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An LMI-Based Approach for Semivalues Constraints in Coalitional Feedback Control ⋆

F. J. Muros * **J. M. Maestre** *

* *Dep. of Systems and Automation Engineering, Univ. of Seville, Spain (e-mail: {franmuros,pepemaestre}@us.es)*

Abstract: In coalitional approaches, communication links are dynamically enabled/disabled by the control scheme to reduce the cooperation burden without compromising the system performance. This problem setup can be interpreted as a cooperative game where solution concepts provide a measure of the impact of links on system behavior. Here, we present how constraints on the set of semivalues can be introduced via linear matrix inequalities (LMIs) to impose design requirements. To this end, an LMI-based iterative method is presented. Finally, an academic example is simulated to illustrate the feasibility of the proposed approach.

Keywords: Semivalues, coalitional control, clustering, distributed control, cooperative game theory, Shapley value, Banzhaf value.

1. INTRODUCTION

In the last two decades, numerous distributed control approaches have been developed due to their well-known benefits such as scalability, modularity and ease of implementation, which make them suitable to control large-scale systems, e.g., traffic, water and power networks (Bakule, 2008; Scattolini, 2009; Maestre and Negenborn, 2014).

Typically, in distributed control, communication links among the controllers are *always* available independently of their contribution to the system performance. To solve this issue, new distributed schemes have appeared under different names such as *coalitional control*. In these approaches, communication links are enabled or disabled depending on the needs of the control scheme at each time instant. Therefore, the communication can be reduced without compromising the overall performance. Examples of these schemes can be found in: (Jilg and Stursberg, 2013; Sadamoto et al., 2014), where the coupling of the plant is utilized to divide it into hierarchically coupled clusters; (Bauso and Notarstefano, 2015; Bauso and Cannon, 2018), where conditions under which the agents reach robust consensus in the presence of disturbances are studied; and (Maestre et al., 2014; Muros et al., 2017b,a), where coalitional control approaches based on cooperative game theory are introduced. Recently, this setting has been extended to an MPC framework by Fele et al. (2017); Maestre and Ishii (2017); Trodden and Baldivieso-Monasterios (2019); Baldivieso-Monasterios et al. (2019).

This work is an extension of (Muros et al., 2017b,a), where constraints on the Shapley (Shapley, 1953) and Banzhaf (Banzhaf, 1965) values were respectively considered in the design of feedback controllers for coalitional control schemes. These values provide a measurement of the links cost both from communicational and system performance viewpoints, so that constraining them can be interesting for the design of the controller. Here, we consider the set of *semivalues* (Dubey et al., 1981), a subgroup of cooperative game theory solution concepts that includes among others both the Shapley and Banzhaf values, thus generalizing our previous results. Some applications of semivalues to political, economic, and sociological problems are detailed by Carreras and Freixas (2002).

With the aim of including constraints on the set of semivalues, an iterative design method based on linear matrix inequalities (LMIs) is considered in this work. LMI-based approaches represent a powerful tool to solve control problems, leading to satisfactory results in terms of stability and computation time (Alamo et al., 2006). Some examples of recent LMI-based methods in the literature have been proposed by: Ebihara et al. (2014), which deal with the analysis and synthesis of linear positive systems; and Witczak et al. (2016), who study robust fault estimation methods for nonlinear systems.

The remaining of the paper is organized as follows. The control problem is stated in a coalitional setting in Section 2. In Section 3, the set of semivalues is formally introduced and properties of interest are presented. Next, an LMI-based design algorithm for the control matrices that includes constraints on the set of semivalues and the control scheme implemented afterwards are presented in Section 4. An academic example that tests the proposed approach is given in Section 5. Finally, concluding remarks and lines of future work are commented in Section 6.

2. PROBLEM SETTING

Let us consider a system composed of a set $\mathcal{N} = \{1, 2, \ldots, |\mathcal{N}|\}$ of interconnected subsystems. The dynamics of subsystem $i \in \mathcal{N}$ are

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$$\begin{aligned}\mathbf{x}_i(k+1) &= \mathbf{A}_{ii}\mathbf{x}_i(k) + \mathbf{B}_{ii}\mathbf{u}_i(k) + \mathbf{d}_i(k),\\ \mathbf{d}_i(k) &= \sum_{j\neq i}[\mathbf{A}_{ij}\mathbf{x}_j(k) + \mathbf{B}_{ij}\mathbf{u}_j(k)].\end{aligned} \tag{1}$$

Here, $\mathbf{x}_i(k) \in \mathbb{R}^{n_{\mathrm{x}_i}}$ and $\mathbf{u}_i(k) \in \mathbb{R}^{n_{\mathrm{u}_i}}$ are respectively the state and input vectors. The state and input-to-state matrices have the corresponding dimensions, i.e., $\mathbf{A}_{ij} \in \mathbb{R}^{n_{\mathrm{x}_i}\times n_{\mathrm{x}_j}}$, $\mathbf{B}_{ij} \in \mathbb{R}^{n_{\mathrm{x}_i}\times n_{\mathrm{u}_j}}$. Finally, $\mathbf{d}_i(k)$ represents the effect of neighbor interactions on the dynamics of subsystem i.

