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Control of An Unstable Reaction-Diffusion PDE with Spatially-Varying Input Delay

Jie Qi *,** **Miroslav Krstic** ***

* *College of Information Science and Technology, Donghua University, Shanghai, China, 201620 .*
** *Engineering Research Center of Ministry of Education for Digitized Textile and Fashion Technology, Donghua University, Shanghai, China, 201620*
*** *Department of Mechanical Aerospace Engineering, University of California, San Diego, USA, CA 92093-0411*

Abstract: We design a predictor-based distributed feedback controller that guarantees exponential stability for a class of reaction-diffusion PDEs with spatially-varying input delay. First, an implicit backstepping transformation is introduced which contains the state of the target system on both sides of the definition and then an additional backstepping transformation is derived by a successive integration approach to arrive at a target system that is a distributed cascade of a 2D transport PDE into a 1D reaction-diffusion PDE. The resulting delay-compensating controller includes spatially-weighted state feedback and feedback of the earlier inputs in four differential spatial regions. The inverse transformation is also derived, to prove L^2 exponential stability.

Keywords: Spatial varying delay, PDE backstepping, Distributed Actuation, Reaction-diffusion PDE.

1. INTRODUCTION

Reaction-diffusion equations have been applied to model the dynamics of numerous spatio-temporal systems, including chemical reactions Aris (1965), lithium-ion batteries Tang et al. (2017); Mahamud and Park (2011), electric and magnetic fieldsBinns and Lawrenson (2013), online social networks, and multi-agent systems Frihauf and Krstic (2010); Meurer and Krstic (2011); Qi et al. (2019, 2015). These systems are often subject to transport phenomena which generate the presence of time delay in their actuation path. In the last few decades, major advances have been made in predictor-feedback control techniques for stabilization of ODEs (Ordinary Differential Equations) with various types of delays, including state-dependent delays Bekiaris-Liberis and Krstic (2013), time-varying delays Krstic (2010), time-varying input and state delays Bekiaris-Liberis and Krstic (2012) and unknown input delays Bresch-Pietri and Krstic (2010).

The stability analysis of PDEs (Partial Differential Equations) with delays are more complicated, so there are fewer results for PDE systems with delay. An example is Hashimoto and Krstic (2016), which considers the stabilization of reaction diffusion PDEs with state delay in the domain. Based on the Lyaponuv-Krasovskii functional method and a linear operator inequality, sufficient delay-dependent conditions of exponential stability are derived for a class of PDE systems subject to unknown and time-varying delay Fridman and Orlov (2009), whereas Solomon and Fridman (2015) applies a similar method to obtain interesting results for a semi-linear case with time-delay. By use of an observer to predict the future state, Selivanov and Fridman (2018) employ the estimated future state by an observer for feedback to compensate the input delay. Delays can be described as a transport PDE Krstic and Smyshlyaev (2008). In this way, a unstable reaction-diffusion system with actuator delay is represented by a transport PDE cascade system in Krstic (2009), where a backstepping approach is developed to stabilize it. A stabilizing feedback boundary control to compensate a constant input delay for reaction-diffusion PDE by decomposition of the state space into a stable part and a finite-dimensional unstable part has been developed in Prieur and Trelat (2019). Our recent work Qi et al. (2018) applies the backstepping technique to design a distributed in-domain actuator for an unstable reaction-diffusion PDE which is subject to a constant and arbitrarily large boundary input delay.

Few research on control design for unstable PDEs with spatially-varying delays. For multi-input finite-dimensional systems with distinct input delays in each input channel, where the distributed delays are modeled by integration. Bekiaris-Liberis and Krstic (2011) introduce an infinite-dimensional forwarding-backstepping transformation of the infinite-dimensional actuator states to compensate the distinct distributed delays. Tsubakino et al. (2016) consider distinct point delays a multi-input linear ODE system with different input delays in each input channel and propose a predictor-based state feedback controller for it via the backstepping technique. These two references for

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Email addresses: jieqi@dhu. edu.cn (Jie Qi), krstic@ucsd.edu

ODE systems are the most related to our work on control design for the reaction-diffusion PDE with spatially-varying delays.

