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Robust Stackelberg Games via Static Output Feedback Strategy for Uncertain Stochastic Systems with State Delay ⋆

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Abstract: In this paper, a robust Stackelberg game for a class of uncertain stochastic systems with state delay is investigated. After introducing some definitions and preliminaries, we derive the conditions for the existence of the robust static output feedback (SOF) Stackelberg strategy set such that the upper bounds of leader's cost function and the weighted cost function of the followers are minimized respectively. In order to obtain the robust SOF Stackelberg strategy set, a heuristic algorithm is proposed based on the stochastic Lyapunov type matrix equations (SLMEs) and the linear matrix inequalities (LMIs). In particular, it is shown that robust convergence is guaranteed by applying the Krasnoselskii-Mann (KM) iterative algorithm. An academic numerical example is presented to demonstrate the effectiveness of the proposed method.

Keywords: Stackelberg games, H_∞ control, stochastic systems, numerical algorithms.

1. INTRODUCTION

Over the last decade, robust Nash and Stackelberg games for linear stochastic systems (LSSs) have been investigated in an H_2/H_∞ framework (see Zhang et al. (2016); Chen and Zhang (2004) and the references therein). The studies has extended from theoretical ones to some practical applications such as multi-agent systems (Moon and Basar (2017)), communication systems (Saksena and Cruz (1985); Mukaidani et al. (2019)), social network systems (Bauso et al. (2016)), and so on. Basically, such studies have been made for systems with either deterministic external disturbances or system uncertainties resulting from un-modeled dynamics (Mukaidani (2013); Mukaidani et al. (2018a)). However, limited efforts have been made on the research of robust Nash and Stackelberg games for the LSSs with both deterministic external disturbances and system uncertainties (Mukaidani et al. (2018c)). Moreover, since complete state information is not always available, it is more realistic for players to construct their strategies based on local or partial state information. A typical information structure with only local or partial state information is the static output feedback (SOF) information structure.

On the other hand, in addition to deterministic external disturbances and system uncertainties, time-delay in state variables and/or control variables of a system can give rise a challenge in the study of dynamic games for LSSs. Very recently, Nash games for uncertain Markov jump delay stochastic systems have been investigated (Mukaidani et al. (2019)). The robust SOF Nash strategies have been constructed. However, it is still an unsolved problem to find a robust SOF Stackelberg strategy set for uncertain delay stochastic systems. Since most social and engineering systems are of hierarchical decision structures with many players (decision makers) and different objectives, it is significant to consider such a decision making problem as a Stackelberg game.

In this paper, a robust Stackelberg game for a class of uncertain delay stochastic systems (UDSSs) with external disturbances and system uncertainties is investigated. Different from the robust Nash game in Mukaidani et al. (2019), this paper studies the Stackelberg game with one leader and N Pareto-cooperative followers. Moreover, in comparison with the existing studies (Mukaidani (2013); Mukaidani et al. (2018a)), this paper focuses on the Stackelberg game for stochastic systems with deterministic external disturbances, system uncertainties, and time-delay in state variables. The main contributions of this paper are as follows. First, using the guaranteed cost control technique (Moheimani and Petersen (1996)), we derive the condition for the existence of the robust SOF Stackelberg strategy set by means of bilinear matrix inequalities (BMIs). In other words, the existence condition is represented as the solvability condition of BMIs in an optimization problem with constraints. Note that BMIs give rise a challenge to construct the robust

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SOF Stackelberg strategy set because they are generally difficult to solve. Upper bounds of the leader's cost function and the weighted cost function of the followers are minimized, respectively, in the presence of system uncertainties. Second, a computational framework for validating heuristic algorithms is proposed to compute the relevant solution set numerically. Instead of solving BMIs, a heuristic algorithm is developed by solving the stochastic Lyapunov type matrix equations (SLMEs) and the linear matrix inequalities (LMIs). It is worth pointing out that the convergence robustness of the proposed algorithm is attained by applying the Krasnoselskii-Mann (KM) iterative algorithm (Yao et al. (2009)). Finally, in order to demonstrate the effectiveness and usefulness of the proposed algorithm, a simple academic example is solved numerically.

Notation: The notations used in this paper are fairly standard: $\mathbb{E}[\ \cdot]$ stands for the conditional expectation operator; $\mathscr{L}_F^2([0,\ \infty),\ \mathbb{R}^k)$ denotes the space of all measurable functions $u(t,\ \omega)\ :\ [0,\ \infty)\times\Omega\to\mathbb{R}^k$, which are F_t-measurable for every $t\geq 0$, and $\mathbb{E}[\int_0^\infty |u(t)|^2 dt] < \infty$, $i\in\mathscr{D}$; $C([-h,0];\mathbb{R}^n)$, $h>0$, denotes the family of continuous functions ϕ from $[-h,0]$ to $\mathbb{R}^n$ with norm $\|\phi\| = \sup_{-h\leq\theta\leq 0}\|\phi(\theta)\|$; $\lambda_{\max}[\cdot]$ and $\lambda_{\min}[\cdot]$ denote its largest and smallest eigenvalues, respectively.

2. PRELIMINARY RESULTS

Consider the following UDSS

$$dx(t) = [A(t)x(t) + A_h x(t-h) + B_v v(t)]dt$$
$$+A_p(t)x(t)dw(t),\ \ x(t) = \phi(t),\ t\in[-h,\ 0], \quad (1a)$$
$$z(t) = Hx(t), \quad (1b)$$

where $x(t)\in\mathbb{R}^n$ denotes the state vector, $v(t)\in\mathbb{R}^{m_v}$ denotes the external disturbance, $z(t)\in\mathbb{R}^{n_z}$ denotes the controlled output, $w(t)\in\mathbb{R}$ denotes a one-dimensional standard Wiener process defined in the filtered probability space, $h\in(0,+\infty)$ denotes the time-delay of the UDSS, and $\phi(t)$ a real-valued initial function. It is assumed that, for all $\delta\in[-h,\ 0]$, there exists a scalar, $\sigma>0$, such that $\|x(t+\delta)\|\leq\sigma\|x(t)\|$ (Wang et al. (2002)).

