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Optimal Sensing Precision in Ensemble and Unscented Kalman Filtering ⋆

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Abstract: We consider the problem of selecting an optimal set of sensor precisions to estimate the states of a non-linear dynamical system using an Ensemble Kalman filter and an Unscented Kalman filter, which uses random and deterministic ensembles respectively. Specifically, the goal is to choose at run-time, a sparse set of sensor precisions for active-sensing that satisfies certain constraints on the estimated state covariance. In this paper, we show that this sensor precision selection problem is a semidefinite programming problem when we use l_1 norm over precision vector as the surrogate measure to induce sparsity. We formulate a sensor selection scheme over multiple time steps, for certain constraints on the terminal estimated state covariance.

Keywords: Non-linear systems, estimation, monitoring, optimal sensing, optimization

1. INTRODUCTION

In this paper, we focus on the problem of sensor design for non-linear stochastic discrete-time systems. The problem of sensor design for a system, primarily addresses questions regarding, a) which type of sensor do we need, b) how accurate sensors do we need, and c) when and d) where do we use them. A sensor design strategy aims to strike a balance between the quality of sensing performance, sensing accuracy choice, and activation over space and (or) time. A considerable work on sensor design for state estimation focuses on addressing sensor selection problem, such as Joshi and Boyd (2009); Zare and Jovanovic (2018); Zare et al. (2018), *when sensor precisions are known.* A typical sensor selection problem either deals with choosing a minimal subset of sensors, from a set of available sensors that guarantees the state estimate covariance to be bounded, as in Tzoumas et al. (2016), or in Zhang et al. (2017) where the authors minimizes the state estimate covariance when the cardinality of the sensor set is bounded. Owing to the combinatorial complexity of the problem, current methods are heuristic and developed in the Kalman filtering framework for linear Gaussian systems. There is also limited work on nonlinear state estimation which includes Chepuri and Leus (2014), which focusses on choosing sparse sensor network with known sensor precisions.

The focus of this paper *is not on sensor selection, but on determining the sensor precisions.* We define *sensor precision* as the inverse of the sensor noise covariance matrix. Particularly, determining the least precise sensors for which state estimation is achieved with desired accuracy. This is achieved by solving an l_1 minimization problem. Once the sensor precisions are determined, existing sensor selection algorithms such as in Joshi and Boyd (2009); Zare and Jovanovic (2018); Zare et al. (2018), can be applied to arrive at a reduced sensor set. In this work, we present sensor precision selection algorithm for nonlinear estimation based on ensemble Kalman filtering (EnKF) as discussed in Evensen (2003) and unscented Kalman filtering (UKF) in Wan and Van Der Merwe (2000).

Specifically, this paper addresses the problem of determining the accuracy (or precision) of a given dictionary of sensors, for a given upper-bound on the estimation accuracy. This problem has been addressed by Li et al. (2008) for a linear continuous-time system, where the sensor precision and the control law were co-designed to achieve a specified closed-loop performance. The paper also presents a state-estimation problem, where sensor precisions were determined to achieve a certain estimation accuracy. In this paper, we look at a similar problem, but for a *nonlinear discrete-time system*, with a user specified upper-bound on the estimation error covariance.

This problem is important in many engineering applications where the choice and precision of sensors for state-estimation is not obvious. Examples of such applications are in Das et al. (2019); da Silva and Colonius (2018); Shi and Chen (2013); Han et al. (2017); Chen et al. (2017) However, due to the constraints on the communication bandwidth and sensor battery life, it may not be desirable to have all the sensors report their measurements at all time instants or use the highest energy settings as in *active sensing* scenario such as in Chepuri and Leus (2015). Therefore, determining what should be the least accuracy of each sensor to achieve a given accuracy in the state estimate becomes important from a practical point-of-view.

Contributions of the Paper: In this paper we formulate a convex optimization problem to determine the optimal sensor precision for a given upper bound on the state estimation error covariance. This is presented for nonlinear discrete-time dynamical systems in EnKF and

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UKF frameworks. To the best of our knowledge, the sensor precision-selection problem for EnKF and UKF has not been addressed before. In this paper, the l_1 norm of the sensor precision is minimized, subjected to a convex constraint that guarantees the desired estimation error. This allows us to start with an over parameterization of the problem and determine a sparse solution via l_1 regularization. Therefore, the system designer can specify a dictionary of sensors with unknown sensor precisions and use the algorithm presented here to determine the optimal precision (and possibly eliminate a few sensors) to achieve the require estimation accuracy.

2. SYSTEM MODEL

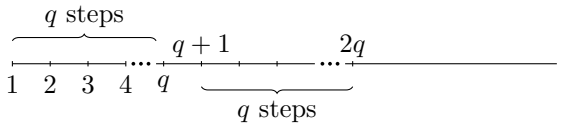
Consider an input/output discrete-time stochastic system modeled by,