In this paper, we design a distributed controller to stabilize a reaction-diffusion system with a spatially-varying input delay. Different delays arise at different spatial positions for distributed actuation. Since the delays depend on the spatial variables, a simple backstepping transformation is not applicable for the system. This is since the resulted predictor-control would contain feedback of future states. Thus, the challenge in the study of stability under spatially-varying input delay, compared to our work on constant input delay Qi et al. (2018), is to construct a new transformation which associates the original system with a stable target system in the form of a distributed cascade of 2D transport PDE into a 1D reaction-diffusion PDE. An intermediate implicit transformation contains the target state on both sides, so an additional challenge is to find the explicit form of the backstepping transformation which maps the original state to the target state. A successive integration approach is used to derive the kernel of the explicit transformation and prove the boundedness of the kernel. Also, the explicit form of the inverse transformation is found. In addition to proving the norm equivalence between the original system and the target one through dealing with the singularity at one boundary of the kernel function by use of the Parsevel's theorem, the exponential stability of the system in the original variables is established. Finally, numerical simulation results are provided to illustrate the theoretical result.

This paper is organized as follows. Section 2 presents the control design for the system with spatial-varying delay. The supportive simulation results are provided in Section 3. The paper ends with concluding remarks and a discussion of future work in Section 4.

2. CONTROL DESIGN

2.1 Problem description

Consider a reaction-diffusion PDE system given by

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t) + U(x, t - D(x)), \quad (1)$$

for $x \in (0,1)$, $t > 0$, with a delay of $D(x)$ time units which depends on x. We make the following two assumptions.

Assumption 1. $D(x)$ is a continuous and invertible function on $[0,1]$, $0 < M_1 \le D(x) \le M_2$ for any $x \in [0,1]$ (without loss of generality assuming $D(x)$ is increasing). The inverse function of $D(x)$ can be expressed as $x = D^{-1}(s)$ on $[0,1]$.

The boundary conditions are

$$u(0,t) = 0, \quad u(1,t) = 0. \quad (2)$$

For $D \equiv 0$, this is a trivial problem, solvable by many different feedback laws, the nominal one being

$$U(x,t) = -(\lambda + c)u(x,t), \quad c > 0, \quad (3)$$

which stabilizes the system (1)–(3) to a zero equilibrium. However, under the occurrence of delay, the system (1) with the nominal feedback law (3) becomes unstable and a delay compensator is needed to stabilize the system.

2.2 PDE representation in 2-D of the 1-D reaction diffusion PDE with delay on distributed input

Introducing a transport equation which alteratively represents the input delay, (1)-(2) is rewritten as

$$\begin{aligned}
&u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t) + v(x,0,t), && (4)\\
&u(t,0) = u(t,1) = 0, && (5)\\
&v_t(x,s,t) = v_s(x,s,t), \quad s \in [0, D(x)) && (6)\\
&v(x, D(x), t) = U(x,t), && (7)\\
&u(x,0) = u_0(x) && (8)\\
&v(x,s,0) = v_0(x,s). && (9)
\end{aligned}$$

where $U(x,t)$ is the input without delay, (u,v) is the state, and $D(x) > 0$ is a delay depending on x. The state of the input delay dynamics (6)-(9) is

$$v(x,s,t) = U(x, s + t - D(x)) \quad (10)$$

as $t \ge D(x) - s$.

We choose a stable target system as:

$$\begin{aligned}
&u_t(x,t) = u_{xx}(x,t) - cu(x,t) + z(x,0,t), && (11)\\
&u(t,0) = 0, \quad u(t,1) = 0, && (12)\\
&z_t(x,s,t) = z_s(x,s,t), \quad s \in [0, D(x)) && (13)\\
&z(x, D(x), t) = 0, && (14)\\
&u(x,0) = u_0(x), && (15)\\
&z(x,s,0) = z_0(x,s). && (16)
\end{aligned}$$

The solution of $z(x,s,t)$ is

$$z(x,s,t) = \begin{cases} z_0(x, s+t) & s+t \leqslant D(x) \\ 0 & s+t > D(x) \end{cases}. \quad (17)$$