Let $A(t)$ and $A_p(t)$ be matrices in the following forms:

$$A(t) = A + D\Theta(t)E_a, \quad (2a)$$
$$A_p(t) = A_p + D_p\Theta_p(t)E_{pa}, \quad (2b)$$

where $\Theta^T(t)\Theta(t)\leq I_{n_a}$, $\Theta_p^T(t)\Theta_p(t)\leq I_{n_a}$.
The coefficient matrices are constant; $\Theta(t)$, $\Theta_p(t)\in\mathbb{R}^{n_p\times n_a}$ are unknown real matrices representing system uncertainties.

To the end of this section, we introduce a definition and some lemmas as the preliminary results for this work. These lemmas are used in the proof of the main results.

Definition 1. (Cao and Lam (2000)) The UDSS is said to be stochastically stable if, when $v(t)\equiv 0$, for all finite $\phi(t)\in\mathbb{R}^n$ defined on $[-h,\ 0]$, there exists an $\tilde{M}>0$ satisfying

$$\lim_{t_f\to\infty}\mathbb{E}\left[\int_0^{t_f} x^T(t,\phi)x(t,\phi)dt\middle|\phi\right]\leq x^T(0)\tilde{M}x(0). \quad (3)$$

Lemma 1. (Mukaidani et al. (2018c)) Let $A\in\mathbb{R}^{n\times n}$, $D\in\mathbb{R}^{n\times p}$, $E\in\mathbb{R}^{q\times n}$, and $\Theta\in\mathbb{R}^{p\times q}$ satisfying $\Theta^T(t)\Theta(t)\leq I_q$ be given matrices. Then, for any matrix $P=P^T>0$, there exist positive scalars $\varepsilon>0$ and $\lambda>0$ such that

$$(A+D\Theta E)^T P(A+D\Theta E)$$
$$\leq A^T PA + \varepsilon^{-1}A^T PDD^T PA + (\varepsilon+\lambda)E^T E, \quad (4a)$$
$$D^T PD\leq\lambda I_p. \quad (4b)$$

The following lemmas have been proved in Mukaidani et al. (2019) as a special case of the Markov jump stochastic system with a single mode.

Lemma 2. (Mukaidani et al. (2019)) Let γ denote the required disturbance attenuation level. Consider a set of symmetric matrices $W\geq 0$ and $U>0$, and positive scalars μ, ε and λ, such that the following LMIs holds for every $i\in\mathscr{D}$:

$$\Lambda(W,\mu,\varepsilon,\lambda) < 0, \quad (5a)$$
$$D_p^T WD_p\leq\lambda I_{n_a}, \quad (5b)$$

where $\Lambda(W,\mu,\varepsilon,\lambda) := \begin{bmatrix}\Phi^{11} & WA_h\\ A_h^T W & -U\end{bmatrix}$, Φ^{11}
$:= WA + A^T W + \mu^{-1}WDD^T W + \mu E_a^T E_a + H^T H + U + A_p^T WA_p$
$+\varepsilon^{-1}A_p^T WD_p D_p^T WA_p + (\varepsilon+\lambda)E_{pa}^T E_{pa} + \gamma^{-2}WB_v B_v^T W$.
Then, we have the following results:

i) The UDSS in (1) is stochastically stable when $v(t)\equiv 0$;
ii) The following inequality holds:

$$\|z\|_2^2 < \gamma^2\|v\|_2^2 + \mathscr{E}(W,U), \quad (6)$$

where

$$\|z\|_2^2 := \mathbb{E}\left[\int_0^\infty\|z(t)\|^2 dt\right],\ \|v\|_2^2 := \mathbb{E}\left[\int_0^\infty\|v(t)\|^2 dt\right],$$

$$\mathscr{E}(W,U) := x^T(0)Wx(0) + \int_{-h}^0\phi^T(s)U\phi(s)ds;$$

iii) The worst-case disturbance is given by

$$v^*(t) = F_\gamma^* x(t) = \gamma^{-2}B_v^T Wx(t). \quad (7)$$

Lemma 3. (Mukaidani et al. (2019)) Define the corresponding cost function for UDSS in (1) with $v(t)\equiv 0$ as follows:

$$\tilde{J} := \mathbb{E}\left[\int_0^\infty x^T(t,\phi)Qx(t,\phi)dt\middle|\phi\right], \quad (8)$$

where $Q=Q^T>0$. Consider a set of symmetric matrices $P\geq 0$ and $V>0$, and positive scalars ν, ε and κ such that the following LMIs holds:

$$\Gamma(P,V,\nu,\varepsilon,\kappa) < 0, \quad (9a)$$
$$D_p^T PD_p\leq\kappa I_{n_a}, \quad (9b)$$

where $\Gamma(P,V,\nu,\varepsilon,\kappa) := \begin{bmatrix}\Psi^{11} & PA_h\\ A_h^T P & -V\end{bmatrix}$, $\Psi^{11} := PA + A^T P$
$+\nu^{-1}PDD^T P + \nu E_a^T E_a + Q + V + A_p^T PA_p + \varepsilon^{-1}A_p^T PD_p D_p^T PA_p$
$+(\varepsilon+\kappa)E_{pa}^T E_{pa}$.
Then, we have the following inequality

$$\tilde{J} < x^T(0)Px(0) + \int_{-h}^0\phi^T(s)V\phi(s)ds. \quad (10)$$

In the subsequent sections, we discuss the main results of this study.

3. PROBLEM FORMULATION

Consider the following UDSS:

$$dx(t) = \Bigg(A(t)x(t) + A_h x(t-h) + B_0 u_0(t) + \sum_{k=1}^{N} B_k u_k(t) + B_v v(t)\Bigg)dt + A_p(t)x(t)dw(t), \tag{11a}$$

$$x(t) = \phi(t),\ t \in [-h,\ 0], \tag{11b}$$

$$y_k(t) = C_k x(t), \tag{11c}$$

$$z(t) = \big[[Hx(t)]^T\ [G_0u_0(t)]^T\ [G_1u_1(t)]^T \cdots [G_Nu_N(t)]^T\big]^T, \tag{11d}$$

where $y_k(t) \in \mathbb{R}^{n_y}$ denotes the measurement output, and $u_k(t) \in \mathbb{R}^{m_k}$, $k = 0,1,...,N$, the k-th control input. It is assumed that $u_0(t)$ is controlled by the leader and $u_k(t)$ is controlled by follower k, $k = 1,...,N$. Without loss of generality, it is assumed that $G_k^T G_k = I_{m_k}$. Furthermore, in order to eliminate the dependence of the cost performance on $x(0)$, it is assumed that $\mathbb{E}[x(0)] = 0$, $\mathbb{E}[x(0)x^T(0)] = M_0 \geq 0$.

