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Stability analysis by averaging: a time-delay approach ⋆

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Abstract: We study stability of linear systems with fast time-varying coefficients. The classical averaging method guarantees the stability of such systems for small enough values of parameter provided the corresponding averaged system is stable. However, it is difficult to find an upper bound on the small parameter by using classical tools for asymptotic analysis. In this paper we introduce an efficient constructive method for finding an upper bound on the value of the small parameter that guarantees a desired exponential decay rate. We transform the system to a model with time-delays of the length of the small parameter. The resulting time-delay system is a perturbation of the averaged LTI system which is assumed to be exponentially stable. The stability of the time-delay system guarantees the stability of the original one. We construct an appropriate Lyapunov functional for finding sufficient stability conditions in the form of linear matrix inequalities (LMIs). The upper bound on the small parameter that preserves the exponential stability is found from LMIs. Two numerical examples (stabilization by vibrational control and by time-dependent switching) illustrate the efficiency of the method.

Keywords: Averaging, linear systems, time-delay systems, Lyapunov-Krasovskii method, LMIs

1. INTRODUCTION

Asymptotic methods for analysis and control of perturbed systems depending on small parameters have led to important qualitative results (Tikhonov, 1952; Kokotovic and Khalil, 1986; Khalil, 2002; Vasilieva and Butuzov, 1973; Bogoliubov and Mitropolsky, 1961; Moreau and Aeyels, 2000; Teel et al., 2003; Cheng et al., 2018). However, by using these methods it is difficult to find an efficient bound on the small parameter that preserves the stability. For singularly perturbed systems, such a bound was presented e.g. in Fridman (2002) by using direct Lyapunov method.

For the sampled-data systems with fast sampling, the time-delay approach was initiated in the framework of asymptotic methods (Mikheev et al., 1988) and averaging (Fridman, 1992). Later the time-delay approach to sampled-data control via direct Lyapunov-Krasovskii method Fridman et al. (2004) led to efficient tools for robust sampled-data and networked control (see e.g. Fridman (2014); Hetel and Fridman (2013); Liu et al. (2019)).

In this paper we consider linear systems with fast varying coefficients. Our objective is to propose a constructive time-delay approach with a corresponding Lyapunov-Krasovskii method to the averaging method for these systems. Differently from the classical results (see Chapter 10 of Khalil (2002)), where the system coefficients are supposed to be at least continuous in time, we assume them to be piecewise-continuous. This allows to apply our results e.g. to fast switching systems. By taking average of the both sides of the system, we present the resulting system as a perturbation of the averaged system, and model it as a system with time-delays of the length of the small parameter. If the transformed time-delayed system is stable, then the original one is also stable. We assume that the averaged LTI system is exponentially stable.

We suggest a direct Lyapunov-Krasovskii approach, and formulate sufficient exponential stability conditions in the form of LMIs. The upper bound on the small parameter that guarantees a desired decay rate for the original system can be found from LMIs. Two numerical examples (stabilization by vibrational control and by time-dependent switching) illustrate the efficiency of the method.

1.1 Necessary notations, definitions and statements

Throughout the paper $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with vector norm $|\cdot|$ and the induced matrix norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices. The superscript T stands for matrix transposition, and the notation $P > 0$, for $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix are denoted by $*$.

We will employ extended Jensen's inequalities (Solomon and Fridman, 2013):

Lemma 1.1. Denote

$$\mathcal{G} = \int_a^b f(s)x(s)ds, \quad \mathcal{Y} = \int_a^b \int_{t-\theta}^t f(\theta)x(s)dsd\theta,$$

where $a \le b$, $f : [a, b] \to \mathbb{R}$, $x(s) \in \mathbb{R}^n$ and the integration concerned is well defined. Then for any $n \times n$ matrix $R > 0$ the following inequalities hold:

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$$\mathcal{G}^T R \mathcal{G} \le \int_a^b |f(\theta)| d\theta \int_a^b |f(s)| x^T(s) R x(s) ds, \tag{1.1}$$

$$\mathcal{Y}^T R \mathcal{Y} \le \int_a^b |f(\theta)| \theta d\theta \int_a^b \int_{t-\theta}^t |f(\theta)| x^T(s) R x(s) ds d\theta. \tag{1.2}$$

2. A TIME-DELAY APPROACH TO STABILITY BY AVERAGING

Consider the fast varying system:

$$\dot{x}(t) = A(\frac{t}{\varepsilon}) x(t), \quad t \ge 0, \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$, $A : [0, \infty) \to \mathbb{R}^{n\times n}$ is piecewise-continuous and $\varepsilon > 0$ is a small parameter. Similar to the case of general averaging in Sect. 10.6 of Khalil (2002), assume the following:

A1 There exist $\varepsilon_1 > 0$ and $t_1 \ge \varepsilon_1$ such that

$$\begin{aligned} &\frac{1}{\varepsilon}\int_{t-\varepsilon}^t A(\frac{s}{\varepsilon}) ds = A_{av} + \Delta A(t,\varepsilon), \quad \forall t \ge t_1, \quad \varepsilon \in (0, \varepsilon_1], \\ &|\Delta A(t,\varepsilon)| \le \sigma(\varepsilon), \end{aligned} \tag{2.2}$$

with Hurwitz constant matrix A_{av}. Here σ is a strictly increasing (of class $\mathcal{K}$) scalar function with $\sigma(0) = 0$.

System (2.1) has almost periodic coefficients if it satifies **A1**. Changing the variable s in (2.2) to $\theta = \frac{t-s}{\varepsilon}$, we can rewrite the first equation in (2.2) as

$$\int_0^1 A(\frac{t}{\varepsilon} - \theta) d\theta = A_{av} + \Delta A(t,\varepsilon), \quad \forall t \ge t_1, \quad \varepsilon \in (0, \varepsilon_1]$$

or, in terms of the fast time $\tau = \frac{t}{\varepsilon}$,

$$\int_0^1 A(\tau - \theta) d\theta = A_{av} + \Delta A(\varepsilon\tau, \varepsilon), \quad \forall \tau \ge \frac{t_1}{\varepsilon_1}. \tag{2.3}$$

Remark 2.1. If $A(\tau)$ is 1-periodic, then in (2.3) we have $\Delta A = 0$. If $A(\tau)$ is T-periodic with $T > 0$, scaling the time $t = T\bar{t}$ and denoting $\bar{x}(\bar{t}) = x(T\bar{t}) = x(t)$, we can present (2.1) as

$$\frac{d}{d\bar{t}} \bar{x}(\bar{t}) = T \cdot A(\frac{T\bar{t}}{\varepsilon}) \bar{x}(\bar{t}) \tag{2.4}$$

with 1-periodic $A(T\bar{\tau})$, where $\bar{\tau} = \frac{\bar{t}}{\varepsilon}$. In general we can consider almost periodic A (in the sense of (2.3) with non-zero ΔA). For example, let A in (2.1) have the form

$$A(\tau) = A_1 \cos(\tau) + A_2 \sin^2(3\tau) + A_3 e^{-\tau}, \quad \tau = \frac{t}{\varepsilon}$$

with constant $n \times n$-matrices A_1, A_2, A_3 and with A_2 Hurwitz. Then, scaling the time $t = 2\pi\bar{t}$ and denoting $\bar{x}(\bar{t}) = x(t)$, we arrive at $\dot{\bar{x}}(\bar{t}) = 2\pi A(\frac{2\pi\bar{t}}{\varepsilon})\bar{x}(\bar{t})$ with

$$\int_0^1 A(2\pi(\tau - \theta)) d\theta = 0.5 A_2 + \Delta A,$$

where

$$\Delta A = A_3 \int_0^1 e^{-2\pi(\tau-\theta)} d\theta \underset{\tau\to\infty}{\longrightarrow} 0.$$

Additionally we assume the following:

A2 All entries $a_{kj}(\tau)$ of $A(\tau)$ are uniformly bounded for $\tau \ge 0$ with the values from some finite intervals $a_{kj}(\tau) \in [a_{kj}^m, a_{kj}^M]$ for $\tau \ge \frac{t_1}{\varepsilon_1}$.

Under **A2**, A can be presented as a convex combination of the constant matrices A_i with the entries a_{kj}^m or a_{kj}^M:

$$\begin{aligned} &A(\tau) = \sum_{i=1}^N f_i(\tau) A_i \quad \forall \tau \ge \frac{t_1}{\varepsilon_1} \\ &f_i \ge 0, \quad \sum_{i=1}^N f_i = 1, \quad 1 \le N \le 2^{n^2}. \end{aligned} \tag{2.5}$$

Note that $f_i \not\equiv 0$. For a constant a_{kj}, we have $a_{kj}^m = a_{kj}^M$. From **A1** we have

$$\sum_{i=1}^N A_i \int_0^1 f_i(\tau - \theta) d\theta = A_{av} + \Delta A, \quad \forall \tau \ge \frac{t_1}{\varepsilon_1}.$$