From a global viewpoint, the dynamics of the system become

$$\mathbf{x}_{\mathcal{N}}(k+1) = \mathbf{A}_{\mathcal{N}}\mathbf{x}_{\mathcal{N}}(k) + \mathbf{B}_{\mathcal{N}}\mathbf{u}_{\mathcal{N}}(k), \tag{2}$$

where $\mathbf{x}_{\mathcal{N}}(k)$ and $\mathbf{u}_{\mathcal{N}}(k)$ aggregate the local state and input vectors, respectively. Also, $\mathbf{A}_{\mathcal{N}}$ and $\mathbf{B}_{\mathcal{N}}$ aggregate the corresponding local matrices, including all mutual interaction between local subsystems.

2.1 Control infrastructure

Subsystems in $\mathcal{N}$ are managed by local controllers connected by a network described by an undirected graph $(\mathcal{N}, \mathcal{E})$, with $\mathcal{E} \subseteq \mathcal{E}^{\mathcal{N}} = \mathcal{N}\times\mathcal{N}$ being edges, i.e., the set of available communication links between different subsystems, which are assumed here to be bidirectional. Each link l can be enabled or disabled at each time instant, assuming a cost per enabled link $c \in \mathbb{R}^{+}\backslash\{0\}, \forall l \in \mathcal{E}$.

Definition 1. Consider a network $(\mathcal{N}, \mathcal{E})$. The set of enabled links in time step k is defined as **network topology** and it is denoted by $\Lambda(k) \subseteq \mathcal{E}$.

Notice that there are $2^{|\mathcal{E}|}$ possible topologies, ranging from decentralized topology Λ_0 (all links disabled) to centralized one $\Lambda_{2^{|\mathcal{E}|}-1}$ (all links enabled). Each topology is related to a set of *communication components*, i.e., disjoint coalitions of agents.

Example 1. Let the 5-link network depicted in Fig. 1. The different network topologies and their corresponding enabled links and communication components are detailed in Table 1.

2.2 Control goal

Local controllers must minimize

$$J(k) = \overbrace{\left(\sum_{j=0}^{\infty}\left[\mathbf{x}_{\mathcal{N}}^{\mathrm{T}}(k+j)\mathbf{Q}_{\mathcal{N}}\mathbf{x}_{\mathcal{N}}(k+j) + \mathbf{u}_{\mathcal{N}}^{\mathrm{T}}(k+j)\mathbf{R}_{\mathcal{N}}\mathbf{u}_{\mathcal{N}}(k+j)\right]\right)}^{J_{\mathrm{s}}(k)} + \overbrace{c|\Lambda(k)|}^{J_{\mathrm{c}}(k)}, \tag{3}$$

at each time step. As can be seen, $J(k)$ is composed of two terms $J_{\mathrm{s}}(k), J_{\mathrm{c}}(k) \in \mathbb{R}^{+}$:

- $J_{\mathrm{s}}(k)$ represents the cost-to-go of the closed-loop system. As usual, $\mathbf{Q}_{\mathcal{N}} \in \mathbb{R}^{n_{\mathrm{x}_{\mathcal{N}}}\times n_{\mathrm{x}_{\mathcal{N}}}}$ and $\mathbf{R}_{\mathcal{N}} \in \mathbb{R}^{n_{\mathrm{u}_{\mathcal{N}}}\times n_{\mathrm{u}_{\mathcal{N}}}}$ are positive semi-definite and definite weighting matrices.
- $J_{\mathrm{c}}(k)$ is related to the cost of communication.

Table 1. Network topologies and components related to the network in Fig. 1

Λ	Links I	II	III	IV	V	Communication Components	Λ	Links I	II	III	IV	V	Communication Components
Λ_0	✗	✗	✗	✗	✗	{1},{2},{3},{4},{5}	Λ_{16}	✓	✗	✗	✗	✗	{1,2},{3},{4},{5}
Λ_1	✗	✗	✗	✗	✓	{1},{2},{3,5},{4}	Λ_{17}	✓	✗	✗	✗	✓	{1,2},{3,5},{4}
Λ_2	✗	✗	✗	✓	✗	{1},{2},{3,4},{5}	Λ_{18}	✓	✗	✗	✓	✗	{1,2},{3,4},{5}
Λ_3	✗	✗	✗	✓	✓	{1},{2},{3,4,5}	Λ_{19}	✓	✗	✗	✓	✓	{1,2},{3,4,5}
Λ_4	✗	✗	✓	✗	✗	{1},{2,3},{4},{5}	Λ_{20}	✓	✗	✓	✗	✗	{1,2,3},{4},{5}
Λ_5	✗	✗	✓	✗	✓	{1},{2,3,5},{4}	Λ_{21}	✓	✗	✓	✗	✓	{1,2,3,5},{4}
Λ_6	✗	✗	✓	✓	✗	{1},{2,3,4},{5}	Λ_{22}	✓	✗	✓	✓	✗	{1,2,3,4},{5}
Λ_7	✗	✗	✓	✓	✓	{1},{2,3,4,5}	Λ_{23}	✓	✗	✓	✓	✓	$\mathcal{N}$
Λ_8	✗	✓	✗	✗	✗	{1,3},{2},{4},{5}	Λ_{24}	✓	✓	✗	✗	✗	{1,2,3},{4},{5}
Λ_9	✗	✓	✗	✗	✓	{1,3,5},{2},{4}	Λ_{25}	✓	✓	✗	✗	✓	{1,2,3,5},{4}
Λ_{10}	✗	✓	✗	✓	✗	{1,3,4},{2},{5}	Λ_{26}	✓	✓	✗	✓	✗	{1,2,3,4},{5}
Λ_{11}	✗	✓	✗	✓	✓	{1,3,4,5},{2}	Λ_{27}	✓	✓	✗	✓	✓	$\mathcal{N}$
Λ_{12}	✗	✓	✓	✗	✗	{1,2,3},{4},{5}	Λ_{28}	✓	✓	✓	✗	✗	{1,2,3},{4},{5}
Λ_{13}	✗	✓	✓	✗	✓	{1,2,3,5},{4}	Λ_{29}	✓	✓	✓	✗	✓	{1,2,3,5},{4}
Λ_{14}	✗	✓	✓	✓	✗	{1,2,3,4},{5}	Λ_{30}	✓	✓	✓	✓	✗	{1,2,3,4},{5}
Λ_{15}	✗	✓	✓	✓	✓	$\mathcal{N}$	Λ_{31}	✓	✓	✓	✓	✓	$\mathcal{N}$