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}_k(\boldsymbol{x}_k, \boldsymbol{w}_k), \tag{1}$$
$$\boldsymbol{y}_k = \boldsymbol{h}_k(\boldsymbol{x}_k) + \boldsymbol{v}_k, \tag{2}$$

where $\boldsymbol{f}_k : \mathbb{R}^n \times \mathbb{R}^{n_w} \to \mathbb{R}^n$ represents the dynamics, $\boldsymbol{h}_k : \mathbb{R}^n \to \mathbb{R}^{n_y}$ is a measurement function, $\boldsymbol{x}_k \in \mathbb{R}^n$ and $\boldsymbol{y}_k \in \mathbb{R}^{n_y}$ are the state vector and the observation vector respectively, whereas $\boldsymbol{w}_k \in \mathbb{R}^{n_w}$ and $\boldsymbol{v}_k \in \mathbb{R}^{n_y}$ are the process noise and measurement noise respectively. We assume that both $\{\boldsymbol{w}_k\}$ and $\{\boldsymbol{v}_k\}$ are zero-mean, Gaussian, independent white random processes $[\boldsymbol{w}_k \sim \mathcal{N}(0, \boldsymbol{Q}_k), \boldsymbol{v}_k \sim \mathcal{N}(0, \boldsymbol{R}_k), \mathbb{E}\left[\boldsymbol{w}_k \boldsymbol{w}_l^T\right] = \boldsymbol{Q}_k \delta_{kl}$, and $\mathbb{E}\left[\boldsymbol{v}_k \boldsymbol{v}_l^T\right] = \boldsymbol{R}_k \delta_{kl}]$. For sake of simplicity, the initial random variable $\boldsymbol{x}_0 \sim \mathcal{N}(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$ is independent of $\{\boldsymbol{w}_k\}$ and $\{\boldsymbol{v}_k\}$. We assume that $\boldsymbol{R}_k$ is a diagonal matrix, representing the measurement noise covariance. The inverse of $\boldsymbol{R}_k$ is the referred to as the *precision matrix*.

In EnKF and UKF, the measurement data $(\boldsymbol{y}_k^o)$ is used to determine the estimate of the state $\boldsymbol{x}_k$, which minimizes the estimation variance. We next introduce an *augmented model*, based on (1) and (2), which aids in formulating a multi-step precision selection problem that satisfies the specified performance criteria.

Augmented Model: We consider each of the q time steps $\{kq - q + 1, ..., kq\}$ of the system defined in (1) & (2) for $k \in \mathbb{Z}$, as a single time step



for the following augmented model:

$$\boldsymbol{X}_{k+1} = \boldsymbol{F}_k(\boldsymbol{X}_k, \boldsymbol{W}_k),\ \boldsymbol{Y}_k = \boldsymbol{H}_k(\boldsymbol{X}_k) + \boldsymbol{V}_k, \tag{3}$$

where,

$$\begin{aligned}
\boldsymbol{X}_k &:= [\boldsymbol{x}_{kq-q+1}^T, ..., \boldsymbol{x}_{kq}^T]^T, \qquad (4)\\
\boldsymbol{Y}_k &:= [\boldsymbol{y}_{kq-q+1}^T, ..., \boldsymbol{y}_{kq}^T]^T,\\
\boldsymbol{W}_k &:= [\boldsymbol{w}_{kq-q+1}^T, ..., \boldsymbol{w}_{kq+q-1}^T]^T \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{Q}}_k),\\
\boldsymbol{V}_k &:= [\boldsymbol{v}_{kq-q+1}^T, ..., \boldsymbol{v}_{kq}^T]^T \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{R}}_k),\\
\boldsymbol{\mathcal{Q}}_k &:= \mathrm{diag}([\boldsymbol{Q}_{kq-q+1}, ..., \boldsymbol{Q}_{kq+q-1}]), \qquad (5)\\
\boldsymbol{\mathcal{R}}_k &:= \mathrm{diag}([\boldsymbol{R}_{kq-q+1}, ..., \boldsymbol{R}_{kq}]),
\end{aligned}$$

denotes stacked random variables. Function $\boldsymbol{F}_k(.)$ can be recursively generated using $\boldsymbol{f}_i(.)$s. It should be noted that the augmented model represents *a q-step q-shift process, rather than a q-step sliding-window process.*

Remark 1: In the rest of the paper, we only use the augmented state model and consequently time step k denotes the batch of q time points $\{kq - q + 1, ..., kq\}$, unless otherwise specified.

3. FILTER MODELS: ENKF AND UKF

The filtering process for the augmented model (3) consists of two sequential steps: dynamics update and measurement update. In EnKF, random samples are generated using Monte Carlo techniques, whereas the state distribution in UKF is represented by a Gaussian random variable (GRV) and is specified using a minimal set of carefully chosen deterministic sample points along with their associated weights, as shown in Wan and Van Der Merwe (2000). The sensor-selection problem for each these filtering frameworks are presented in the next section.

3.1 Dynamic Update for EnKF Model

Let $\boldsymbol{\mathcal{X}}_k^+ = \left[\boldsymbol{X}_k^{1+}\ \boldsymbol{X}_k^{2+}\ \cdots\ \boldsymbol{X}_k^{N+}\right] \in \mathbb{R}^{nq \times N}$ be the matrix with N number of *posterior* samples $\boldsymbol{X}_k^{i+}$ at time k. The posterior mean from the samples is approximated as,

$$\boldsymbol{\mu}_k^+ := \mathbb{E}\left[\boldsymbol{X}_k^+\right] \approx \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{X}_k^{i+} = \frac{1}{N}\boldsymbol{\mathcal{X}}_k^+ \boldsymbol{1}_N,$$

where $\boldsymbol{1}_N \in \mathbb{R}^N$ is a column vector of N ones. We define,

$$\bar{\boldsymbol{\mathcal{X}}}_k^+ := \left[\boldsymbol{\mu}_k^+\ \cdots\ \boldsymbol{\mu}_k^+\right] = \boldsymbol{\mu}_k^+ \boldsymbol{1}^T = \frac{1}{N}\boldsymbol{\mathcal{X}}_k^+ \boldsymbol{1}\boldsymbol{1}^T,$$