The robust SOF Stackelberg game with one leader and N followers is formulated as follows.

Problem formulation : (i) *For a given $\gamma > 0$, find a robust SOF Stackelberg strategy set $(u_0^*, u_1^*, ..., u_N^*)$ and a worst case disturbance $v(t) = v^*(t)$,*

$$u_k(t) = u_k^*(t) = F_k^* y_k(t) = F_k^* C_k x(t),\ k = 0,1,...,N, \tag{12a}$$

$$v(t) = v^*(t) = F_\gamma^* x(t) \tag{12b}$$

such that $u_k(t) = u_k^*(t)$, $k = 0,1,...,N$, make UDSS (11) *stochastically stable when $v(t) = 0$ and the following inequality holds:*

$$\|z\|_2^2 < \gamma^2 \|v\|_2^2 + \mathscr{G}(\tilde{W}, \tilde{U}), \tag{13}$$

where $\mathscr{G}(\tilde{W}, \tilde{U}) := x^T(0)\tilde{W}x(0) + \int_{-h}^{0} \phi^T(s)\tilde{U}\phi(s)ds$.

(ii) *When $v(t) = v^*(t) = F_\gamma^* x(t)$ is applied, let us consider the weighted cost function of the followers, given below:*

$$J_\rho(u_0, u_1(u_0), ..., u_N(u_0)) := \sum_{k=1}^{N} \rho_k \tilde{J}_k(u_1, ..., u_N, v^*, i)$$
$$\sum_{k=1}^{N} \rho_k = 1,\ 0 < \rho_k < 1,\ k = 1,...,N. \tag{14}$$

For a leader's fixed strategy $u_0 = u_0(t)$, a follower's strategy set $(\bar{u}_1^0(u_0), ..., \bar{u}_N^0(u_0))$ minimizes $J_\rho(u_0, u_1^0(u_0), ..., u_N^0(u_0))$. That is,

$$J_\rho(u_0, \bar{u}_1^0(u_0), ..., \bar{u}_N^0(u_0)) = \min_{u_1(u_0),...,u_N(u_0)} J_\rho(u_0, u_1(u_0), ..., u_N(u_0)), \tag{15}$$

where for $Q_k = Q_k^T > 0$ and $R_k = R_k^T > 0$,

$$\tilde{J}_k(u_1, ..., u_N, v^*, i) = \sup_{\Theta, \Theta_p} J_k(u_1, ..., u_N, v^*, i), \tag{16a}$$

$$J_k(u_1, ..., u_N, v^*, i) = \mathbb{E}\Bigg[\int_0^\infty \Big(x^T(t,\phi)Q_k x(t,\phi) + C_k^T F_k^T(F_0)R_k F_k(F_0)C_k\Big)dt\,\Big|\,\phi\Bigg]. \tag{16b}$$

(iii) *For any mapping $\mathscr{T}_k$ such that $u_k^0 = \mathscr{T}_k u_0 = u_k(u_0) \in \mathbb{R}^{m_k}$, $k = 1,...,N$, the following inequality holds:*

$$J_0(\bar{u}_0, \bar{u}_1, ..., \bar{u}_N, v^*) \leq J_0(u_0, \bar{u}_1^0(u_0), ..., \bar{u}_N^0(u_0), v^*),\ \forall u_0, \tag{17}$$

where

$$\bar{u}_k = \bar{u}_k^0(\bar{u}_0),\ k = 1,...,N. \tag{18}$$

In the following subsections, the existence conditions of the robust SOF Stackelberg strategy set are established based on a constraint optimization problem.

3.1 H_∞ CONSTRAINT

First, the disturbance attenuation condition is investigated. Consider the closed-loop UDSS and the cost functions. For arbitrary $u_k(t) = F_k C_k x(t)$, $k = 0,1,...,N$, the closed-loop UDSS is established as

$$dx(t) = \Big[\big(\bar{A} + D\Theta(t)E_a\big)x(t) + A_h x(t-h) + B_v v(t)\Big]dt + A_p(t)x(t)dw(t), \tag{19a}$$

$$z(t) = \tilde{H}x(t), \tag{19b}$$

where $\bar{A} := A + \sum_{k=0}^{N} B_k F_k C_k$,
$\tilde{H} := \big[H^T\ [G_0F_0C_0]^T\ [G_1F_1C_1]^T \cdots [G_NF_NC_N]^T\big]^T$.
Thus, using Lemma 2, we have the following LMIs:

$$\tilde{\Lambda}(\tilde{W}, \tilde{\mu}, \tilde{\varepsilon}, \tilde{\lambda}) < 0, \tag{20a}$$

$$D_p^T \tilde{W} D_p \leq \tilde{\lambda} I_{n_a}, \tag{20b}$$

where $\tilde{\Lambda}(\tilde{W}, \tilde{\mu}, \tilde{\varepsilon}, \tilde{\lambda}) := \begin{bmatrix} \tilde{\Phi}^{11} & \tilde{W}A_h \\ A_h^T \tilde{W} & -\tilde{U} \end{bmatrix}$, $\tilde{\Phi}^{11} := \tilde{W}\bar{A} + \bar{A}^T\tilde{W} + \tilde{\mu}^{-1}\tilde{W}DD^T\tilde{W} + \tilde{\mu}E_a^TE_a + \tilde{H}^T\tilde{H} + \tilde{U} + A_p^T\tilde{W}A_p + \tilde{\varepsilon}^{-1}A_p^T\tilde{W}D_pD_p^T\tilde{W}A_p + (\tilde{\varepsilon} + \tilde{\lambda})E_{pa}^TE_{pa} + \gamma^{-2}\tilde{W}B_vB_v^T\tilde{W}$.
Then, the following worst-case disturbance can be obtained.