Unfortunately, the minimization of (3) is a mixed-integer NP-complete problem (Bemporad and Morari, 1999), because it involves finding the optimal topology that leads to the optimal trajectories for states and inputs. To avoid this issue, the problem is simplified below so that a suboptimal solution can be obtained.

Assumption 1. If topology $\Lambda(k)$ is assumed to be fixed, the control law can be calculated as a feedback $\mathbf{K}_{\Lambda} \in \mathbb{R}^{n_{\mathrm{u}_{\mathcal{N}}}\times n_{\mathrm{x}_{\mathcal{N}}}}$, i.e.,

$$\mathbf{u}_{\mathcal{N}}(k) = \mathbf{K}_{\Lambda}\mathbf{x}_{\mathcal{N}}(k), \tag{4}$$

which is associated with a Lyapunov function

$$f(\mathbf{x}_{\mathcal{N}}(k)) = \mathbf{x}_{\mathcal{N}}^{\mathrm{T}}(k)\mathbf{P}_{\Lambda}\mathbf{x}_{\mathcal{N}}(k),$$

with $\mathbf{P}_{\Lambda} \in \mathbb{R}^{n_{\mathrm{x}_{\mathcal{N}}}\times n_{\mathrm{x}_{\mathcal{N}}}}$ being a positive definite matrix.

Assumption 2. The Lyapunov function $f(\mathbf{x}_{\mathcal{N}}(k))$ provides a bound on the cost-to-go of the closed-loop system. In particular, if the overall state and the topology are respectively given by $\mathbf{x}_{\mathcal{N}}(k)$ and Λ, we have

$$\mathbf{x}_{\mathcal{N}}^{\mathrm{T}}(k)\mathbf{P}_{\Lambda}\mathbf{x}_{\mathcal{N}}(k) \geq J_{\mathrm{s}}(k). \tag{5}$$

Note that Assumption 2 is equivalent to impose

$$\begin{aligned}&\overbrace{\mathbf{x}_{\mathcal{N}}^{\mathrm{T}}(k+1)\mathbf{P}_{\Lambda}\mathbf{x}_{\mathcal{N}}(k+1)}^{\geq J_{\mathrm{s}}(k+1)} + \overbrace{\mathbf{x}_{\mathcal{N}}^{\mathrm{T}}(k)\mathbf{Q}_{\mathcal{N}}\mathbf{x}_{\mathcal{N}}(k) + \mathbf{x}_{\mathcal{N}}^{\mathrm{T}}(k)\mathbf{K}_{\Lambda}^{\mathrm{T}}\mathbf{R}_{\mathcal{N}}\mathbf{K}_{\Lambda}\mathbf{x}_{\mathcal{N}}(k)}^{\text{stage cost}}\\ &\leq \overbrace{\mathbf{x}_{\mathcal{N}}^{\mathrm{T}}(k)\mathbf{P}_{\Lambda}\mathbf{x}_{\mathcal{N}}(k)}^{\geq J_{\mathrm{s}}(k)}.\end{aligned} \tag{6}$$

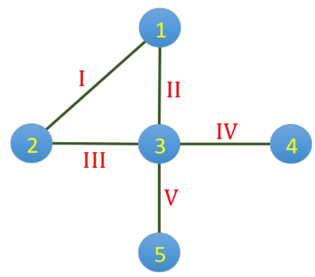

Fig. 1. An example of a 5-link network

In order to stress that a different feedback control law $\mathbf{K}_\Lambda$ needs to be calculated for each Λ, the subscript indicating the topology has been introduced in the notation. Likewise, the same holds for $\mathbf{P}_\Lambda$, i.e., a different Lyapunov function is computed offline for each topology. In this work, both matrices will be calculated using the following LMI, which is derived from (6) by applying recursively the Schur complement (Zhang, 2005), as done in (Maestre et al., 2014):