2.3 Backstepping transformation in implicit form

We seek a backstepping transformation in the form

$$\begin{aligned}
z(x,s,t) &= v(x,s,t) + (c+\lambda)\int_0^1 \gamma(x,s,y)u(y,t)dy \\
&+ (c+\lambda)\int_x^1 \int_0^s \gamma(x, s-r, y)v(y,r,t)drdy \\
&+ (c+\lambda)\int_0^x \int_0^{\phi(y,s)} \gamma(x, s-r, y)z(y,r,t)drdy, \quad (18)
\end{aligned}$$

where

$$\phi(y,s) = \min\{D(y), s\}. \quad (19)$$

and kernel function $\gamma(x,s,y)$ is defined on $\mathcal{T} = \{[0,1] \times [0, D(x)] \times [0,1]\}$. The integration area (y,r) of the last term of (18) is shown in Figure 1.

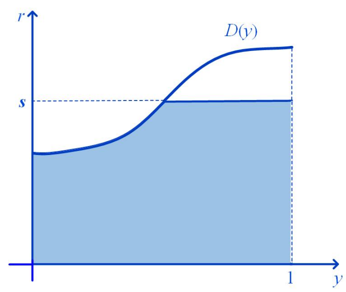


Fig. 1. The area of integration (y,r) for the last term of (18).

The transformation can be rewritten as

$$\begin{aligned} z(x,s,t) &= v(x,s,t) + (c+\lambda)\int_0^1 \gamma(x,s,y)u(y,t)dy \\ &+ (c+\lambda)\int_x^1\int_0^s \gamma(x,s-r,y)v(y,r,t)drdy \\ &+ (c+\lambda)\int_0^{D^{-1}(s)}\int_0^{D(y)} \gamma(x,s-r,y)z(y,r,t)drdy \\ &+ (c+\lambda)\int_{D^{-1}(s)}^x\int_0^s \gamma(x,s-r,y)z(y,r,t)drdy. \end{aligned} \tag{20}$$

Take the time and space derivative of $z(x,s,t)$,

$$\begin{aligned} z_t(x,s,t) &= v_s(x,s,t) + (c+\lambda)\gamma(x,s,1)u_x(1,t) \\ &- (c+\lambda)\gamma(x,s,0)u_x(0,t) \\ &+ (c+\lambda)\int_0^1 (\gamma_{yy}+\lambda\gamma)u(y,t)dy \\ &+ (c+\lambda)\int_0^x \gamma(x,s,y)v(y,0,t)dy \\ &+ (c+\lambda)\int_x^1 \gamma(x,0,y)v(y,s,t)dy \\ &+ (c+\lambda)\int_x^1\int_0^s \gamma_s(x,s-r,y)v(y,r,t)drdy \\ &+ (c+\lambda)\int_0^{D^{-1}(s)} \gamma(x,s-D(y),y)z(y,D(y),t)dy \\ &- (c+\lambda)\int_0^x \gamma(x,s,y)z(y,0,t)dy \\ &+ (c+\lambda)\int_0^{D^{-1}(s)}\int_0^{D(y)} \gamma_s(x,s-r,y)z(y,r,t)drdy \\ &+ (c+\lambda)\int_{D^{-1}(s)}^x \gamma(x,0,y)z(y,s,t)dy \\ &+ (c+\lambda)\int_{D^{-1}(s)}^x\int_0^s \gamma_s(x,s-r,y)z(y,r,t)drdy \end{aligned} \tag{21}$$

where we use integration by parts. And then we calculate

$$\begin{aligned} z_s(x,s,t) &= v_s(x,s,t) + (c+\lambda)\int_0^1 \gamma_s(x,s,y)u(y,t)dy \\ &+ (c+\lambda)\int_x^1 \gamma(x,0,y)v(y,s,t)dy \\ &+ (c+\lambda)\int_x^1\int_0^s \gamma_s(x,s-r,y)v(y,r,t)drdy \\ &+ (c+\lambda)\frac{d}{ds}D^{-1}(s)\int_0^s \gamma(x,s-r,D^{-1}(s))z(D^{-1}(s),r,t)dr \\ &+ (c+\lambda)\int_0^{D^{-1}(s)}\int_0^{D(y)} \gamma_s(x,s-r,y)z(y,r,t)drdy \\ &- (c+\lambda)\frac{dD^{-1}(s)}{ds}\int_0^s \gamma(x,s-r,D^{-1}(s))z(D^{-1}(s),r,t)dr \\ &+ (c+\lambda)\int_{D^{-1}(s)}^x \gamma(x,0,y)z(y,s,t)dy \\ &+ (c+\lambda)\int_{D^{-1}(s)}^x\int_0^s \gamma_s(x,s-r,y)z(y,r,t)drdy. \end{aligned} \tag{22}$$