$$v^*(t) = F_\gamma^* x(t) = \gamma^{-2}B_v^T\tilde{W}x(t). \tag{21}$$

3.2 STACKELBERG GAME

Second, the Stackelberg game for the UDSS is considered. Let us consider the following UDSS and the cost functions of the followers

$$u_k(t) = F_k(F_0)y_k(t) = F_k(F_0)C_kx(t),\ k = 1,...,N \tag{22}$$

with the fixed leader's strategy $u_0(t) = F_0y_0(t) = F_0C_0x(t)$ and worst case disturbance $v(t) = v^*(t)$:

$$dx(t) = \Bigg(\Big[A_\gamma + D\Theta(t)E_a + B_0F_0C_0 + \sum_{k=1}^{N} B_kF_k(F_0)C_k\Big]x(t) + A_hx(t-h)\Bigg)dt + A_p(t)x(t)dw(t), \tag{23a}$$

$$J_k(u_1(u_0), ..., u_N(u_0), v^*, i) = \mathbb{E}\Bigg[\int_0^\infty x^T(t,\phi)\Big(Q_k + C_k^TF_k^T(F_0)R_kF_k(F_0)C_k\Big)x(t,\phi)dt\,\Big|\,\phi\Bigg], \tag{23b}$$

where $A_\gamma := A + B_vF_\gamma^*$.

In this case, for arbitrary follower's strategy set $u_k(t) = F_k(F_0)C_kx(t)$, $k = 1,...,N$, with the fixed gain F_0, the following LMIs can be obtained by applying Lemma 3:

$$\Gamma_\rho(P_\rho, V_\rho, \nu_\rho, \varepsilon_\rho, \kappa_\rho) < 0, \quad (24a)$$

$$D_p^T P_\rho D_p \le \kappa_\rho I_{n_a}, \quad (24b)$$

where $\Gamma_\rho(P_\rho, V_\rho, \nu_\rho, \varepsilon_\rho, \kappa_\rho) := \begin{bmatrix} \tilde{\Psi}^{11} & P_\rho A_h \\ A_h^T P_\rho & -V_\rho \end{bmatrix}$,
$\tilde{\Psi}^{11} := P_\rho \tilde{A}_\gamma + \tilde{A}_\gamma^T P_\rho + \nu_\rho^{-1} P_\rho DD^T P_\rho + \nu_\rho E_a^T E_a + Q_\rho$ $+ \sum_{k=1}^N \rho_k C_k^T F_k^T(F_0) R_k F_k(F_0) C_k + V_\rho + A_p^T P_\rho A_p$ $+\varepsilon_\rho^{-1} A_p^T P_\rho D_p D_p^T P_\rho A_p + (\varepsilon_\rho + \kappa_\rho) E_{pa}^T E_{pa}$, $Q_\rho := \sum_{k=1}^N \rho_k Q_k$, $\tilde{A}_\gamma := A + B_0F_0C_0 + \sum_{k=1}^N B_k F_k(F_0) C_k + B_v F_\gamma^*$.
Consequently, the following optimization problem related to Pareto suboptimal strategy for the UDSS can be defined:

$$\begin{aligned} &\min_{u_1(u_0),...,u_N(u_0)} J_\rho(u_0, u_1(u_0), ..., u_N(u_0)) \\ &\quad = \min_{\Sigma_\rho} \mathbf{Tr}\left[M_0 P_\rho + LL^T V_\rho\right], \quad (25) \\ &\text{s.t.} \Sigma_\rho := (P_\rho, F_1, ..., F_N, V_\rho, \nu_\rho, \varepsilon_\rho, \kappa_\rho) \text{ satisfies (24)} \end{aligned}$$

where $u_0 = u_0(t)$ is the fixed leader's strategy and $LL^T := \int_{-h}^0 \phi(s)\phi^T(s)ds$.

In order to calculate $(F_1(F_0), ..., F_N(F_0))$ in (22), the Karush-Kuhn-Tucker (KKT) conditions are derived. Define the following Lagrangian:

$$\mathscr{L}_\rho = \mathbf{Tr}\left[M_0 P_\rho + LL^T V_\rho\right] + \mathbf{Tr}\left[S_\rho \Delta_\rho\right], \quad (26)$$

where S_ρ is the symmetric matrix of the Lagrange multiplier. Furthermore, we have

$$\begin{aligned} \Delta_\rho &:= \Delta_\rho(P_\rho, V_\rho, F_\gamma, F_0, F_1, ..., F_N, \nu_\rho, \varepsilon_\rho, \kappa_\rho) \\ &= \tilde{\Psi}^{11} + P_\rho A_h V_\rho^{-1} A_h^T P_\rho. \quad (27) \end{aligned}$$

In this case, we have the following stochastic Lyapunov type matrix equations (SLMEs):

$$\frac{\partial \mathscr{L}_\rho}{\partial P_\rho} = \Delta_\rho^0 = \Delta_\rho^0(S_\rho, P_\rho, V_\rho, F_\gamma, F_0, F_1, ..., F_N, \nu_\rho, \varepsilon_\rho) = 0, \quad (28a)$$

$$\frac{\partial \mathscr{L}_\rho}{\partial S_\rho} = \Delta_\rho = 0, \quad (28b)$$

$$\frac{1}{2} \cdot \frac{\partial \mathscr{L}_\rho}{\partial F_k(F_0)} = \Delta_\rho^k = \Delta_\rho^k(F_k, S_\rho, P_\rho, \nu_\rho, \varepsilon_\rho, \kappa_\rho) = 0, \quad (28c)$$

where $k = 1, ..., N$, and

$$\begin{aligned} \Delta_\rho^0 &= M_0 + S_\rho \tilde{A}_\gamma^T + \tilde{A}_\gamma S_\rho + \nu_\rho^{-1}\left[S_\rho P_\rho DD^T + DD^T P_\rho S_\rho\right] \\ &\quad + A_p S_\rho A_p^T + \varepsilon_\rho^{-1}[A_p S_\rho A_p^T P_\rho D_p D_p^T + D_p D_p^T P_\rho A_p S_\rho A_p^T] \\ &\quad + S_\rho P_\rho A_h V_\rho^{-1} A_h^T + A_h V_\rho^{-1} A_h^T P_\rho S_\rho \\ \Delta_\rho^k &= \rho_k R_k F_k(F_0) C_k S_\rho C_k^T + B_k^T P_\rho S_\rho C_k^T, \ k = 1, \dots, N. \end{aligned}$$

It should be noted that the derivative with respect to V_ρ, ν_ρ, ε_ρ and κ_ρ is not needed because this optimization part can be performed by means of the LMI instead of the KKT condition.