$$\begin{bmatrix} \mathbf{W}_\Lambda & \mathbf{W}_\Lambda\mathbf{A}_\mathcal{N}^\mathrm{T} + \mathbf{Y}_\Lambda^\mathrm{T}\mathbf{B}_\mathcal{N}^\mathrm{T} & \mathbf{W}_\Lambda\mathbf{Q}_\mathcal{N}^{1/2} & \mathbf{Y}_\Lambda^\mathrm{T}\mathbf{R}_\mathcal{N}^{1/2} \\ \mathbf{A}_\mathcal{N}\mathbf{W}_\Lambda + \mathbf{B}_\mathcal{N}\mathbf{Y}_\Lambda & \mathbf{W}_\Lambda & 0 & 0 \\ \mathbf{Q}_\mathcal{N}^{1/2}\mathbf{W}_\Lambda & 0 & \mathbf{I} & 0 \\ \mathbf{R}_\mathcal{N}^{1/2}\mathbf{Y}_\Lambda & 0 & 0 & \mathbf{I} \end{bmatrix} > 0, \tag{7a}$$

$$i \overset{\Lambda}{\nleftrightarrow} j \Longrightarrow \begin{cases} \mathbf{Y}_\Lambda^{ij} = \mathbf{Y}_\Lambda^{ji} = 0, \\ \mathbf{W}_\Lambda^{ij} = \mathbf{W}_\Lambda^{ji} = 0, \end{cases} \tag{7b}$$

where $i \overset{\Lambda}{\nleftrightarrow} j$ denotes that there are no communication paths between agents i and j when topology $\Lambda \subseteq \mathcal{E}$ is enabled, and with $\mathbf{W}_\Lambda = \mathbf{P}_\Lambda^{-1}$ and $\mathbf{Y}_\Lambda = \mathbf{K}_\Lambda\mathbf{P}_\Lambda^{-1}$. Note that constraints (7b) impose a sparsity pattern in the controller design to account for the communication constraints of the network topology.

Once matrices $\mathbf{P}_\Lambda$ are computed, we consider here the *link-game* introduced in (Maestre et al., 2014), which is defined by pair $(\mathcal{E}, \boldsymbol{r^v})$, with $\mathcal{E}$ as the set of players and

$$r^v(\Lambda, \mathbf{x}_\mathcal{N}(k)) = \mathbf{x}_\mathcal{N}^\mathrm{T}(k)\mathbf{P}_\Lambda\mathbf{x}_\mathcal{N}(k) + c|\Lambda|, \tag{8}$$

as its characteristic function, which assigns a value to each possible topology Λ. Since (5) holds, note that $r^v(\Lambda, \mathbf{x}_\mathcal{N}(k))$ corresponds to an upper bound of (3) [1].

3. THE SET OF SEMIVALUES

In (Maestre et al., 2014; Muros et al., 2017b,a), different game theoretical tools were used to find out which links are the most relevant/dispensable for the control scheme. We will consider in this paper the set of semivalues, which is characterized by the *null player*, *symmetry* and *additivity* properties (Dubey et al., 1981). In particular, given a link-game $(\mathcal{E}, \boldsymbol{r^v})$, vector of semivalues $\boldsymbol{\psi}(\mathcal{E}, \boldsymbol{r^v})$ assigns to each player $l \in \mathcal{E}$ a weighted average of its *marginal contribution* to any coalition it belongs to, by means of the following expression:

$$\psi_l(\mathcal{E}, \boldsymbol{r^v}) = \sum_{\Lambda \subseteq \mathcal{E}: l \notin \Lambda} \zeta(|\Lambda|)[r^v(\Lambda \cup \{l\}) - r^v(\Lambda)], \tag{9a}$$

$$\text{with} \quad \sum_{|\Lambda|=0}^{|\mathcal{E}|-1} \binom{|\mathcal{E}|-1}{|\Lambda|} \zeta(|\Lambda|) = 1. \tag{9b}$$

In the literature, it is possible to find several solution concepts that belong to the set of semivalues. Undoubtedly, the most well-known semivalues are the Shapley and Banzhaf values (Shapley, 1953; Banzhaf, 1965). They are respectively denoted by $\boldsymbol{\phi}(\mathcal{E}, \boldsymbol{r^v})$ and $\boldsymbol{\beta}(\mathcal{E}, \boldsymbol{r^v})$, verifying $\forall |\Lambda| \in [0, |\mathcal{E}| - 1]$

$$\zeta(|\Lambda|)|_{\boldsymbol{\phi}} = \frac{|\Lambda|!(|\mathcal{E}| - |\Lambda| - 1)!}{|\mathcal{E}|!}, \quad \zeta(|\Lambda|)|_{\boldsymbol{\beta}} = \frac{1}{2^{|\mathcal{E}|-1}}, \tag{10}$$

which, as can be checked, satisfy condition (9b).

3.1 Matrix notation

Taking into account (9), it is possible to derive a matrix notation for the set of semivalues, which is introduced below:

Definition 2. Consider matrix $\boldsymbol{\Psi} \in \mathbb{R}^{|\mathcal{E}| \times 2^{|\mathcal{E}|}}$, denoted as **semivalues standard matrix**, where the rows correspond to each link $l \in \mathcal{E}$ and the columns to the different topologies $\Lambda \subseteq \mathcal{E}$. The $s_{l\Lambda}$ element of $\boldsymbol{\Psi}$ is given as

$$s_{l\Lambda} = \begin{cases} \zeta(|\Lambda| - 1), & l \in \Lambda, \\ -\zeta(|\Lambda|), & l \notin \Lambda, \end{cases} \tag{11}$$

with terms $\zeta(|\Lambda|)$ in $s_{l\Lambda}$ satisfying (9b).