From $z_t = z_s$ (13) and also from substituting $s = 0$ into transformation (18) and combining (4) and (11), we get

$$\begin{aligned} \gamma_s(x,s,y) &= \gamma_{yy}(x,s,y) + \lambda\gamma(x,s,y) \\ &- (c+\lambda)\int_0^x \delta(\zeta-y)\gamma(x,s,\zeta)d\zeta, \\ & x\in[0,1],\quad y\in(0,1),\quad 0\le s\le D(x) \end{aligned} \tag{23}$$

$$\gamma(x,s,0) = \gamma(x,s,1) = 0, \tag{24}$$

$$\gamma(x,0,y) = \delta(x-y). \tag{25}$$

The kernel equation has a solution as follows

$$\gamma(x,s,y) = \begin{cases} 2\mathrm{e}^{-cs}\Psi(x,s,y) & 0\le y\le x \\ 2\mathrm{e}^{\lambda s}\Psi(x,s,y) & x<y\le 1 \end{cases}, \tag{26}$$

where

$$\Psi(x,s,y) = \sum_{n=1}^{\infty} \mathrm{e}^{-n^2\pi^2 s}\sin(n\pi y)\sin(n\pi x). \tag{27}$$

Note that $\gamma(x,s,y)$ is bounded except at singular points $s = 0$ although it is not continuous at $x = y$.

Remark 1. Denote

$$R[u(t)](x) \triangleq (c+\lambda)\int_0^1 \gamma(x,s,y)u(y,t)dy,$$

$$P[v(t)](x,s) \triangleq (c+\lambda)\int_x^1\int_0^s \gamma(x,s-r,y)\cdot v(y,r,t)drdy,$$

$$S[z(t)](x,s) \triangleq (c+\lambda)\int_0^x\int_0^{\phi(y,s)} \gamma(x,s-r,y)\cdot z(y,r,t)drdy.$$

Transformation (18) has the structure of

$$z = v + R[u] + P[v] + S[z]. \tag{28}$$

It is clear that (28) contains z on both sides, so it is an implicit transformation. In this sense, it is necessary to find a transformation from v and u to z through an inverse operator $(I-S)^{-1}$ as follows

$$z = (I-S)^{-1}(v + P[v] + R[u]). \tag{29}$$

We find (29) in the next subsection.

2.4 Explicit form of backstepping transformation and controller

Apply the successive approximation to transformation (18) in form of (28), which gives

$$z^{n+1} = v + P[v] + R[u] + S[z^n]. \tag{30}$$

By letting the initial guess

$$z^0 = 0, \tag{31}$$

then

$$z^1 = v + P[v] + R[u], \tag{32}$$

and

$$z^{n+1} = z^1 + S[z^n]. \tag{33}$$

Begin with denoting the difference between two consecutive forms as

$$\Delta z^n = z^{n+1} - z^n, \tag{34}$$

and recall the initial state

$$\Delta z^0 = z^1, \tag{35}$$

which gives the following iteration formula

$$\Delta z^n = S[z^n - z^{n-1}] = S^n[\Delta z^0] = S^n[z^1]. \tag{36}$$

If the limit as follows exists

$$\lim_{n\to\infty} z^n = z, \tag{37}$$

which is equivalent to the following series converging,

$$z(x,s,t)=\sum_{n=0}^{\infty}\Delta z^n(x,s,t), \tag{38}$$

then the inverse operator $(I-S)^{-1}$ exists and z can be written as $z=(I-S)^{-1}z^1$. Next, we employ successive approximation on (36) so as to compute the series (38). First, for notational convenience, we rewrite (26) as

$$\gamma(x,s,y)=\begin{cases}\gamma_1(x,s,y) & 0\le y\le x\\ \gamma_2(x,s,y) & x<y\le 1\end{cases}. \tag{39}$$