then variance from the samples $\boldsymbol{\Sigma}_{xx,k}^+$ is,

$$\mathbb{E}\left[(\boldsymbol{X}_k^{i+} - \boldsymbol{\mu}_k^+)(\boldsymbol{X}_k^{i+} - \boldsymbol{\mu}_k^+)^T\right] \approx \boldsymbol{\mathcal{X}}_k^+ \boldsymbol{A} \boldsymbol{\mathcal{X}}_k^{+T}. \tag{6}$$

where

$$\boldsymbol{A} := \left[\frac{1}{N-1}\left(\boldsymbol{I}_N - \frac{\boldsymbol{1}\boldsymbol{1}^T}{N}\right)\left(\boldsymbol{I}_N - \frac{\boldsymbol{1}\boldsymbol{1}^T}{N}\right)\right]$$

The state of each ensemble member at the next time step is estimated using the dynamics model:

$$\boldsymbol{X}_{k+1}^{i-} = \boldsymbol{F}_k(\boldsymbol{X}_k^{i+}, \boldsymbol{W}_k^i), \tag{7}$$

If applied to a linear system, this ensemble approach reduces the cost associated with the time propagation of the covariance matrix from $\mathcal{O}(n^3q^3)$ (classical KF) to $\mathcal{O}(n^2q^2N)$ (EnKF).

3.2 Measurement update for EnKF Model

The ensemble members are corrected to minimize the error with respect to the measurements in the presence of noise and model uncertainties. A measurement update formulation proposed by Evensen and Van Leeuwen (1996) is:

$$\begin{aligned}
\boldsymbol{X}_{k+1}^{i+} =& \boldsymbol{X}_{k+1}^{i-} + \boldsymbol{\Sigma}_{xy,k+1}^{-}(\boldsymbol{\Sigma}_{yy,k+1}^{-} + \boldsymbol{\mathcal{R}}_k)^{-1}\\
&\times (\boldsymbol{Y}_{k+1}^o - \boldsymbol{H}_{k+1}(\boldsymbol{X}_{k+1}^{i-}) + \boldsymbol{\epsilon}_k^i), \qquad (8)
\end{aligned}$$

where $\boldsymbol{\epsilon}_k^i$ is sampled from $\mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{R}}_k)$. We define $\boldsymbol{\Sigma}_{xy,k+1}^{-}$ as:

$$\begin{aligned}
\boldsymbol{\Sigma}_{xy,k+1}^{-} =& \frac{1}{N-1}(\boldsymbol{\mathcal{X}}_{k+1}^{-} - \bar{\boldsymbol{\mathcal{X}}}_{k+1}^{-}) \times\\
&(\boldsymbol{H}_{k+1}(\boldsymbol{\mathcal{X}}_{k+1}^{-}) - \boldsymbol{H}_{k+1}(\bar{\boldsymbol{\mathcal{X}}}_{k+1}^{-}))^T, \qquad (9)
\end{aligned}$$

and $\boldsymbol{\Sigma}^{-}_{yy,k+1}$ is defined as:

$$\boldsymbol{\Sigma}^{-}_{yy,k+1} = \frac{1}{N-1}\{\boldsymbol{H}_{k+1}(\boldsymbol{\mathcal{X}}^{-}_{k+1}) - \boldsymbol{H}_{k+1}(\bar{\boldsymbol{\mathcal{X}}}^{-}_{k+1}))\times (\boldsymbol{H}_{k+1}(\boldsymbol{\mathcal{X}}^{-}_{k+1}) - \boldsymbol{H}_{k+1}(\bar{\boldsymbol{\mathcal{X}}}^{-}_{k+1}))^T\}. \quad (10)$$

Remark 2: Equation (9) and (10) allows for direct evaluation of the nonlinear measurement function $\boldsymbol{H}_k(\boldsymbol{x})$ in calculating the Kalman gain, which is shown in Tang et al. (2014) to hold for unbiased measurement forecasts $\{\boldsymbol{H}_k(\boldsymbol{X}^{i-}_k)\}$, which we assume to be true in our work.

The covariance update equation of the augmented model is:

$$\boldsymbol{\Sigma}^{+}_{xx,k+1} = \boldsymbol{\Sigma}^{-}_{xx,k+1} - \boldsymbol{\Sigma}^{-}_{xy,k+1}(\boldsymbol{\Sigma}^{-}_{yy,k+1} + \boldsymbol{\mathcal{R}}_{k+1})^{-1} \times \boldsymbol{\Sigma}^{-T}_{xy,k+1}, \quad (11)$$

where $\boldsymbol{\Sigma}^{-}_{xx,k+1} = \boldsymbol{\mathcal{X}}^{-}_{k+1}\boldsymbol{A}\boldsymbol{\mathcal{X}}^{-T}_{k+1}$.