Matrix $\boldsymbol{\Psi}$, with its elements $s_{l\Lambda}$ defined by (11), is unique for any link-game with $|\mathcal{E}|$ links, and verifies

$$\boldsymbol{\psi}(\mathcal{E}, \boldsymbol{r^v}) = \begin{bmatrix} \psi_\mathrm{I} \\ \psi_\mathrm{II} \\ \vdots \\ \psi_{|\mathcal{E}|} \end{bmatrix} = \boldsymbol{\Psi} \begin{bmatrix} r^v(\Lambda_0, \mathbf{x}_\mathcal{N}) \\ r^v(\Lambda_1, \mathbf{x}_\mathcal{N}) \\ \vdots \\ r^v(\Lambda_{2^{|\mathcal{E}|}-1}, \mathbf{x}_\mathcal{N}) \end{bmatrix} = \boldsymbol{\Psi}\boldsymbol{r^v}. \tag{12}$$

Next, a property needed to obtain a closed expression for the semivalue of a link is presented. This expression will be used to apply constraints on the semivalues afterwards.

Property 1. Let $(\mathcal{N}, \mathcal{E})$, $(\mathcal{E}, \boldsymbol{r^v})$ be a network and a game, respectively. Let also $s_{l\Lambda}$ be the elements of matrix $\boldsymbol{\Psi}$. The following expressions are satisfied, $\forall l, \Lambda$:

$$\sum_{\Lambda \subseteq \mathcal{E}} s_{l\Lambda} = 0, \qquad \sum_{\Lambda \subseteq \mathcal{E}} s_{l\Lambda}|\Lambda| = 1. \tag{13}$$

Note that the characteristic function of a *game* has to be necessarily zero for empty set Λ_0. Hence, the next redefinition may be required:

$$r^{v\prime}(\Lambda, \mathbf{x}_\mathcal{N}) = r^v(\Lambda, \mathbf{x}_\mathcal{N}) - r^v(\Lambda_0, \mathbf{x}_\mathcal{N}), \quad \forall \Lambda \subseteq \mathcal{E}. \tag{14}$$

Then, combining (8), (11) and (14), the semivalue of a link $l \in \mathcal{E}$ for the redefined game can be computed by

$$\begin{aligned} \psi_l(\mathcal{E}, \boldsymbol{r^{v\prime}}) &= \sum_{\Lambda \subseteq \mathcal{E}} s_{l\Lambda} \left[\mathbf{x}_\mathcal{N}^\mathrm{T}\mathbf{P}_\Lambda\mathbf{x}_\mathcal{N}\right] + c \sum_{\Lambda \subseteq \mathcal{E}} s_{l\Lambda}|\Lambda| \\ &- \overbrace{\left[\mathbf{x}_\mathcal{N}^\mathrm{T}\mathbf{P}_{\Lambda_0}\mathbf{x}_\mathcal{N}\right]}^{\text{constant}} \sum_{\Lambda \subseteq \mathcal{E}} s_{l\Lambda} + 0 \cancelto{0}{\sum_{\Lambda \subseteq \mathcal{E}} s_{l\Lambda}|\Lambda|}, \end{aligned} \tag{15}$$

Finally, taking into account Property 1, it is possible to derive the following closed expression:

$$\psi_l(\mathcal{E}, \boldsymbol{r^{v\prime}}) = \psi_l(\mathcal{E}, \boldsymbol{r^v}) = c + \sum_{\Lambda \subseteq \mathcal{E}} s_{l\Lambda} \left[\mathbf{x}_\mathcal{N}^\mathrm{T}\mathbf{P}_\Lambda\mathbf{x}_\mathcal{N}\right], \tag{16}$$

which coincides for original (8) and redefined (14) games.

[1] From now on, the dependence on time step k will be omitted for the sake of clarity.

Example 2. The semivalues standard matrix can be easily obtained by using (11), e.g., the expression for *any* 3-link network is given by

$$\mathbf{\Psi}_3 = \begin{bmatrix} -\zeta(0) & \zeta(0) & -\zeta(1) & -\zeta(1) & \zeta(1) & \zeta(1) & -\zeta(2) & \zeta(2) \\ -\zeta(0) & -\zeta(1) & \zeta(0) & -\zeta(1) & \zeta(1) & -\zeta(2) & \zeta(1) & \zeta(2) \\ -\zeta(0) & -\zeta(1) & -\zeta(1) & \zeta(0) & -\zeta(2) & \zeta(1) & \zeta(1) & \zeta(2) \end{bmatrix}, \tag{17}$$

with $\zeta(|\Lambda|)$ verifying (9b). Notice that Property 1 is trivially satisfied.

4. A CONTROLLER WITH CONSTRAINTS ON THE SET OF SEMIVALUES

Semivalues can be interpreted as *a posteriori* cost of the links, which is clearly seen in (16). Indeed, its computation involves both their *a priori* cost, which in this work is simply given by c, and their average impact on the control performance in the different topologies. For this reason, the introduction of constraints on semivalues can be relevant from a design viewpoint to promote/discourage the use of communication resources from a holistic perspective.

