(\xi_1,\tau)=\frac{1}{\gamma_1}\left[r_1^+(\xi_1,\tau)+r_1^-(\xi_1,\tau)\right].
\end{aligned}
\tag{20}
$$

Substituting in (19) we obtain two BVPs in the Riemann invariants

$$
\begin{aligned}
&\gamma_1\theta_1\partial_\tau r_1^{\pm}\pm\partial_{\xi_1}r_1^{\pm}=0\ ,\ r_1^+(0,\tau)+r_1^-(0,\tau)=0\\
&(1+\gamma_1\lambda_s')r_1^-(1,\tau)+(1-\gamma_1\lambda_s')r_1^+(1,\tau)=\gamma_1\zeta(\tau)-\gamma_1\lambda_s'\Phi_2(\tau)\\
&\theta_s\frac{\mathrm{d}\zeta}{\mathrm{d}\tau}=r_1^+(1,\tau)-r_1^-(1,\tau)-\Phi_2\\
&\theta_2\frac{\mathrm{d}\Phi_2}{\mathrm{d}\tau}=\lambda_s'(r_1^+(1,\tau)-r_1^-(1,\tau))+\zeta-\\
&\qquad-\left(\lambda_s'\pm 2/\sqrt[4]{\underline{f}_\theta}\right)\Phi_2-\left(1/\sqrt[4]{\underline{f}_\theta}\right)\Phi_2^2.
\end{aligned}
\tag{21}
$$

Consider now the two families of characteristics defined by the differential equations

$$
\frac{\mathrm{d}\tau}{\mathrm{d}\xi_1}=\pm\gamma_1\theta_1\Longrightarrow\tau^{\pm}(\sigma;\xi_1,\tau)=\tau+(\sigma-\xi_1)\gamma_1\theta_1. \tag{22}
$$

We take into account that $r^+(\xi_1,\tau)$ is constant along $\tau^+(\sigma;\xi_1,\tau)$ while $r^-(\xi_1,\tau)$ is constant along $\tau^-(\sigma;\xi_1,\tau)$ to obtain the representation formulae

$$
\begin{aligned}
&r_1^+(\xi_1,\tau)=r_1^+(1,\tau+(1-\xi_1)\gamma_1\theta_1)\\
&r_1^-(\xi_1,\tau)=r_1^-(0,\tau+\xi_1\gamma_1\theta_1).
\end{aligned}
\tag{23}
$$

In this way the solution of the PDEs in the Riemann invariants are represented using their boundary values.

In particular, if $\tau^+(\cdot;\xi_1,\tau)$ can be extended backwards up to $\xi_1=0$ and $\tau^-(\cdot;\xi_1,\tau)$ forwards up to $\xi_1=1$, we can deduce from (23)

$$
\begin{aligned}
&r_1^+(0,\tau)=r_1^+(1,\tau+\gamma_1\theta_1)\\
&r_1^-(1,\tau)=r_1^-(0,\tau+\gamma_1\theta_1).
\end{aligned}
\tag{24}
$$

Denoting $w_1^+(\tau) := r_1^+(1,\tau)$, $w_1^-(\tau) := r_1^-(0,\tau)$, taking into account (24) and substituting into the boundary conditions of (21), the following system is obtained

$$\begin{aligned}
&\theta_s \frac{d\zeta}{d\tau} = w_1^+(\tau) - w_1^-(\tau+\gamma_1\theta_1) - \Phi_2 \\
&\theta_2 \frac{d\Phi_2}{d\tau} = \lambda_s'(w_1^+(\tau) - w_1^-(\tau+\gamma_1\theta_1)) + \zeta - \\
&\quad - \left(\lambda_s' \pm 2/\sqrt[4]{\underline{f}_\theta}\right)\Phi_2 - \left(1/\sqrt[4]{\underline{f}_\theta}\right)\Phi_2^2 \\
&w_1^-(\tau) + w_1^+(\tau+\gamma_1\theta_1) = 0 \\
&(1+\gamma_1\lambda_s')w_1^-(\tau+\gamma_1\theta_1) + (1-\gamma_1\lambda_s')w_1^+(\tau) \\
&\quad = \gamma_1\zeta(\tau) - \gamma_1\lambda_s'\Phi_2(\tau).
\end{aligned} \tag{25}$$

With a final notation $\eta_1^\pm(\tau) := w_1^\pm(\tau+\gamma_1\theta_1)$, system (25) can be given a form which is suitable for the construction of its solution by steps on the intervals $(k\gamma_1\theta_1, (k+1)\gamma_1\theta_1)$