From (36),

$$\begin{aligned}\Delta z^1(x,s,t)=&(c+\lambda)\int_0^x\int_0^{\phi(y,s)}\gamma_1(x,s-r,y)\\ &\cdot z^1(y,r,t)drdy,\end{aligned} \tag{40}$$

and by exchanging the integration orders of $drdy$ and dr_1dy_1 and variables replacing with $y\Leftrightarrow y_1$ and $r\Leftrightarrow r_1$, we rewrite Δz^2 as

$$\Delta z^2=\int_0^x\int_0^{\phi(y,s)}K^2(x,s,y,r)z^1(y,r,t)drdy \tag{41}$$

where

$$\begin{aligned}K^2(x,s,y,r)=&(c+\lambda)^2\int_y^x\int_r^{\phi(y_1,s)}\gamma_1(x,s-r_1,y_1)\\ &\cdot\gamma_1(y_1,r_1-r,s)dr_1dy_1.\end{aligned}$$

Continue iteration and exchanging the integration order, we observe the pattern as

$$\begin{aligned}\Delta z^n=&\int_0^x\int_0^{\phi(y,s)}K^n(x,s,y,r)z^1(y,r,t)drdy,\\ &n=2,3,\cdots\end{aligned} \tag{42}$$

where

$$K^1=(c+\lambda)\gamma_1(x,s-r,y), \tag{43}$$

$$\begin{aligned}K^n=&(c+\lambda)^n\int_y^x\int_r^{\phi(y_1,s)}\int_y^{y_1}\int_r^{\phi(y_2,r_1)}\cdots\\ &\int_y^{y_{n-2}}\int_r^{\phi(y_{n-1},r_{n-2})}\gamma_1(x,s-r_1,y_1)\gamma_1(y_1,r_1-r_2,y_2)\\ &\cdots\gamma_1(y_{n-2},r_{n-2}-r_{n-1},y_{n-1})\gamma_1(y_{n-1},r_{n-1}-r,y)\\ &dr_{n-1}dy_{n-1}\cdots dr_1dy_1 \qquad n=2,3,\cdots\end{aligned} \tag{44}$$

The expression (44) also can be rewritten recursively as

$$\begin{aligned}K^n(x,s,y,r)=&(c+\lambda)^n\int_y^x\int_r^{\phi(y_1,s)}\gamma_1(x,s-r_1,y_1)\\ &\cdot G^{n-1}(y_1,r_1)dr_1dy_1,\end{aligned} \tag{45}$$

$$\begin{aligned}G^{n-1}(y_1,r_1)=&\int_y^{y_1}\int_r^{\phi(y_2,r_1)}\gamma_1(y_1,r_1-r_2,y_2)\\ &\cdot G^{n-2}(y_2,r_2)dr_2dy_2,\end{aligned} \tag{46}$$

$$\vdots$$

$$\begin{aligned}G^2(y_{n-2},r_{n-2})=&\int_y^{y_{n-2}}\int_r^{\phi(y_{n-1},r_{n-2})}\\ &\gamma_1(y_{n-2},r_{n-2}-r_{n-1},y_{n-1})\\ &\cdot G^1(y_{n-1},r_{n-1})dr_{n-1}dy_{n-1},\end{aligned} \tag{47}$$

$$G^1(y_{n-1},r_{n-1})=\gamma_1(y_{n-1},r_{n-1}-r,y). \tag{48}$$

Assembling all the Δz^n, we obtain

$$\begin{aligned}z(x,s,t)=&z^1(x,s,t)\\ &+\int_0^x\int_0^{\phi(y,s)}Q(x,s,y,r)z^1(y,r,t)drdy,\end{aligned} \tag{49}$$

where

$$\begin{aligned}Q(x,s,y,r)=&\sum_{n=1}^{\infty}K^n(x,s,y,r)\\ :=&K^1(x,s,y,r)+H(x,s,y,r),\end{aligned} \tag{50}$$