3.3 Dynamic Update for UKF Model

The dynamic update step from $k \rightarrow k+1$ starts with generating deterministic points called σ points. To capture the mean ${}_a\boldsymbol{\mu}^{+}_k$ of the augmented state vector ${}_a\boldsymbol{X}^{+}_k := [\boldsymbol{X}^{+}_k\ \boldsymbol{W}_k]^T$, where ${}_a\boldsymbol{X}^{+}_k \in \mathbb{R}^{n_a}$ and $n_a = nq + n_w q$, as well as the augmented error covariance ${}_a\boldsymbol{\Sigma}^{+}_{xx,k} = \text{diag}([\boldsymbol{\Sigma}^{+}_{xx,k}\ \boldsymbol{\mathcal{Q}}_k])$ the sigma points are chosen as

$${}_a\boldsymbol{X}^{0+}_k = {}_a\boldsymbol{\mu}^{+}_k,$$

$${}_a\boldsymbol{X}^{i+}_k = {}_a\boldsymbol{\mu}^{+}_k + \left(\sqrt{(n_a+\rho){}_a\boldsymbol{\Sigma}^{+}_{xx,k}}\right)_i, i = 1,...,n_a,$$

$${}_a\boldsymbol{X}^{i+}_k = {}_a\boldsymbol{\mu}^{+}_k - \left(\sqrt{(n_a+\rho){}_a\boldsymbol{\Sigma}^{+}_{xx,k}}\right)_{i-nq}, i = n_a+1,...,2n_a,$$

with associated weights as

$$\boldsymbol{\omega}^{(m)}_0 = \rho/(n_a+\rho),\ \boldsymbol{\omega}^{(c)}_0 = \rho/(n_a+\rho) + (1-\alpha^2+\beta),$$
$$\boldsymbol{\omega}^{(m)}_i = \boldsymbol{\omega}^{(c)}_i = 1/\{2(n_a+\rho)\}.$$

The weight vectors are:

$$\boldsymbol{\mathcal{W}}^m = [\boldsymbol{\omega}^{(m)}_0 \boldsymbol{\omega}^{(m)}_1 ... \ \boldsymbol{\omega}^{(m)}_{2n_a+1}]^T,$$
$$\boldsymbol{\mathcal{W}}^c = [\boldsymbol{\omega}^{(c)}_0 \boldsymbol{\omega}^{(c)}_1 ... \ \boldsymbol{\omega}^{(c)}_{2n_a+1}]^T,$$

where $\rho = \alpha^2(n_a+\kappa) - n_a$ is the scaling parameter, α is set to 0.001, κ is set to 0, and β is 2 in this work. The term $\left(\sqrt{(n_a+\rho){}_a\boldsymbol{\Sigma}^{+}_{xx,k}}\right)_i$ represents ith row of the matrix square root. The propagated state of each ensemble member at time $k+1$ is generated exactly as EnKF by using $\boldsymbol{X}^{i-}_{k+1} = \boldsymbol{F}_k(\boldsymbol{X}^{i+}_k, \boldsymbol{W}^i_k)$, where ${}_a\boldsymbol{X}^{i+}_k := [\boldsymbol{X}^{i+}_k\ \boldsymbol{W}^i_k]^T$.

But unlike EnKF, the corresponding prior mean and covariance at time $k+1$ are:

$$\boldsymbol{\mu}^{-}_{k+1} = \boldsymbol{\mathcal{X}}^{-}_{k+1}\boldsymbol{\mathcal{W}}^m,\ \boldsymbol{\Sigma}^{-}_{xx,k+1} = \boldsymbol{\mathcal{X}}^{-}_{k+1}\boldsymbol{B}_k\boldsymbol{\mathcal{X}}^{-T}_{k+1}$$

where $\boldsymbol{B}_k := \boldsymbol{L}\boldsymbol{L}^T$, $\boldsymbol{L} := \left(\text{diag}(\boldsymbol{\mathcal{W}}^c) - \boldsymbol{\mathcal{W}}^c\boldsymbol{1}^T_{2nq+1}\right)$.

We define the following terms which we use in the following measurement update phase of UKF.

$$\boldsymbol{\mathcal{Y}}^{-}_{k+1} = \boldsymbol{H}(\boldsymbol{\mathcal{X}}^{-}_{k+1}),\ \bar{\boldsymbol{\mathcal{Y}}}^{-}_{k+1} = \boldsymbol{\mathcal{Y}}^{-}_{k+1}\boldsymbol{\mathcal{W}}^m\boldsymbol{1}^T_{2nq+1},$$
$$\bar{\boldsymbol{\mathcal{X}}}^{-}_{k+1} = \boldsymbol{\mathcal{X}}^{-}_{k+1}\boldsymbol{\mathcal{W}}^m\boldsymbol{1}^T_{2nq+1},$$

where $\boldsymbol{\mathcal{Y}}^{-}_{k+1} = \left[\boldsymbol{Y}^{1-}_{k+1}\ \boldsymbol{Y}^{2-}_{k+1}\ \cdots\ \boldsymbol{Y}^{(2nq+1)-}_{k+1}\right]$ and $\boldsymbol{\mathcal{X}}^{-}_{k+1} = \left[\boldsymbol{X}^{1-}_{k+1}\ \boldsymbol{X}^{2-}_{k+1}\ \cdots\ \boldsymbol{X}^{(2nq+1)-}_{k+1}\right]$

3.4 Measurement Update for UKF

We calculate $\boldsymbol{\Sigma}^{-}_{xy,k+1}$ and $\boldsymbol{\Sigma}^{-}_{yy,k+1}$ as:

$$\boldsymbol{\Sigma}^{-}_{xy,k+1} = (\boldsymbol{\mathcal{X}}^{-}_{k+1} - \bar{\boldsymbol{\mathcal{X}}}^{-}_{k+1}) \times \text{diag}(\boldsymbol{\mathcal{W}}^c) \times (\boldsymbol{\mathcal{Y}}^{-}_{k+1} - \bar{\boldsymbol{\mathcal{Y}}}^{-}_{k+1})^T \quad (12)$$

$$\boldsymbol{\Sigma}^{-}_{yy,k+1} = (\boldsymbol{\mathcal{Y}}^{-}_{k+1} - \bar{\boldsymbol{\mathcal{Y}}}^{-}_{k+1}) \times \text{diag}(\boldsymbol{\mathcal{W}}^c) \times (\boldsymbol{\mathcal{Y}}^{-}_{k+1} - \bar{\boldsymbol{\mathcal{Y}}}^{-}_{k+1})^T \quad (13)$$

The covariance update equation is exactly same as (11).