From $\Delta_\rho^0 = 0$, we have $S_\rho > 0$. Therefore, from $\Delta_\rho^k = 0$, if $C_k S_\rho C_k^T$ is nonsingular, each follower has the following strategy:

$$\begin{aligned} u_k(t) &= F_k^*(F_0) y_k(t) = F_k^*(F_0) C_k x(t) = F_k^* C_k x(t) \\ &= -[\rho_k R_k]^{-1} B_k^T P_\rho S_\rho C_k^T [C_k S_\rho C_k^T]^{-1} y_k(t). \quad (29) \end{aligned}$$

Second, the leader's strategy is established. The cost, J_0, can be obtained by

$$J_0(u_0, \vec{u}_1^0(u_0), ..., \vec{u}_N^0(u_0), v^*) = \mathbf{Tr}\left[M_0 P_0 + LL^T V_0\right], \quad (30)$$

where P_0 is the solution of the following LMIs

$$\hat{\Gamma}_0(P_0, V_0, \nu_0, \varepsilon_0, \kappa_0) < 0, \quad (31a)$$

$$D_p^T P_0 D_p \le \kappa_0 I_{n_a}, \quad (31b)$$

where $\hat{\Gamma}_0(P_0, V_0, \nu_0, \varepsilon_0, \kappa_0) := \begin{bmatrix} \hat{\Psi}^{11} & P_0 A_h \\ A_h^T P_0 & -V_0 \end{bmatrix}$,
$\hat{\Psi}^{11} := P_0 \hat{A}_\gamma + \hat{A}_\gamma^T P_0 + \nu_0^{-1} P_0 DD^T P_0 + \nu_0 E_a^T E_a$ $+ Q_0 + C_0^T F_0^T R_0 F_0 C_0 + V_0 + A_p^T P_0 A_p + \varepsilon_0^{-1} A_p^T P_0 D_p D_p^T P_0 A_p$ $+ (\varepsilon_0 + \kappa_0) E_{pa}^T E_{pa}$, $\hat{A}_\gamma := A + B_0 F_0 C_0 + \sum_{k=1}^N B_k F_k^* C_k + B_v F_\gamma^*$.
Hence, the following optimization problem related to the leader's strategy can be defined:

$$\min_{u_0} J_0(u_0, \vec{u}_1^0(u_0), ..., \vec{u}_N^0(u_0)) = \min_{\Sigma_0} \mathbf{Tr}\left[M_0 P_0 + LL^T V_\rho\right], \quad (32)$$

s.t. $\Sigma_0 := (P_0, F_0, V_0, \nu_0, \varepsilon_0, \kappa_0)$ satisfies (31).

In order to solve the preceding optimization problem, let us consider the following Lagrangian:

$$\begin{aligned} \mathscr{L}_0 &= \mathbf{Tr}\left[M_0 P_0 + LL^T V_0\right] + \mathbf{Tr}\left[S_0 \Delta_0 + T_0 \Delta_\rho\right] \\ &\quad + \mathbf{Tr}\left[Z_0 \Delta_\rho^0 + \sum_{k=1}^N Z_k \Delta_\rho^k\right], \quad (33) \end{aligned}$$

where $\Delta_0 := \Delta_0(P_0, V_0, \nu_0, \varepsilon_0, \kappa_0) = \hat{\Psi}^{11} + P_0 A_h V_0^{-1} A_h^T P_0$.
As a necessary condition, the following equations can be derived by using the KKT condition:

$$\frac{\partial \mathscr{L}_0}{\partial P_0} = \Delta_0^1 = \Delta_0^1(S_0, P_0, V_0, F_\gamma, F_0, F_1, ..., F_N, \nu_0, \varepsilon_0) = 0, \quad (34a)$$

$$\begin{aligned} \frac{\partial \mathscr{L}_0}{\partial P_\rho} &= \Delta_0^2 = \Delta_0^2(T_0, P_\rho, V_\rho, S_\rho, Z_0, Z_1, ..., Z_N, F_\gamma, F_0, \\ &\qquad F_1, ..., F_N, \nu_\rho, \varepsilon_\rho) = 0, \quad (34b) \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathscr{L}_0}{\partial S_\rho} &= \Delta_0^3 = \Delta_0^3(Z_0, P_\rho, V_\rho, Z_1, ..., Z_N, F_\gamma, F_0, \\ &\qquad F_1, ..., F_N, \nu_\rho, \varepsilon_\rho) = 0, \quad (34c) \end{aligned}$$

$$\frac{1}{2} \cdot \frac{\partial \mathscr{L}_0}{\partial F_0} = \Delta_0^4 = \Delta_0^4(F_0, P_0, S_0, P_\rho, S_\rho, T_0, Z_0,) = 0, \quad (34d)$$

$$\frac{1}{2} \cdot \frac{\partial \mathscr{L}_0}{\partial F_k} = \Delta_0^k = \Delta_0^k(Z_k, P_0, S_0, P_\rho, S_\rho, T_0, Z_0, F_k) =, 0 \quad (34e)$$