with

$$H(x,s,y,r)=\sum_{n=2}^{\infty}K^n(x,s,y,r). \tag{51}$$

The Q are defined in (50), using γ defined in (26), γ_1, γ_2 defined in (39), K^1 defined in (43), K^n defined in (45) which in turn are defined using $G^{n-1},\cdots,G^1$ defined in (46)-(48). According to (32), z^1 only depends upon u and v. Recalling (29), we rewrite (49) as

$$z=\mathcal{T}[v+R[u]+P[v]], \tag{52}$$

where

$$\begin{aligned}\mathcal{T}[z]=&(I-S)^{-1}[z]\\ =&z+(c+\lambda)\int_0^x\int_0^{\phi(y,s)}Q(x,s,y,r)z(y,r)drdy.\end{aligned} \tag{53}$$

It is worth noting that the right-hand side of (49) does not contain z since $z^1(x,s,t)$ only depends upon $u(\cdot,t)$ and $v(\cdot,\cdot,t)$. Thus (49) is a mapping which transforms the original system (4)-(9) to the target system (11)-(16) though the transformations $(v,u)\mapsto(z_1,u)$ and $(z_1,u)\mapsto(z,u)$.

So far, we have computed the expression of the inverse operator of $(I-S)^{-1}$ in (49). We can apply successive integration approach to prove the convergence of the series (51). Due to limited space, we only give a sketch of the proof. Substituting (26) into (47) with (48), we find

$$\begin{aligned}&|G^2(y_{n-2},r_{n-2})|\\ \le&\Big(\int_r^{r_{n-2}}\int_0^1 4e^{-2c(r_{n-2}-r_{n-1})}\\ &\cdot\Psi^2(y_{n-1},r_{n-2}-r_{n-1},y_{n-2})dy_{n-1}dr_{n-1}\Big)^{\frac{1}{2}}\cdot\\ &\Big(\int_r^{r_{n-2}}\int_0^1 4e^{-2c(r_{n-1}-r)}\\ &\cdot\Psi^2(y,r_{n-1}-r,y_{n-1})dy_{n-1}dr_{n-1}\Big)^{\frac{1}{2}}\\ \le&\sum_{m=1}^{\infty}\frac{1}{c+m^2\pi^2}\le\frac{1}{6},\end{aligned} \tag{54}$$

where we apply the Cauchy Schwarz inequality and Parseval's theorem. In a similar way, we obtain

$$|G^3(y_{n-3},r_{n-3})|\le\left(\frac{1}{6}\right)^{\frac{3}{2}}(r_{n-3}-r)^{\frac{1}{2}}(y_{n-3}-y)^{\frac{1}{2}} \tag{55}$$

and, in turn,

$$|G^4(y_{n-3},r_{n-3})|\le\left(\frac{1}{6}\right)^2\frac{(r_{n-4}-r)(y_{n-4}-y)}{2} \tag{56}$$

Continuing with such calculations gives

$$|G^{n-1}(x,s)|\le\left(\frac{1}{6}\right)^{\frac{n-1}{2}}\frac{(r_1-r)^{\frac{n-3}{2}}(y_1-y)^{\frac{n-3}{2}}}{(n-3)!}. \tag{57}$$

Finally, we obtain

$$|K^n(x,s)| \leq (c+\lambda)^n \left(\frac{1}{6}\right)^{\frac{n}{2}} \frac{(s-r)^{\frac{n-2}{2}}(x-y)^{\frac{n-2}{2}}}{(n-2)!}. \quad (58)$$

Hence,

$$|H(x,s,y,r)| = \frac{(c+\lambda)^2}{6} \mathrm{e}^{\frac{(c+\lambda)\sqrt{(s-r)(x-y)}}{\sqrt{6}}}, \quad (59)$$

which implies that the series of $|H|$ converges.