We will further integrate (2.1) on $[t - \varepsilon, t]$ for $t \ge t_1$. Note that similar to Fridman and Shaikhet (2016), we can present

$$\frac{1}{\varepsilon}\int_{t-\varepsilon}^t \dot{x}(s) ds = \frac{x(t) - x(t-\varepsilon)}{\varepsilon} = \frac{d}{dt}[x(t) - G], \tag{2.6}$$

where

$$G \triangleq \frac{1}{\varepsilon}\int_{t-\varepsilon}^t (s - t + \varepsilon)\dot{x}(s) ds. \tag{2.7}$$

Then, integrating (2.1) and taking into account (2.6) we arrive at

$$\begin{aligned} \frac{d}{dt}[x(t) - G] = &\frac{1}{\varepsilon}\int_{t-\varepsilon}^t A(\frac{s}{\varepsilon}) ds \cdot x(t) \\ &+ \frac{1}{\varepsilon}\int_{t-\varepsilon}^t A(\frac{s}{\varepsilon})[x(s) - x(t)] ds, \quad t \ge t_1. \end{aligned}$$

For shortness we will omit arguments of ΔA. By changing variable $\varepsilon\theta = t - s$ in the last integral, we have

$$\begin{aligned} &\frac{1}{\varepsilon}\int_{t-\varepsilon}^t A(\frac{s}{\varepsilon})[x(s) - x(t)] ds \\ &= \int_0^1 A(\frac{t}{\varepsilon} - \theta)[x(t - \varepsilon\theta) - x(t)] d\theta \\ &= -\int_0^1 A(\frac{t}{\varepsilon} - \theta) \int_{t-\varepsilon\theta}^t \dot{x}(s) ds d\theta. \end{aligned}$$

Finally, denoting

$$z(t) = x(t) - G \tag{2.8}$$

and employing (2.2), we transform (2.1) to a time-delay system for $\varepsilon \in (0, \varepsilon^*]$ and $t \ge t_1$

$$\dot{z}(t) = (A_{av} + \Delta A)x(t) - \int_0^1 A(\frac{t}{\varepsilon} - \theta) \int_{t-\varepsilon\theta}^t \dot{x}(s) ds d\theta. \tag{2.9}$$

System (2.9) is a kind of a neutral type system that depends on the past values of $\dot{x}$. However, this is not a neutral system in Hale's form (Hale and Lunel, 1993) because G depends on $\dot{x}$ and not on x.

Summarizing, if $x(t)$ is a solution to (2.1), then it satisfies the time-delay system (2.9). Therefore, the stability of the time-delay system guarantees the stability of the original system. We will derive the stability conditions for the time-delay system via direct Lyapunov-Krasovskii method.

Given $\varepsilon^* \in (0, \varepsilon_1]$, denote by $f_i^* > 0$ $(i = 1, ..., N)$ the following bound:

$$\int_0^1 \varepsilon |f_i(\frac{t}{\varepsilon} - \theta)| \theta d\theta \le f_i^*, \quad t \ge t_1, \quad \varepsilon \in (0, \varepsilon^*]. \tag{2.10}$$

Note that since $f_i \in [0,1]$ and $\varepsilon \in (0, \varepsilon^*]$, we can always choose $f_i^* \leq \frac{\varepsilon^*}{2}$.

Theorem 2.1. Let **A1** and **A2** hold. Given matrices A_{av}, A_i $(i = 1, \dots, N)$ and constants $\alpha > 0$ and $\varepsilon^* \in (0, \varepsilon_1]$, let there exist $n \times n$ matrices $P > 0$, $R > 0$, $H_i > 0$ $(i = 1, \cdots, N)$ and a scalar $\lambda > 0$ that satisfy the following LMIs:

$$\left[\begin{array}{c|c} \Phi & \begin{array}{c} \sqrt{\varepsilon^*} A_i^T (R + \sum_{j=1}^N H_j) \\ 0_{(N+2)n,n} \end{array} \\ \hline * \; * & -R - \sum_{j=1}^N H_j \end{array}\right] < 0, \quad i = 1, \dots, N. \quad (2.11)$$