Remark 3: Since the covariance update equation for EnKF and UKF are identical, this allows us to formulate a common precision selection algorithm that is presented next.

4. PROBLEM FORMULATION

We define precision matrix of measurement ($\boldsymbol{\mathcal{S}}_k$) as the inverse of the covariance matrix of the augmented measurement noise ($\boldsymbol{\mathcal{R}}_k$). We assume that the precision matrix $\boldsymbol{\mathcal{S}}_k$ is a diagonal matrix, with diagonal elements $\{\lambda_i\}$, where $i \in [1,...,qn_y]$. The sensor precision associated with the i^{th} sensor is λ_i. Equation (11) for time step k, can be written as:

$$\boldsymbol{\Sigma}^{+}_{xx,k} = \boldsymbol{\Sigma}^{-}_{xx,k} - \boldsymbol{\Sigma}^{-}_{xy,k}(\boldsymbol{\Sigma}^{-}_{yy,k} + \boldsymbol{\mathcal{S}}^{-1}_k)^{-1}\boldsymbol{\Sigma}^{-T}_{xy,k}$$
$$= \boldsymbol{\Sigma}^{-}_{xx,k} - \boldsymbol{\Sigma}^{-}_{xy,k}\{\boldsymbol{\Sigma}^{-}_{yy,k} + \text{diag}([\lambda_1,...,\lambda_{qn_y}])^{-1}\}^{-1}\boldsymbol{\Sigma}^{-T}_{xy,k}$$

The λ_i's are the control variables, that regulate the estimation error covariance matrix $\boldsymbol{\Sigma}^{+}_{xx,k}$, when we have the prior ensemble $\boldsymbol{\mathcal{X}}^{-}_k$ which is generated from $\boldsymbol{\mathcal{X}}^{+}_{k-1}$ using (7). The augmented process noise $\boldsymbol{W}_k$ is generated by sampling from $\boldsymbol{\mathcal{Q}}_k$ in (5).

Our objective is to design $\{\lambda_i\}$ such that $\boldsymbol{M}_q\boldsymbol{\Sigma}^{+}_{xx,k}\boldsymbol{M}^T_q$ is upper-bounded by a prescribed positive definite matrix $\boldsymbol{P}^d_{kq}$, where the matrix $\boldsymbol{M}_q := [\boldsymbol{0}^1_{n\times n}, \boldsymbol{0}^2_{n\times n}, ..., \boldsymbol{0}^{q-1}_{n\times n}, \boldsymbol{I}_{n\times n}]$, is utilized to extract error covariance matrix of posterior estimate of $\boldsymbol{x}_{kq}$ from $\boldsymbol{\Sigma}^{+}_{xx,k}$. The matrix $\boldsymbol{P}^d_{kq}$ is the performance bound based on which we select sensor precisions. Although we use the augmented model in (3), the performance bound is on the covariance of the estimate of $\boldsymbol{x}_{kq}$.

4.1 Optimal Sensor Precision

The solution to the sensor selection problem for EnKF and UKF is presented as the following theorem:

Theorem 1. The optimal precision of each of the sensors, $\boldsymbol{\lambda}_k := [\lambda_1,...,\lambda_{qn_y}]$ at time k, which guarantees $\boldsymbol{M}_q\boldsymbol{\Sigma}^{+}_{xx,k}\boldsymbol{M}^T_q \preceq \boldsymbol{P}^d_{kq}$, for given prior ensemble $\boldsymbol{\mathcal{X}}^{+}_{k-1}$, is obtained by solving the following semidefinite programming (SDP) problem,

$$\boldsymbol{\lambda}^*_k = \min_{\boldsymbol{\lambda_k}:=[\lambda_1,...,\lambda_{qn_y}]^T} ||\boldsymbol{\lambda}_k||_1, \quad (14)$$

subject to,

$$\begin{bmatrix} \boldsymbol{P}^d_{kq} + \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^T & \boldsymbol{D} \end{bmatrix} \succeq 0, \quad \lambda_i \geq 0,\ \forall i \in [1,...,qn_y], \quad (15)$$

where

$$\boldsymbol{A} := -\boldsymbol{M}_q\boldsymbol{\Sigma}^{-}_{xx,k}\boldsymbol{M}^T_q + \boldsymbol{M}_q\boldsymbol{\Sigma}^{-}{}_{xy,k}\boldsymbol{\mathcal{S}}_k\boldsymbol{\Sigma}^{-T}_{xy,k}\boldsymbol{M}^T_q,$$

$$\boldsymbol{B} := \boldsymbol{M}_q \boldsymbol{\Sigma}^-_{xy,k} \boldsymbol{\mathcal{S}}_k,\ \boldsymbol{D} := (\boldsymbol{\Sigma}^-{}_{yy,k})^{-1} + \boldsymbol{\mathcal{S}}_k,$$
$$\boldsymbol{\mathcal{S}}_k := \text{diag}([\lambda_1, ..., \lambda_{qn_y}]).$$

The matrices $\boldsymbol{\Sigma}^-_{xx,k}, \boldsymbol{\Sigma}^-_{xy,k}, \boldsymbol{\Sigma}^-_{yy,k}$ are calculated using the prior ensemble $\boldsymbol{\mathcal{X}}^-_k$ and the expected observations, $\boldsymbol{H}_k(\boldsymbol{\mathcal{X}}^-_k)$ using (9) & (10) for EnKF, or (12) & (13) for UKF. We calculate $\boldsymbol{\mathcal{X}}^-_k$ from $\boldsymbol{\mathcal{X}}^+_{k-1}$ using (7) both for EnKF and UKF.