where $k = 1, \dots, N$,
$\frac{\partial \mathscr{L}_0}{\partial S_0} = \Delta_0 = 0$, $\frac{\partial \mathscr{L}_0}{\partial T_0} = \Delta_\rho = 0$, $\frac{\partial \mathscr{L}_0}{\partial Z_0} = \Delta_\rho^0 = 0$, $\frac{\partial \mathscr{L}_0}{\partial Z_k} = \Delta_\rho^k = 0$,
$\Delta_0^1 = M_0 + S_0 \hat{A}_\gamma^T + \hat{A}_\gamma S_0 + \nu_0^{-1}\left[S_0 P_0 DD^T + DD^T P_0 S_0\right]$ $+ A_p S_0 A_p^T + \varepsilon_0^{-1}[A_p S_0 A_p^T P_0 D_p D_p^T + D_p D_p^T P_0 A_p S_0 A_p^T]$ $+ S_0 P_0 A_h V_0^{-1} A_h^T + A_h V_0^{-1} A_h^T P_0 S_0$, $\Delta_0^2 = T_0 \hat{A}_\gamma^T + \hat{A}_\gamma T_0$ $+ \nu_\rho^{-1}\left[T_0 P_\rho DD^T + DD^T P_\rho T_0\right] + A_p T_0 A_p^T$ $+ \varepsilon_\rho^{-1}[A_p T_0 A_p^T P_\rho D_p D_p^T + D_p D_p^T P_\rho A_p T_0 A_p^T] + T_0 P_\rho A_h V_\rho^{-1} A_h^T$ $+ A_h V_\rho^{-1} A_h^T P_\rho T_0 + \nu_\rho^{-1}\left[S_\rho Z_0 DD^T + DD^T Z_0 S_\rho\right]$ $+ \varepsilon_\rho^{-1}[A_p S_\rho A_p^T Z_0 D_p D_p^T + D_p D_p^T Z_0 A_p S_\rho A_p^T] + S_\rho Z_0 A_h V_\rho^{-1} A_h^T$ $+ A_h V_\rho^{-1} A_h^T Z_0 S_\rho + \frac{1}{2} \sum_{k=1}^N \left[B_k Z_k^T C_k S_\rho + S_\rho C_k^T Z_k B_k^T\right]$, $\Delta_0^3 = Z_0 \hat{A}_\gamma$

$+\hat{A}_\gamma^T Z_0 + \nu_\rho^{-1}\left[Z_0 DD^T P_\rho + P_\rho DD^T Z_0\right] + A_p^T Z_0 A_p$
$+\varepsilon_\rho^{-1}\left[A_p^T Z_0 D_p D_p^T P_\rho A_p + A_p^T P_\rho D_p D_p^T Z_0 A_p\right] + Z_0 A_h V_\rho^{-1} A_h^T P_\rho$
$+P_\rho A_h V_\rho^{-1} A_h^T Z_0 + \frac{1}{2}\sum_{k=1}^N \left[(\rho_k C_k^T F_k^T R_k + P_\rho B_k) Z_k^T C_k\right.$
$\left.+C_k^T Z_k(\rho_k C_k^T F_k^T R_k + P_\rho B_k)^T\right]$, $\Delta_0^4 = R_0 F_0 C_0 S_0 C_0^T + B_0^T(P_0 S_0,$
$+P_\rho T_0 + Z_0 S_\rho) C_0^T$, $\Delta_0^k = B_k^T(P_0 S_0 + P_\rho T_0 + Z_0 S_\rho) C_k^T$
$+\rho_k R_k\left(F_k C_k T_0 + \frac{1}{2} Z_k^T C_k S_\rho\right) C_k^T$.

From (34d), if $C_0 S_0 C_0^T$ is nonsingular, the gain of the leader's strategy F_0 can be computed as follows, because the SLME (34a) has the unique solution $S_0 > 0$:

$$F_0 = -R_0^{-1} B_0^T [P_0 S_0 + P_\rho T_0 + Z_0 S_\rho] C_0^T [C_0 S_0 C_0^T]^{-1}. \qquad (35)$$

Summarizing what we have discussed so far, we are now in a position to state the main results of this work.

Theorem 1. Consider the UDSS (11) controlled by one leader and N followers with $u_i(t) = F_k x(t)$, $k = 0, 1, \ldots, N$, and the deterministic disturbance $v(t) = F_\gamma x(t)$. Suppose that,
(i) for a given attenuation performance level, $\gamma > 0$, there exists a feasible solution set $\tilde{W}$ such that LMIs (20) is satisfied;
(ii) there exist feasible solution sets such that two optimization problems (25) and (32) with constraints LMIs (24) and (31) are solved, respectively;
(iii) there exist solution sets to SLMEs (28a), (34a) and (34b).
Then, (29) and (35) constitute the robust SOF Stackelberg strategy set which satisfies conditions (13)–(17).

4. HEURISTIC ALGORITHM

In order to compute the robust SOF Stackelberg strategy set, the optimization problems (25) and (32), SLME (28a), (34a) and (34b) should be solved. However, it is difficult to obtain the solution set. Hence, the following heuristic algorithm based on the KM iterations (Yao et al. (2009)) is proposed.

Step 1. *Set the initial values: choose $F_k^{(0)}$, $k = 0, 1, \ldots, N$, and $F_\gamma^{(0)}$, such that closed-loop* UDSS (11a) *is stochastically stable; choose appropriate $Z_k^{(0)}$, $k, 1, \ldots, N$ and compute $F_\gamma^{(0)} = \gamma^{-2} B_v^T \tilde{W}^{(0)}$ with $\tilde{W}^{(0)} = I_n$;*