Two different sets of constraints were introduced in (Muros et al., 2017b,a) and will also be considered here, namely, *absolute* constraints, if the value of a player l is set lower/higher than a constant threshold $\mathscr{V}_l/\mathscr{W}_l$, i.e.,

$$\psi_l(\mathcal{E}, \boldsymbol{r^v}) < \mathscr{V}_l, \qquad \psi_l(\mathcal{E}, \boldsymbol{r^v}) > \mathscr{W}_l, \tag{18}$$

and *relative* constraints, if the value of a player l_p is set higher than (or equal to) that of another player l_q

$$\psi_{l_\mathrm{p}}(\mathcal{E}, \boldsymbol{r^v}) \geq \psi_{l_\mathrm{q}}(\mathcal{E}, \boldsymbol{r^v}). \tag{19}$$

These constraints are related to the following LMI conditions (Muros et al., 2017b,a):

$$\begin{aligned} &\mathbf{S_a} > 0, \quad \text{with } \mathbf{S_a} = \begin{bmatrix} \mathscr{V}_l - c & 0 \\ 0 & -\sum\limits_{\Lambda\subseteq\mathcal{E}} s_{l\Lambda}\mathbf{P}_\Lambda \end{bmatrix}, \\ &\mathbf{S_b} > 0, \quad \text{with } \mathbf{S_b} = \begin{bmatrix} c - \mathscr{W}_l & 0 \\ 0 & \sum\limits_{\Lambda\subseteq\mathcal{E}} s_{l\Lambda}\mathbf{P}_\Lambda \end{bmatrix}, \end{aligned} \tag{20a}$$

$$\mathbf{S_c} \geq 0, \quad \text{with } \mathbf{S_c} = \sum_{\Lambda\subseteq\mathcal{E}} \left(s_{l_\mathrm{p}\Lambda} - s_{l_\mathrm{q}\Lambda}\right)\mathbf{P}_\Lambda, \tag{20b}$$

where the LMI setting requires the first principal minors of (20a) to be nonnegative, i.e.,

$$\mathscr{V}_l - c \geq 0, \qquad c - \mathscr{W}_l \geq 0, \tag{21}$$

which in the limit case, i.e., with principal minors equal to zero, reduce the LMI conditions to

$$\mathbf{S_a^0} > 0, \text{ with } \mathbf{S_a^0} = -\sum_{\Lambda\subseteq\mathcal{E}} s_{l\Lambda}\mathbf{P}_\Lambda, \quad \mathbf{S_b^0} > 0, \text{ with } \mathbf{S_b^0} = \sum_{\Lambda\subseteq\mathcal{E}} s_{l\Lambda}\mathbf{P}_\Lambda. \tag{22}$$

Definition 3. The **semivalues constraint set**, denoted by $\mathcal{S}$, is the set of different LMI conditions (20) imposed in the design.

4.1 Design algorithm and control scheme

As stated previously, feedbacks and Lyapunov functions need to be computed for each topology Λ. This process is performed offline and must account for the desired constraints on the semivalues included in set $\mathcal{S}$. To this end, the iterative design method proposed by Muros et al. (2017b) can be used. Basically, the optimization will be alternated with respect to $\mathbf{K}_\Lambda$ and $\mathbf{P}_\Lambda$, keeping the other fixed, following a *D-K procedure* (Skogestad and Postlethwaite, 2001). This method will be repeated until a prefixed number of iterations t is achieved or until convergence has been attained. Once the design stage is over, $\mathbf{K}_\Lambda$ and $\mathbf{P}_\Lambda$ are available for each topology. Then, the following hierarchical-coalitional control architecture can be implemented:

Control Scheme

Let $k_\mathrm{s} \in \mathbb{N}^+$ be a sampling time of the upper control layer (measured as a multiple of time steps k of the lower control layer). Also, let $\mathbf{x}_\mathcal{N}(0) = \mathbf{x}^0_\mathcal{N}$ be the initial state of the overall system. The control algorithm is executed as follows:

- Upper control layer: every k_s time instants, cost function (8) is minimized to obtain the most suitable topology, which will be enabled the current and the next $k_\mathrm{s} - 1$ steps. To this end, it is communicated to the lower layer.
- Lower control layer:
  - (I) At time step k, each agent i measures and broadcasts its state. Note that the set of active links given by topology Λ limits the agents that can receive the information to those connected either directly or indirectly with agent i.
  - (II) Each agent i uses the state information received to update its control action. Globally, this fact implies that feedback $\mathbf{u}_\mathcal{N} = \mathbf{K}_\Lambda\mathbf{x}_\mathcal{N}$ is applied.

Stability of this method is proven in (Maestre et al., 2014).