Proof.

$$\boldsymbol{\Sigma}^+_{xx,k} = \boldsymbol{\Sigma}^-_{xx,k} - \boldsymbol{\Sigma}^-_{xy,k}(\boldsymbol{\Sigma}^-_{yy,k} + \boldsymbol{\mathcal{R}}_k)^{-1}\boldsymbol{\Sigma}^-{}^T_{xy,k}$$
$$= \underbrace{\boldsymbol{\Sigma}^-_{xx,k} - \boldsymbol{\Sigma}^-{}_{xy,k}\boldsymbol{\mathcal{R}}_k^{-1}\boldsymbol{\Sigma}^-{}^T_{xy,k}}_{-\hat{\boldsymbol{A}}} + \underbrace{\boldsymbol{\Sigma}^-_{xy,k}\boldsymbol{\mathcal{R}}_k^{-1}}_{\hat{\boldsymbol{B}}}$$
$$\times \underbrace{[(\boldsymbol{\Sigma}^-{}_{yy,k})^{-1} + \boldsymbol{\mathcal{R}}_k^{-1}]^{-1}}_{\boldsymbol{D}^{-1}} \underbrace{\boldsymbol{\mathcal{R}}_k^{-1}\boldsymbol{\Sigma}^-{}^T_{xy,k}}_{\hat{\boldsymbol{B}}^T}$$
$$\boldsymbol{\Sigma}^+_{xx,k} = -\hat{\boldsymbol{A}} + \hat{\boldsymbol{B}}\boldsymbol{D}^{-1}\hat{\boldsymbol{B}}^T,$$

The posterior error covariance of $\boldsymbol{x}_{kq}$, i.e. $\boldsymbol{M}_q\boldsymbol{\Sigma}^+_{xx,k}\boldsymbol{M}_q^T \preceq \boldsymbol{P}^d_{kq}$. We have

$$\boldsymbol{M}_q\boldsymbol{\Sigma}^+_{xx,k}\boldsymbol{M}_q^T = -\boldsymbol{M}_q\hat{\boldsymbol{A}}\boldsymbol{M}_q^T + \boldsymbol{M}_q\hat{\boldsymbol{B}}\boldsymbol{D}^{-1}\hat{\boldsymbol{B}}^T\boldsymbol{M}_q^T$$

Hence,

$$\boldsymbol{P}^d_{kq} + \boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{B}^T \succeq 0, \tag{16}$$

where $\boldsymbol{M}_q\hat{\boldsymbol{A}}\boldsymbol{M}_q^T := \boldsymbol{A}$ and $\boldsymbol{M}_q\hat{\boldsymbol{B}} := \boldsymbol{B}$. Since $\boldsymbol{D} \succ 0$ and $\boldsymbol{P}^d_{kq} + \boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{B}^T \succeq 0$, using Schur's complement we get the following,

$$\begin{bmatrix} \boldsymbol{P}^d_{kq} + \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^T & \boldsymbol{D} \end{bmatrix} \succeq 0. \tag{17}$$

Matrices $\boldsymbol{A}, \boldsymbol{B}$, and $\boldsymbol{D}$ are linear in $\boldsymbol{\mathcal{R}}_k^{-1}$ or $\boldsymbol{\mathcal{S}}_k$. Equation (17) is a linear matrix inequality (LMI) over $\boldsymbol{\mathcal{S}}_k$. The fact that the precision values are non-negative introduces the linear constraint $\lambda_i \geq 0$. The optimal precision is thus obtained by minimizing $\|\boldsymbol{\lambda}_k\|_1$, subject to the above LMI and linear constraint on $\boldsymbol{\lambda}_k$.

Algorithm 1 Precision Selection

Input: $\boldsymbol{f}_k(.), \boldsymbol{g}_k(.), \boldsymbol{h}_k(.), q, k, \boldsymbol{\mathcal{X}}^+_{k-1}, \boldsymbol{Q}_k, \boldsymbol{P}^d_{kq}$

Output: Set $\boldsymbol{\lambda} \in \{\mathbb{R}^{+^{(qn_y \times 1)}} \cup \boldsymbol{0}_{qn_y \times 1}\}$.