Step 2-1. *Solve the following optimization problem for $P_\rho^{(n+1)}$ and $V_\rho^{(n+1)}$ for variable Ξ_ρ:*

$$\min_{\Xi_\rho} \mathbf{Tr}\left[M_0 P_\rho^{(n+1)} + LL^T V_\rho^{(n+1)}\right], \qquad (36a)$$

$$\Xi_\rho := (P_\rho^{(n+1)}, V_\rho^{(n+1)}, \nu_\rho^{(n+1)}, \varepsilon_\rho^{(n+1)}, \kappa_\rho^{(n+1)}),$$

s.t. Ξ_ρ satisfies (36b) and (36c)

$$\Gamma_\rho(P_\rho, V_\rho, \nu_\rho, \varepsilon_\rho, \kappa_\rho) := \begin{bmatrix} \Psi^{11} & P_\rho A_h & P_\rho D & A_p^T P_\rho D_p \\ A_h^T P_\rho & -V_\rho & 0 & 0 \\ D^T P_\rho & 0 & -\nu_\rho I_{n_p} & 0 \\ D_p^T P_\rho A_p & 0 & 0 & -\varepsilon_\rho I_{n_p} \end{bmatrix} < 0, \qquad (36b)$$

$$D_p^T P_\rho D_p \leq \kappa_\rho I_{n_a}, \qquad (36c)$$

where $\Psi^{11} := P_\rho \hat{A}_\gamma^{(n)} + \hat{A}_\gamma^{(n)T} P_\rho + \nu_\rho E_a^T E_a + Q_\rho$
$+\sum_{k=0}^N \rho_k C_k^T F_k^{(n)T} R_k F_k^{(n)} C_k + V_\rho + A_p^T P_\rho A_p + (\varepsilon_\rho + \kappa_\rho) E_{pa}^T E_{pa}$,
$\hat{A}_\gamma^{(n)} := A + \sum_{k=0}^N B_k F_k^{(n)} C_k + B_v F_\gamma^{(n)}$;

Step 2-2. *Solve the following* SLME *for $S_\rho^{(n+1)}$:*

$$\Delta_\rho^0(S_\rho^{(n+1)}, P_\rho^{(n+1)}, V_\rho^{(n+1)}, F_\gamma^{(n)}, F_0^{(n)}, F_1^{(n)}, \ldots, F_N^{(n)}, \nu_\rho^{(n+1)}, \varepsilon_\rho^{(n+1)}) = 0; \qquad (37)$$

Step 2-3. *Compute $F_k^{(n+1)}$, $k = 1, \ldots, N$:*

$$F_k^{(n+1)} = -[\rho_k R_k]^{-1} B_k^T P_\rho^{(n+1)} S_\rho^{(n+1)} C_k^T [C_k S_\rho^{(n+1)} C_k^T]^{-1}; \qquad (38)$$

Step 3-1. *Solve the following optimization problem for $P_0^{(n+1)}$ and $V_0^{(n+1)}$ for variable Ξ_0:*

$$\min_{\Xi_0} \mathbf{Tr}\left[M_0 P_0^{(n+1)} + LL^T V_0^{(n+1)}\right], \qquad (39a)$$

$$\Xi_0 := (P_0^{(n+1)}, V_0^{(n+1)}, \nu_0^{(n+1)}, \varepsilon_0^{(n+1)}, \kappa_0^{(n+1)}),$$

s.t. Ξ_0 satisfies (39b) and (39c)

$$\Gamma_0(P_0, V_0, \nu_0, \varepsilon_0, \kappa_0) := \begin{bmatrix} \Psi^{11} & P_0 A_h & P_0 D & A_p^T P_0 D_p \\ A_h^T P_0 & -V_0 & 0 & 0 \\ D^T P_0 & 0 & -\nu_0 I_{n_p} & 0 \\ D_p^T P_0 A_p & 0 & 0 & -\varepsilon_0 I_{n_p} \end{bmatrix} < 0, \qquad (39b)$$

$$D_p^T P_0 D_p \leq \kappa_0 I_{n_a}, \qquad (39c)$$

where $\Psi^{11} := P_0 \hat{A}_\gamma^{(n)} + \hat{A}_\gamma^{(n)T} P_0 + \nu_0 E_a^T E_a + Q_0 + C_0^T F_0^T R_0 F_0 C_0$
$+V_0 + A_p^T P_0 A_p + (\varepsilon_0 + \kappa_0) E_{pa}^T E_{pa}$;

Step 3-2. *Solve the following* SLME *for $S_0^{(n+1)}$:*

$$\Delta_0^1(S_0^{(n+1)}, P_0^{(n+1)}, V_0^{(n+1)}, F_\gamma^{(n)}, F_0^{(n)}, F_1^{(n+1)}, \ldots, F_N^{(n+1)}, \nu_0^{(n+1)}, \varepsilon_0^{(n+1)}) = 0; \qquad (40)$$

Step 3-3. *Solve the following* SLME *for $Z_0^{(n+1)}$:*

$$\Delta_0^3(Z_0^{(n+1)}, P_\rho^{(n+1)}, V_\rho^{(n+1)}, Z_1^{(n)}, \ldots, Z_N^{(n)}, F_\gamma^{(n)}, F_0^{(n)}, F_1^{(n+1)}, \ldots, F_N^{(n+1)}, \nu_\rho^{(n+1)}, \varepsilon_\rho^{(n+1)}) = 0; \qquad (41)$$

Step 3-4. *Solve the following* SLME *for $T_0^{(n+1)}$:*

$$\Delta_0^2(T_0^{(n+1)}, P_\rho^{(n+1)}, V_\rho^{(n+1)}, S_\rho^{(n+1)}, Z_0^{(n+1)}, Z_1^{(n)}, \ldots, Z_N^{(n)}, F_\gamma^{(n)}, F_0^{(n)}, F_1^{(n+1)}, \ldots, F_N^{(n+1)}, \nu_\rho^{(n+1)}, \varepsilon_\rho^{(n+1)}) = 0; \qquad (42)$$

Step 3-5. *Compute $Z_k^{(n+1)}$, $k = 1, \ldots, N$:*

$$Z_k^{(n+1)} = -2[\rho_k R_k]^{-1}\left[B_k^T\left(P_0^{(n+1)} S_0^{(n+1)} + P_\rho^{(n+1)} T_0^{(n+1)} + Z_0^{(n+1)} S_\rho^{(n+1)}\right) + \rho_k R_k F_k^{(n+1)} C_k T_0^{(n+1)}\right] \times C_k^T [C_k S_\rho^{(n+1)} C_k^T]^{-1}; \qquad (43)$$

Step 3-6. *Compute $F_0^{(n+1)}$:*

$$F_0^{(n+1)} = -R_0^{-1} B_0^T\left[P_0^{(n+1)} S_0^{(n+1)} + P_\rho^{(n+1)} T_0^{(n+1)} + Z_0^{(n+1)} S_\rho^{(n+1)}\right] C_0^T [C_0 S_0^{(n+1)} C_0^T]^{-1}; \qquad (44)$$