5. SIMULATION RESULTS

In this section, we consider an academic network composed of five agents $\mathcal{N} = \{1, 2, 3, 4, 5\}$ and five links $\mathcal{E} = \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}\}$, corresponding to the scheme depicted in Fig. 1. The different network topologies and components are specified in Table 1. The dynamics are described by the following overall matrices:

$$\mathbf{A}_\mathcal{N} = \begin{bmatrix} 1 & 0.0314 & 0 & 0 & 0 \\ 0.0271 & 1 & 0.1055 & 0 & 0 \\ 0 & 0.0599 & 1 & 0.3948 & 0 \\ 0 & 0 & 0.4076 & 1 & 0.3298 \\ 0 & 0 & 0 & 0.1527 & 1 \end{bmatrix}, \tag{23}$$

$$\mathbf{B}_\mathcal{N} = \begin{bmatrix} 1 & 0.15 & 0.15 & 0.15 & 0.15 \\ 0.15 & 1 & 0.15 & 0.15 & 0.15 \\ 0.15 & 0.15 & 1 & 0.15 & 0.15 \\ 0.15 & 0.15 & 0.15 & 1 & 0.15 \\ 0.15 & 0.15 & 0.15 & 0.15 & 1 \end{bmatrix}, \tag{24}$$

where $\mathbf{x}_i, \mathbf{u}_i \in \mathbb{R}$ are respectively the state and input of each subsystem $i \in \mathcal{N}$. Stage cost J_s is defined by matrices $\mathbf{Q}_{\mathcal{N}} = \mathbf{I} \in \mathbb{R}^{5\times5}$ and $\mathbf{R}_{\mathcal{N}} = 10 \cdot \mathbf{I} \in \mathbb{R}^{5\times5}$. We also assume a cost per enabled link $c = 1.5$. The Shapley and Banzhaf values, with their terms $\zeta(|\Lambda|)$ defined by (10), will be analyzed here. In this regard, the following semivalues constraints are considered:

$$\psi_{\mathrm{I}}(\mathcal{E}, \boldsymbol{r^v}) < 2, \quad \psi_{\mathrm{V}}(\mathcal{E}, \boldsymbol{r^v}) > 1.2, \tag{25a}$$

$$\begin{aligned} \psi_{\mathrm{I}}(\mathcal{E}, \boldsymbol{r^v}) &\geq \psi_{\mathrm{II}}(\mathcal{E}, \boldsymbol{r^v}), \\ \psi_{\mathrm{III}}(\mathcal{E}, \boldsymbol{r^v}) \geq \psi_{\mathrm{IV}}&(\mathcal{E}, \boldsymbol{r^v}) \geq \psi_{\mathrm{V}}(\mathcal{E}, \boldsymbol{r^v}), \end{aligned} \tag{25b}$$

where all constraints defined in (25) can be rewritten as in (20), with that of (20a) satisfying (21).

The design algorithm proposed in Section 4.1 has been implemented using the Matlab® tools `LMI Control Toolbox` (Gahinet et al., 1995) and `NetV0`, a class for coalitional control (Maestre et al., 2015), in a 2.2 GHz quad-core Intel® Core™ i7/16 GB RAM computer. As stopping criterion we have considered whichever comes first from: a maximum number of iterations $t_{\max} = 20$, $\eta(t) < \eta_{\max} = 1.5$, and $\eta(t+1) - \eta(t) < 0.001$, with $\eta(t)$ being an efficiency index given by (Muros et al., 2017b)

$$\eta(t) = \frac{\sum_{\Lambda \subseteq \mathcal{E}} \mathrm{tr}(\mathbf{P}_{\Lambda}^{(t)})}{2^{|\mathcal{E}|} \cdot \mathrm{tr}(\mathbf{P}_{\mathrm{LQR}})}, \tag{26}$$

where $\mathbf{P}_{\mathrm{LQR}}$ is the matrix corresponding to the LQR solution for the centralized case, i.e., that with full communication. The evolution of $\eta(t)$ with the number of iterations t is shown in Fig. 2 for both semivalues analyzed. As a result of the considered algorithm, matrices $\mathbf{K}_{\Lambda}, \mathbf{P}_{\Lambda} \in \mathbb{R}^{5\times5}$, $\forall \Lambda \subseteq \mathcal{E}$, have been obtained. Once the design problem is solved, the control scheme is executed along a simulation time $T_{\mathrm{sim}} = 50$, taking $k_s = 3$ and considering the initial state

$$\mathbf{x}_{\mathcal{N}}^0 = [\,1.4455 \quad 5.8405 \quad -1.6053 \quad 0.6507 \quad 8.5141\,]. \tag{27}$$

The topology evolution of the scheme for the unconstrained scenario, and also for considering constraints (25) on the Shapley and Banzhaf values, is represented in Fig. 3. Likewise, the specific change on the semivalues before and after considering constraints (25) is detailed in Fig. 4.

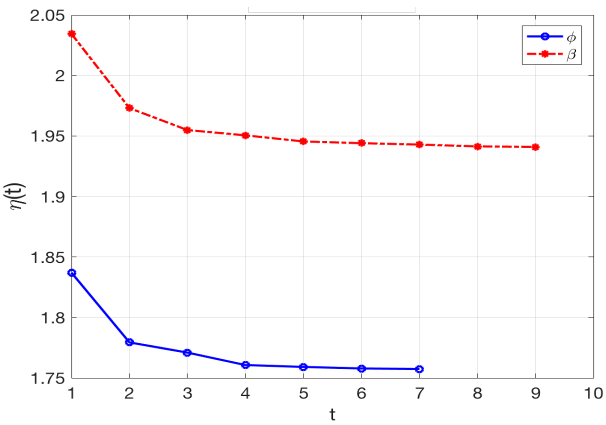

Fig. 2. Efficiency index $\eta(t)$ for the semivalues analyzed. The convergence is achieved with $t = 7$ for the Shapley value and with $t = 9$ for the Banzhaf value.