$$\begin{aligned}
&(1+\gamma_1\lambda_s')\theta_s \frac{d\zeta}{d\tau} = -\gamma_1\zeta(\tau) - \Phi_2 + 2\eta_1^+(\tau-\gamma_1\theta_1) \\
&(1+\gamma_1\lambda_s')\theta_2 \frac{d\Phi_2}{d\tau} = \zeta - \left[\lambda_s' \pm \left(2/\sqrt[4]{\underline{f}_\theta}\right)(1+\gamma_1\lambda_s')\right]\Phi_2 \\
&\quad - \left(1/\sqrt[4]{\underline{f}_\theta}\right)\Phi_2^2 + 2\lambda_s'\eta_1^+(\tau-\gamma_1\theta_1) \\
&\eta_1^+(\tau) = -\eta_1^-(\tau-\gamma_1\theta_1) \\
&\eta_1^-(\tau) = -\frac{1-\gamma_1\lambda_s'}{1+\gamma_1\lambda_s'}\eta_1^+(\tau-\gamma_1\theta_1) \\
&\quad + \frac{\gamma_1}{1+\gamma_1\lambda_s'}\zeta(\tau) - \frac{\gamma_1\lambda_s'}{1+\gamma_1\lambda_s'}\Phi_2.
\end{aligned} \tag{26}$$

The definitions of $\eta_1^\pm$ and the successive substitutions in (21) show that a solution of (26) is associated to a solution of (21). The initial conditions for (26) are also obtained from those of (21) as follows: $\zeta(0), \Phi_2(0)$ migrate while $\eta_1^\pm(\tau)$ on $(-\gamma_1\theta_1, 0)$ are obtained from the initial conditions $r_1^\pm(\xi_1, 0)$. The following result can be easily proven by applying Theorem of Răsvan (2014)

Theorem 1. Let $(r_1^\pm(\xi_1,\tau), \zeta(\tau), \Phi_2(\tau))$ be a classical solution of (21) defined by certain initial conditions. Then, $(\zeta(\tau), \Phi_2(\tau), \eta_1^\pm(\tau))$ is a solution of (26) with the initial conditions suitably computed starting from those of (21). Conversely, let $(\zeta(\tau), \Phi_2(\tau), \eta_1^\pm(\tau))$ be a solution of (26). Then, $(r_1^\pm(\xi_1,\tau), \zeta(\tau), \Phi_2(\tau))$ where

$$\begin{aligned}
r_1^+(\xi_1,\tau) &= \eta_1^+(\tau - \xi_1\gamma_1\theta_1) \\
r_1^-(\xi_1,\tau) &= \eta_1^-(\tau - (1-\xi_1)\gamma_1\theta_1)
\end{aligned} \tag{27}$$

is a solution of (21) with the resulting initial conditions.

Theorem 1 established a one-to-one correspondence between the solutions of the two mathematical objects, (21) and (26) - in fact of (19) and (26). In this way, all mathematical results obtained for one mathematical object are projected back onto the other one. This will be the case with the stability results which follow.

5. STABILITY BY THE FIRST APPROXIMATION

In this section we shall consider systems (19), (21) and (26) without the quadratic term Φ_2^2. For the resulting linear(ized) systems, the exponential stability property follows from the location of the roots of a certain characteristic equation in $\mathbb{C}^-$. The characteristic equations coincide for the three mathematical objects up to a multiplier which is, nevertheless, an entire function of the complex variable. Therefore, the set of the roots is the same for the aforementioned characteristic equations. We focus on (26) and observe that stability is a qualitative property concerning behavior for large $t > 0$. It is thus reasonable to "eliminate" $\eta_1^+(\tau)$ by using the first difference equation and to deal with the system

$$\begin{aligned}
&(1+\gamma_1\lambda_s')\theta_s \frac{d\zeta}{d\tau} = -\gamma_1\zeta(\tau) - \Phi_2 - 2\eta_1^-(\tau-2\gamma_1\theta_1) \\
&(1+\gamma_1\lambda_s')\theta_2 \frac{d\Phi_2}{d\tau} = \zeta - \lambda_s''\Phi_2 - 2\lambda_s'\eta_1^-(\tau-2\gamma_1\theta_1) \\
&\eta_1^-(\tau) = \frac{\gamma_1}{1+\gamma_1\lambda_s'}\zeta(\tau) - \frac{\gamma_1\lambda_s'}{1+\gamma_1\lambda_s'}\Phi_2 + \frac{1-\gamma_1\lambda_s'}{1+\gamma_1\lambda_s'}\eta_1^-(\tau-2\gamma_1\theta_1)
\end{aligned} \tag{28}$$

where $\lambda_s'' = \left[\lambda_s' \pm \left(2/\sqrt[4]{\underline{f}_\theta}\right)(1+\gamma_1\lambda_s')\right]$.