1: **procedure**
2: Calculate $\boldsymbol{\mathcal{Q}}_k$
3: $\boldsymbol{\mathcal{X}}^+_{k-1} \rightarrow \boldsymbol{\mathcal{X}}^-_k$ using $\boldsymbol{F}_k(.)$, $\boldsymbol{W}_k \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\mathcal{Q}}_k)$
4: Calculate $\boldsymbol{\Sigma}^-_{xx,k}, \boldsymbol{\Sigma}^-_{xy,k}, \boldsymbol{\Sigma}^-_{yy,k}$
5: Calculate $\boldsymbol{M}_q$
6: Construct $\boldsymbol{\mathcal{S}}_k := \text{diag}([\lambda_1, ..., \lambda_{qn_y}])$
7: Construct $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{D}$ matrices
8: Solve SDP problem in (14), (15)

Remark 4: Theorem 1 determines the optimal set of sensor precisions. The l_1 regularization induces sparseness in the solution. Therefore, if the optimization is performed on an over parameterized problem, i.e. with a large dictionary of sensors that considers all possible sensor choices, we expect the optimal solution $\boldsymbol{\lambda}^*$ to have entries that are zero. This indicates that those sensors are not needed to achieve the require state estimation accuracy and can be removed. Dimension of each sensor is assumed to be one.

However, numerical solution to the sensor precision problem will result in small precision values that are not exactly zero. Those sensors can then be eliminated iteratively using theorem 1 with the reduced dictionary of sensors, discussed later in this paper. An upper bound on the precision for sensor(s) can also be incorporated in the optimization problem as a convex constraint over the argument space. For EnKF, covariance inflation Wu and Zheng (2018) technique is used while calculating $\boldsymbol{\Sigma}^-_{xx,k}, \boldsymbol{\Sigma}^-_{xy,k}, \boldsymbol{\Sigma}^-_{yy,k}$ matrices.

Remark 5: Before solving the SDP problem, it is recommended to formulate the optimization problem with respect to normalized variables. This improves the numerical accuracy of the solution and make the optimal solution meaningful. For instance, in sensor precision selection for improving space-situational awareness, certain states are in the order of 10^3 km and others are in radians. Therefore, normalization with respect to dynamics, error covariance, and sensor noise is required to avoid ill conditioning of covariance matrices and improve the efficacy of the proposed optimal sensor precision algorithm.

4.2 Discussion: Sensor Pruning

As mentioned earlier, numerical solution of the l_1 regularization problem may assign small precision to certain $\{\lambda_i\}$s of $\boldsymbol{\lambda}$, which are not exactly zero and can possibly be removed without affecting the estimation quality. Therefore, a separate pruning process is required to reduce the set of sensors in the system. We define sensor pruning as choosing a subset of available sensors, which ensures that the covariance bound $\boldsymbol{P}^d_{kq}$ is satisfied. The rationale behind choosing a subset of the sensors is to eliminate the sensors whose precision requirement is too low compared to other sensors.

An adhoc algorithm to address this has been presented in Li et al. (2008), where the calculated sensor precision vector $\boldsymbol{\lambda}^*$ is first sorted in ascending order. Iteratively, the smallest precision sensor are removed and the precisions are recalculated. This is continued till the problem becomes infeasible. The work of Li et al. (2008) focusses on integrated design of controller and sensing architecture, without taking observability into consideration. However, in the context of state-estimation, observability condition must be addressed while sensor pruning. Other sensor selection algorithms proposed in Tzoumas et al. (2016) and Zhang et al. (2017) can also be investigated. However, the problem of sensor pruning becomes challenging for nonlinear system, and is the focus of our future work.

5. NUMERICAL EXPERIMENT

In this section, we provide simulation results for the sensor precision selection algorithm for single time step update ($q = 1$) and multiple time step update ($q = 3$); also including the case where sensor precisions are constrained.

5.1 Test Problem: The Lorenz (1996) model

The sensor precision selection scheme is applied to the Lorenz-96 (L96) model to test its validity, when an EnKF and an UKF filter are used for state estimation. The

L96 model consists of N_x equally spaced variables, x_i for $i = 1, ..., N_x$, which are evolved in time using the set of differential equations:

$$\frac{dx_i}{dt} = (x_{i+1} - x_{i-2})x_{i-1} - x_i + F, \tag{18}$$

with cyclic boundaries: $x_{i+N} = x_i$ and $x_{i-N} = x_i$. The three terms in (18) are analogous to advection, damping, and forcing. The system exhibits varying degrees of chaotic behaviors depending on the choices of F and N_x. We fix N_x and F at 20 and 8 respectively, which leads to chaotic behavior in the system dynamics as shown in Lorenz (1996), and Lorenz and Emanuel (1998).

5.2 Experimental Setup

In this experiment we consider no process noise, i.e. $\boldsymbol{\mathcal{Q}}_k = 0$, but with initial condition uncertainty. Forward integration of (18) is performed numerically using the fourth-order Runge-Kutta method with 20 internal stages for $k \to k+1$, with a time step of 0.05 time units for each stage as shown in Lorenz (1996). We assume the following non-linear measurement model:

$$y_{i,k} = \frac{1}{1 + e^{-x_{i,k}}} + v_{i,k} \tag{19}$$

where $(.)_{i,k}$ denotes i^{th} component of a vector at time point k, with measurement noise $\boldsymbol{v}_k \sim \mathcal{N}(0, \boldsymbol{R}_k)$. The initial ensemble is generated from a multivariate Gaussian distribution with mean vector of size qN_x, whose elements are chosen randomly from $[0, F]$ and a random positive definite matrix ($\boldsymbol{\Sigma}_{init}$) of size $qN_x \times qN_x$ as the covariance, where qN_x denotes the size of the augmented state vector. We use $2qN_x + 1$ number of samples for both EnKF and UKF, for $q = 1$ and 3.