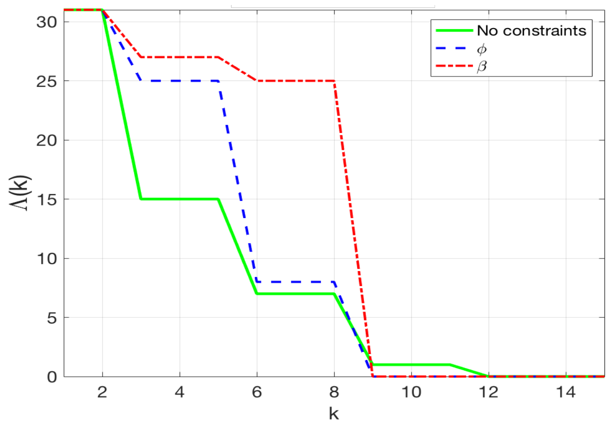

Fig. 3. Network topology evolution

It is interesting to observe that each semivalue evolves in its own way to satisfy the specifications, which in turn may lead to a different network topology evolution depending on the semivalue. In any case, the chosen topologies deactivate links III and IV, and also enable link II, respectively the most expensive and the cheapest ones according to constraints (25). Note also that all semivalues tend to $c = 1.5$ independently of considering or not constraints, as expected from (16), when the origin is reached in steady state. Finally, cumulated cost $J_{\mathrm{cum}} = \sum_{k=0}^{T_{\mathrm{sim}}} [\mathbf{x}_{\mathcal{N}}^{\mathrm{T}}(k)\mathbf{Q}_{\mathcal{N}}\mathbf{x}_{\mathcal{N}}(k) + \mathbf{u}_{\mathcal{N}}^{\mathrm{T}}(k)\mathbf{R}_{\mathcal{N}}\mathbf{u}_{\mathcal{N}}(k) + c|\Lambda(k)|]$ computed for each semivalue in the proposed coalitional scheme is given by $J_{\mathrm{cum}}^{\phi} = 423.3$ and $J_{\mathrm{cum}}^{\beta} = 342$.

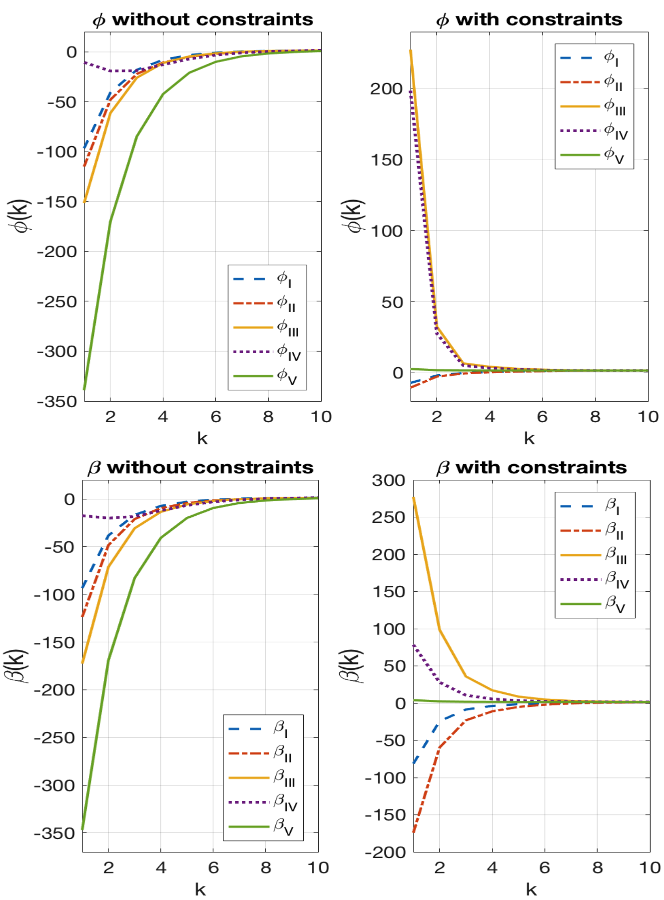

Fig. 4. Semivalues evolution with and without considering constraints (25)

6. CONCLUSIONS

This paper expands the framework of (Muros et al., 2017b,a) for the Shapley and Banzhaf values to deal with the set of semivalues. In this sense, a property that leads to a closed expression of the semivalues as a function of the game has been presented. Likewise, semivalues constraints are considered and the corresponding linear matrix inequalities (LMIs) have been derived and included in a design algorithm in order to foster/penalize the use of the communication links inside the network. On top of that, a coalitional control scheme is proposed and executed in a simulation example to show that the semivalues analyzed satisfy their specifications and the links are enabled/disabled accordingly in the topology trajectories.

Future work should include a deeper analysis of the connections between overall cost and semivalue formulation. Also, other semivalues in the literature different from the Shapley and Banzhaf values could be explored. Finally, the application of the presented results to real large-scale systems, exploring ways to mitigate the combinatorial explosion, will also be subject of further research.

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