Consequently, the transformation (29) is bounded. Substitute $s = D(x)$ into transformation (49) with (32), and combine (7), (14) and (10). Exchanging the integration order, after a lengthy computation, the control is given as

$$\begin{aligned} U(x,t) = -(c+\lambda)\Bigg(&\int_0^1 R(x,D(x),y)u(y,t)dy \\ &+\int_x^1\int_{t-D(y)}^{t-(D(x)-D(y))} \gamma_2(x,D(x)-D(y)+t-\tau,y)U(y,\tau)d\tau dy \\ &+\int_0^x\int_{t-D(y)}^{t} \frac{1}{c+\lambda}Q(x,D(x),y,\tau+D(y)-t)U(y,\tau)d\tau dy \\ &+\int_0^1\int_{t-D(y)}^{t-(D(y)-D(0))} P_1(x,D(x),y,\tau+D(y)-t)U(\eta,\tau))d\tau d\eta \\ &+\int_0^1\int_{t-(D(y)-D(0))}^{t-\min\{0,D(x)-D(y)\}} P_2(x,D(x),y,\tau+D(y)-t)U(\eta,\tau)d\tau d\eta\Bigg) \end{aligned} \quad (60)$$

where

$$R(x,D(x),y) = \gamma(x,D(x),y) - \int_0^x\int_0^{D(\zeta)} Q(x,D(x),\zeta,r)\gamma(\zeta,r,y)d\zeta dr, \quad (61)$$

$$P_1(x,D(x),y,r) = \int_0^{\min\{y,x\}}\int_r^{D(\zeta)} Q(x,D(x),\zeta,\iota)\gamma_2(\zeta,\iota-r,y)d\iota d\zeta, \quad (62)$$

$$P_2(x,D(x),y,r) = \int_{D^{-1}(r)}^{\min\{y,x\}}\int_r^{D(\zeta)} Q(x,D(x),\zeta,\iota)\gamma_2(\zeta,\iota-r,y)d\iota d\zeta, \quad (63)$$

which are defined, respectively, using γ defined in (26), γ_1, γ_2 defined in (39) and Q defined in (50), which in turn are defined using H defined in (51) and (45)-(48). The controller (60) contains the feedback of the states (first term) and feedback of the past control (last four terms) which compensates the delay.

2.5 Inverse backstepping transformation

The proposed transformation (18) is invertible. To see this, we postulate the inverse transformation which transforms the target system back to the original system as follows

$$\begin{aligned} v(x,s,t) = z(x,s,t) &- (c+\lambda)\int_0^1 \eta(x,s,y)u(y,t)dy \\ &- (c+\lambda)\int_0^1\int_0^{\phi(y,s)} \eta(x,s-r,y)z(y,r,t)drdy, \end{aligned} \quad (64)$$

where $\eta(x,s,y)$ defined on $[0,1]\times[0,D]\times[0,1]$ is a scalar kernel function and $\phi(y,s)$ is defined in (19). From the equivalence between the target and the original systems, the kernel function satisfies

$$\eta_s(x,s,y) = \eta_{yy}(x,s,y) - c\eta(x,s,y), \quad (65)$$

$$\eta(x,s,0) = 0, \quad \eta(x,s,1) = 0, \quad (66)$$

$$\eta(x,0,y) = \delta(x-y). \quad (67)$$

The solution of the kernel equation (65)-(67) is expressed as

$$\eta(x,s,y) = \sum_{n=1}^{\infty} 2\mathrm{e}^{-(c+n^2\pi^2)s}\sin(n\pi y)\sin(n\pi x).$$

Remark 2. The kernel of the inverse transformation can be expressed in an explicit equation which is the same as the form with constant delay Qi et al. (2018). It is because the boundary condition $z(x,D(x),t) = 0$ is used in the derivation of the inverse kernel equation.

Since there exists the inverse transformation, the L^2 norm equivalence between the original system (4)-(8) and (11)-(16) can be established. Furthermore, the target system (11)-(16) is exponentially stable in L^2 norm , which implies that the original system (4)-(8) is also exponentially stable in L^2 norm and thus gives the following theorem:

Theorem 1. Consider the system consisting of the plant (4)-(9) and the control law (60). Let $H_E^1[0,1] = \{f \in H^1(0,1), f(0) = f(1) = 0\}$. For any initial conditions $u_0 \in H_E^1[0,1]$, $v_0(x,s) \in L^2([0,1]\times[0,D(x)])$ are compatible such that $u_0(0) = u_0(1) = 0$, $v_0(x,D) = U(x,0)$, then the system is exponentially stable, i.e.,

$$V_1(t) \leq M\mathrm{e}^{-\alpha t}V_1(0), \quad (68)$$

where M, α are positive constants.