Step 4. *Solve the following optimization problem for $\tilde{W}^{(n+1)}$ for the variables Σ_2:*

$$\min_{\Sigma_2} \mathbf{Tr}[\tilde{W}^{(n+1)} + LL^T \tilde{U}^{(n+1)}], \tag{45a}$$

$$\Sigma_2 := (\tilde{W}^{(n+1)}, \tilde{U}^{(n+1)}, \tilde{\mu}^{(n+1)}, \tilde{\varepsilon}^{(n+1)}, \tilde{\lambda}^{(n+1)}),$$

s.t. Σ_2 satisfies (45b) and (45c)

$$\hat{\Lambda}(\tilde{W}, \tilde{\mu}, \tilde{\varepsilon}, \tilde{\lambda}) := \begin{bmatrix} \Phi^{11} & \tilde{W}A_h & \tilde{W}D & A_p^T\tilde{W}D_p & \tilde{W}B_v \\ A_h^T\tilde{W} & -\tilde{U} & 0 & 0 & 0 \\ D^T\tilde{W} & 0 & -\tilde{\mu}I_{n_p} & 0 & 0 \\ D_p^T\tilde{W}A_p & 0 & 0 & -\tilde{\varepsilon}I_{n_p} & 0 \\ B_v^T\tilde{W} & 0 & 0 & 0 & -\gamma^2 I_{m_v} \end{bmatrix} < 0, \tag{45b}$$

$$D_p^T\tilde{W}D_p \le \tilde{\lambda}I_{n_a}, \tag{45c}$$

where $\Phi^{11} := \tilde{W}\bar{A}^{(n)} + \bar{A}^{(n)T}\tilde{W} + \tilde{\mu}E_a^TE_a + H^TH + \tilde{F}^T\tilde{F} + \tilde{U} + A_p^T\tilde{W}A_p + (\tilde{\varepsilon} + \tilde{\lambda})E_{pa}^TE_{pa}$, $\bar{A}^{(n)} := A + \sum_{k=0}^{N} B_kF_k^{(n+1)}$;

Step 5. *Set* $Z^{(n+1)} \leftarrow \theta^{(n)}Z^{(n+1)} + (1-\theta^{(n)})Z^{(n)}$, *where* $Z^{(n)} := \begin{bmatrix} P_\rho^{(n)} & S_\rho^{(n)} & P_0^{(n)} & S_0^{(n)} & T_0^{(n)} & Z_0^{(n)} & Z_1^{(n)T} & Z_2^{(n)T} & \tilde{W}^{(n)} \end{bmatrix}$. *Furthermore,* $\theta^{(n)} \in (0,1]$ *is chosen at each iteration to ensure that* $\mathscr{J}^{(n)} > \mathscr{J}^{(n+1)}$ *with* $\mathscr{J}^{(n)} = \mathbf{Tr}[P_\rho^{(n)} + V_\rho^{(n)} + S_\rho^{(n)} + P_0^{(n)} + V_0^{(n)} + S_0^{(n)} + \tilde{W}^{(n)} + \tilde{U}^{(n)}]$;

Step 6. *If the iterative algorithm consisting of Steps 2 to 5 converges, we have obtained the iterative solutions; otherwise, if the number of iterations reaches a preset threshold, declare that there is no strategy set. Stop.*

Finally, the robust convergence property can be stated.

Theorem 2. The algorithm achieves the convergence if there exists $\theta^{(n)} \in (0,1]$ such that for all $n \in \mathbb{N}$, $\mathscr{J}^{(n)} > \mathscr{J}^{(n+1)}$.

5. NUMERICAL EXAMPLE

Consider the UDSS in (11) with one leader and two followers modified from Wu and Grigoriadis (2001) and the following coefficient matrices:

$$N = 2,\ A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix},\ A_h = 0.1A,\ A_p = 0.2A,$$

$$B_0 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},\ B_1 = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix},\ B_2 = \begin{bmatrix} -0.2 \\ 0 \end{bmatrix},\ B_v = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix},$$

$$H = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ D = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix},\ D_p = 0.1D,\ C_0 = [\,1\ \ 0\,],$$

$$C_1 = C_2 = [\,0\ \ 1\,],\ E_a = [\,0\ \ 1\,],\ E_{pa} = 0.1E_a,$$

$$Q_0 = 1.5I_2,\ Q_1 = 1.2I_2,\ Q_2 = 2I_2,\ R_0 = 0.8,\ R_1 = 0.6,$$

$$R_2 = 0.4,\ \rho_1 = \rho_2 = 0.5,\ L := [\,-0.5\ \ 1\,],\ h = 0.1.$$

The value of parameter γ related to the H_∞ constraint is set to $\gamma = 3$. The gains of the robust SOF Stackelberg strategy set (29), (35), and the worst case disturbance (21) are obtained below:

$$F_0^* = \left[\, -1.1154 \times 10^{-2} \,\right],\ F_1^* = \left[\, 6.3295 \times 10^{-1} \,\right],$$

$$F_2^* = [\, -1.3172 \,],\ F_\gamma^* = \left[\, 2.8468 \times 10^{-2}\ \ 2.1747 \times 10^{-2} \,\right].$$

In Step 5 of the heuristic algorithm, $\theta^{(n)} = 0.7$ is chosen. It is noted that the proposed algorithm with KM iterative technique converges after 24 iterations with the order of error as 10^{-7}. Furthermore, the convergence property of $\mathscr{J}^{(n)} > \mathscr{J}^{(n+1)}$ can be verified.

6. CONCLUSION

In this paper, the robust Stackelberg game for the UDSSs has been studied. As the result, the condition for the existence of the robust SOF Stackelberg strategy set is established, which relies on the solution of the BMIs in the optimization problem with constraints. Since the BMIs are difficult to solve, the heuristic algorithm is proposed to solve the SLMEs and LMIs instead of solving the BMIs. The convergence robustness of the proposed algorithm is attained by applying the KM iterative algorithm. Finally, the simple example demonstrated the effectiveness and usefulness of the proposed algorithm.

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