To study the effects of state estimate covariance bounds on the optimal sensor precision values, we linearly vary the required error covariance bound from a factor of 0.9 to 0.6 of the initial covariance $\boldsymbol{\Sigma}_{init}$ as shown along the x-axis of the figures. For $q = 1$ shown in fig.(1) and fig.(3), 21 linearly varying bounds are considered within the interval of $[0.9, 0.6]$, where as for $q = 3$ shown in fig.(2), fig.(4), 7 linearly varying bounds are chosen from the same interval. Measurement model in (19) shows 20 different sensors, whose indices are plotted along the y-axis of each of the figures.

5.3 Solving for Sensor Precisions

We use CVX software of Grant and Boyd (2014) to solve our SDP problem. CVX internally calls SeDuMi solver of Sturm (1999), a MATLAB implementation of the second-order interior-point methods. The l_1 norm minimization problem with LMI constraint yields desired precision values for the sensors shown in the figures as heat maps, with sensor precision ranges shown in the right y-axis. Fig.(1) and fig.(2) show optimal precisions required for EnKF for $q = 1$ and $q = 3$ respectively, satisfying prescribed covariance bounds. Fig.(3) and fig.(4), are the corresponding plots for UKF. For $q = 1$, sensor precisions are calculated to satisfy the covariance bound at the immediate next time instant. When $q = 3$, sensor precision are calculated for consecutive 3 time instants to satisfy covariance bound on the state variable at the end of the time horizon. For EnKF, the sensor precisions are restricted to be below 15, whereas for UKF precisions are bounded above by 3, while solving the SDP problem. We see that the optimal solution results in high accuracy sensing only at the end of the time interval, with poor (or no) sensing within the interval. However, this changes when upper limit on the precisions are reduced. In that case, we will see higher precision within the interval.

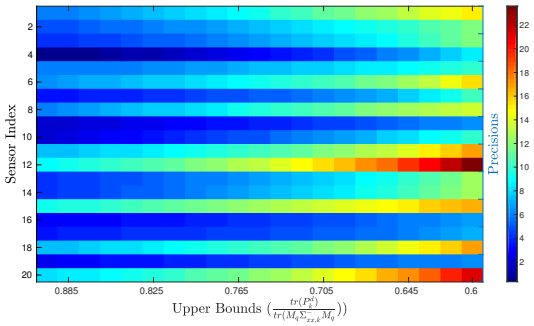


Fig. 1. Precision of sensors updated at each time step ($q = 1$) for EnKF without precision bounds

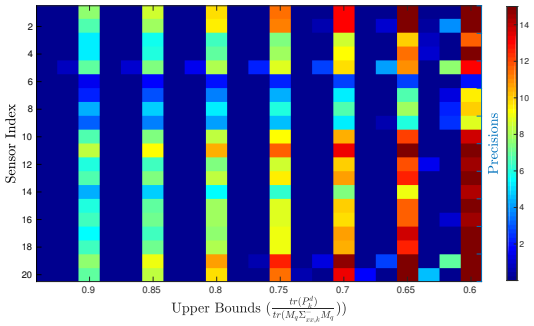


Fig. 2. Precision of sensors updated for 3 consecutive time step ($q = 3$), with precision bounds for EnKF

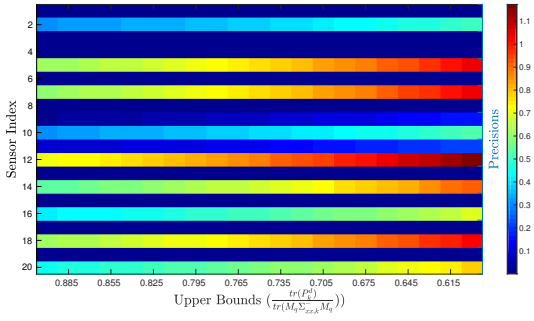


Fig. 3. Precision of sensors updated at each time step ($q = 1$) for, for UKF without precision bounds

Note that we get different values for the optimal precisions for EnKF and UKF. These also depend on the sample size and covariance inflation parameter for EnKF, and choices of α, β, κ for UKF. It will be interesting to investigate the impact of these two frameworks on the sensor precision problem, and determine if one requires more precision than the other to arrive at the same estimation accuracy. These important questions will be addressed in our future work.

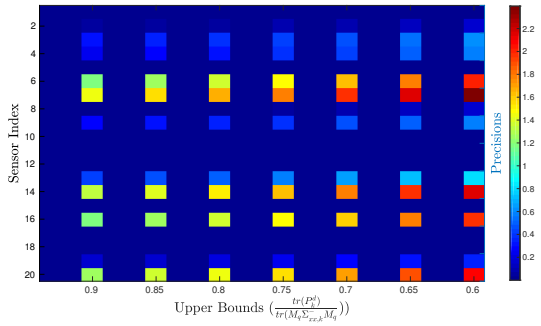


Fig. 4. Precision of sensors updated for 3 consecutive time step ($q = 3$), for UKF with precision bounds

6. CONCLUSION

In this paper, a new sensor precision selection problem for non-linear dynamical systems is presented in the framework of EnKF and UKF. The problem is shown to be convex, which can be easily solved using standard software such as CVX. The algorithm is applied to the Lorenz 1996 model of order 20 and results from both EnKF and UKF framework are presented. Sensor pruning, in the event of very small precisions in the optimal solution, is also discussed and methods to solve them are presented. Future work involves developing new sensor pruning algorithms for nonlinear systems, and also investigating impact of EnKF and UKF framework, along with other norm minimizations, on optimality and practicality.

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