This system of coupled delay-differential and difference equations belongs to the general class described by

$$\begin{aligned}
&\dot{x}(t) = A_0x(t) + A_1y(t-\tau),\ \dim x = n,\ \dim y = m \\
&y(t) = Cx(t) + Dy(t-\tau).
\end{aligned} \tag{29}$$

Here $n = 2$, $m = 1$. For these systems a necessary condition of exponential stability is the location of the eigenvalues of D inside the unit disk of $\mathbb{C}$; this condition accounts for the strong stability of the difference operator. In the case of (28) this condition reads

$$0 < |(1-\gamma_1\lambda_s')(1+\gamma_1\lambda_s')^{-1}| < 1 \tag{30}$$

which holds for $\gamma_1\lambda_s' > 0$. Now the characteristic equation is deduced in the easiest way directly from (19) viewed without the nonlinear quadratic term. Skipping the computational details, we obtain the characteristic quasi-polynomial as

$$\begin{aligned}
p(z) = \gamma_1 &\left[\frac{\lambda_s'\theta_s\theta_2}{(\gamma_1\theta_1)^2}z^2 + \frac{1}{\gamma_1\theta_1}(\theta_2 + (\lambda_s'' - \lambda_s')\lambda_s'\theta_s)z + \right. \\
&\left. + (\lambda_s'' - \lambda_s')\right]\cosh z + \left[\frac{\theta_s\theta_2}{(\gamma_1\theta_1)^2}z^2 + \frac{\lambda_s''\theta_s}{\gamma_1\theta_1}z + 1\right]\sinh z
\end{aligned} \tag{31}$$

Observe that (31) has exactly the form

$$p(z) = (a_2z^2 + a_1z + a_0)\cosh z + (b_2z^2 + b_1z + b_0)\sinh z \tag{32}$$

used in Čebotarev and Meiman (1949) to obtain Routh Hurwitz like criterion for quasi-polynomials. The purely algebraic results - given parameters subject to inequalities - were obtained by *strongly relying on function behavior in the frequency domain as stated in the very first result of Pontryagin (1942).* The aforementioned results of Čebotarev and Meiman (1949) are summarized in some "cases" - various combinations of coefficient signs; they have been completed recently by adding the so called "lost cases" Răsvan (2007).

In order to use them we remark that all coefficients in (31) are positive except $a_0 = \lambda_s'' - \lambda_s'$. The necessary stability conditions of Stodola type require however *positiveness of all coefficients*.

It follows that, as expected, the equilibrium with $\bar{q} < 0$ (corresponding to $\lambda_s'' - \lambda_s' < 0$ is unstable - as physical interpretation suggests. Therefore we shall focus on the other - "normal situation" with $\bar{q} > 0$ and $\lambda_s'' - \lambda_s' = +(2/\sqrt[4]{\underline{f}_\theta})(1+\gamma_1\lambda_s') > 0$.

Since now all coefficients are strictly positive, all Stodola-like conditions are fulfilled and, moreover, we are in the so called Case I - Čebotarev and Meiman (1949). Therefore the next step will be checking the sign of the expression $\sqrt{a_1b_1} - |\sqrt{a_0b_2} - \sqrt{a_2b_0}|$. The calculations are tedious but not quite straightforward. Observe first that

$$a_0b_2 - a_2b_0 = \frac{\gamma_1\theta_s\theta_2}{(\gamma_1\theta_1)^2}\left[\frac{2(1+\gamma_1\lambda_s')}{\sqrt[4]{\underline{f}_\theta}} - \lambda_s'\right] > 0$$

since $\gamma_1 > 1$ and $\underline{f}_\theta < 1$. Then another simple manipulation reduces the aforementioned inequality to

$$\left[1+\frac{2(1+\gamma_1\lambda_s')\lambda_s' - \theta_s}{\theta_2\sqrt[4]{\underline{f}_\theta}}\right]^{1/2}\left[1+\frac{2(1+\gamma_1\lambda_s')}{\lambda_s'\sqrt[4]{\underline{f}_\theta}}\right]^{1/2} > \\ > \left[\left(\frac{2(1+\gamma_1\lambda_s')}{\lambda_s'\sqrt[4]{\underline{f}_\theta}}\right)^{1/2} - 1\right] \tag{33}$$

which is nothing more but $\sqrt{(1+Y)(1+X^2)} > X-1$ which is straightforward. But if (33) holds, the so called *auxiliary equation* in Čebotarev and Meiman (1949) has no real roots and the positiveness of the coefficients is necessary and sufficient for root location in $\mathbb{C}^-$. Due to the neutral character of system (28), exponential stability would require root location in $\Re e(z) < -\alpha < 0$. But we have however inequality (30) i.e. the *difference operator strongly stable*. Asymptotic analysis of the zeros of the quasi-polynomials Bellman and Cooke (1963) show that the roots are well delimited from the imaginary axis $\imath\mathbb{R}$. Stability is indeed exponential.

6. CONCLUSIONS AND PERSPECTIVE

We have tackled in this paper the stability problem under water hammer for a hydroelectric power plant with surge tank. Taking into account the several time scales, the model was reduced to distributed parameters tunnel and lumped parameters penstock. The resulting non-standard BVP for hyperbolic PDEs has been discussed from the point of view of model validation and stability. Since the boundary conditions contained a nonlinear function of quadratic type, there were obtained two equilibria, one of them (the "normal") being exponentially stable by the first approximation while, the other one (the "abnormal") is unstable. However, this last aspect suggest to pursue the research towards global behavior where such phenomena like dissipativeness (the neglected nonlinear terms are very much alike to nonlinear damping) or even self-sustained oscillations (supposing that Poincaré-Bendixson type results for FDE - in particular NFDE – do exist) are quite probable to be met.

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