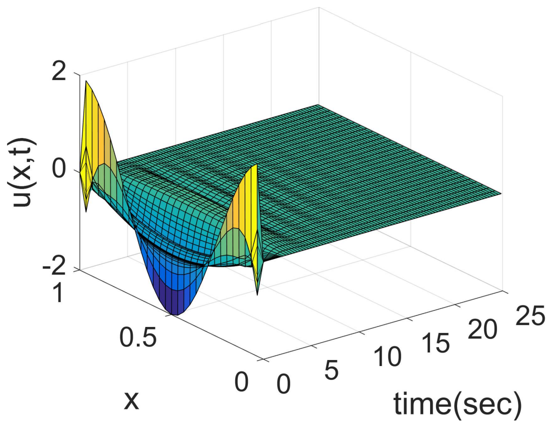
3. SIMULATION

To illustrate the feasibility of the proposed control law for the reaction-diffusion PDE system with spatially-varying input delay, we provide an example for the PDE system (4)-(9) with spatially-varying input delay. In the numerical example, we set reaction coefficient $\lambda = 10$, delay function is $D(x) = x+1$. The initial conditions are chosen as $u_0 = 2\cos(2\pi x)$, $v_0 = \sin(\pi x)\cos(\pi s)$, and then apply the controller (60). We discrete the partial differential equation by the finite difference method using the Crank-Nicolson scheme. The step sizes for discretization of x, t and s are denoted by Δx, Δt and Δs, respectively. In simulation let $\Delta t = 0.01$, $\Delta x = \frac{1}{M}$, and $\Delta s(x) = \frac{D(x)}{M}$ with $M = 31$. Applying the finite difference method, the dynamics of the state $u(x,t)$ is obtained which are illustrated in Fig.2. It is shown that the state will converge after about $5s$.

Fore more clear illustration, the norm of $u(x,t)$ and control effort are shown in Fig.3 (a) and Fig.3 (b), respectively. Since in-domain control applied, the value of the control effort is not as large as that of the boundary control.

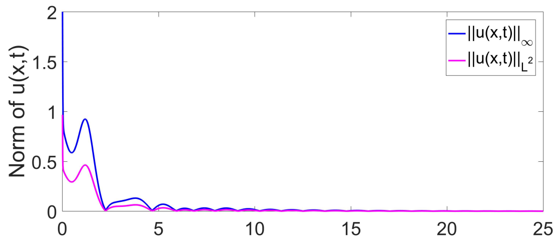
4. CONCLUSION

In this paper, we design a compensated distributed controller which stabilizes a reaction-diffusion system subject to spatially-varying input delay. First, we introduce an implicit backstepping transformation which results in well-posed kernel equation. Based the implicit transformation, an explicit form of backstepping transformation with a bounded kernel is derived by the successive approach. The control is obtained by combining the transformation and the boundary condition at $s = D(x)$. The explicit inverse transformation is obtained, which establishes the

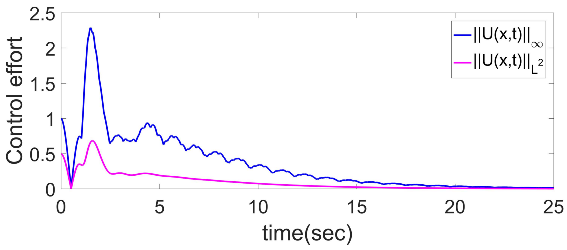


(a)

Fig. 2. The dynamics of the state $u(x,t)$.



(a)



(b)

Fig. 3. (a) The 2-norm and ∞-norm of the dynamical state $u(x,t)$, (b) The 2-norm and ∞-norm of the control effort.

exponential stability of the system in the original variables. The numerical simulations are presented to support the theoretical statements. Further research includes trying to design an observer for the system with sensor delays.

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