Here

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & P \\ * & \Phi_{22} & \Phi_{23} & -P \\ * & * & \Phi_{33} & 0_{Nn,n} \\ * & * & * & -\lambda I_n \end{bmatrix} \quad (2.12)$$

with

$$\begin{aligned} &\Phi_{11} = P A_{av} + A_{av}^T P + 2\alpha P + \lambda \sigma^2 I_n, \\ &\Phi_{12} = -A_{av}^T P - 2\alpha P, \\ &\Phi_{22} = -\frac{4}{\varepsilon^*} e^{-2\alpha\varepsilon^*} R + 2\alpha P, \\ &\Phi_{13} = -\Phi_{23} = -P[A_1, \cdots, A_N], \\ &\Phi_{33} = -2e^{-2\alpha\varepsilon^*} diag\{\frac{1}{f_1^*} H_1, \cdots, \frac{1}{f_N^*} H_N\}, \\ &\Phi_{14} = -\Phi_{24} = P, \qquad \Phi_{44} = -\lambda I_n. \end{aligned}$$

Then system (2.1) is exponentially stable with a decay rate α for all $\varepsilon \in (0, \varepsilon^*]$, meaning that there exists $M_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$ the solutions of (2.1) initialized by $x(0) \in \mathbb{R}^n$ satisfy the following inequality:

$$|x(t)|^2 \leq M_0 e^{-2\alpha t} |x(0)|^2, \quad \forall t \geq 0. \quad (2.13)$$

Moreover, if the LMIs (2.11) hold with $\alpha = 0$, then system (2.1) is exponentially stable with a small enough decay rate $\alpha = \alpha_0 > 0$ for all $\varepsilon \in (0, \varepsilon^*]$.

Proof : Choose

$$V_P = z^T(t) P z(t), \; P > 0. \quad (2.14)$$

Then

$$\begin{aligned} \frac{d}{dt} V_P = 2[x(t) - G]^T P \Big[&(A_{av} + \Delta A) x(t) \\ &- \sum_{i=1}^N A_i \int_0^1 f_i\left(\frac{t}{\varepsilon} - \theta\right) \int_{t-\varepsilon\theta}^t \dot{x}(s) ds d\theta \Big]. \end{aligned} \quad (2.15)$$

To compensate G-term we will use as in Fridman and Shaikhet (2016)

$$V_G = \frac{1}{\varepsilon} \int_{t-\varepsilon}^t e^{-2\alpha(t-s)} (s - t + \varepsilon)^2 \dot{x}^T(s) R \dot{x}(s) ds, \; R > 0. \quad (2.16)$$

We have

$$\begin{aligned} &\frac{d}{dt} V_G + 2\alpha V_G = \varepsilon \dot{x}^T(t) R \dot{x}(t) \\ &-\frac{2}{\varepsilon} \int_{t-\varepsilon}^t e^{-2\alpha(t-s)} (s - t + \varepsilon) \dot{x}^T(s) R \dot{x}(s) ds. \end{aligned} \quad (2.17)$$

By Jensen's inequality (1.1)

$$2G^T R G \leq \int_{t-\varepsilon}^t (s - t + \varepsilon) \dot{x}^T(s) R \dot{x}(s) ds.$$

Then

$$\frac{d}{dt} V_G + 2\alpha V_G \leq \varepsilon \dot{x}^T(t) R \dot{x}(t) - \frac{4}{\varepsilon} e^{-2\alpha\varepsilon} G^T R G. \quad (2.18)$$

To compensate the Y_i-terms (distributed delay)

$$Y_i = \int_0^1 f_i\left(\frac{t}{\varepsilon} - \theta\right) \int_{t-\varepsilon\theta}^t \dot{x}(s) ds d\theta \quad (2.19)$$

in (2.15), we employ as in Solomon and Fridman (2013)

$$\begin{aligned} &V_H = \sum_{i=1}^N V_{H_i}, \\ &V_{H_i} = 2 \int_0^1 \int_{t-\varepsilon\theta}^t e^{-2\alpha(t-s)} (s - t + \varepsilon\theta) \dot{x}^T(s) H_i \dot{x}(s) ds d\theta \end{aligned} \quad (2.20)$$

with $H_i > 0$. Differentiating V_{H_i}, we have

$$\begin{aligned} &\frac{d}{dt} V_{H_i} + 2\alpha V_{H_i} = \varepsilon \dot{x}^T(t) H_i \dot{x}(t) \\ &\quad -2 \int_0^1 \int_{t-\varepsilon\theta}^t e^{-2\alpha(t-s)} \dot{x}^T(s) H_i \dot{x}(s) ds d\theta \\ &\leq \varepsilon \dot{x}^T(t) H_i \dot{x}(t) - 2e^{-2\alpha\varepsilon} \int_0^1 \int_{t-\varepsilon\theta}^t \dot{x}^T(s) H_i \dot{x}(s) ds d\theta. \end{aligned}$$

Applying further Jensen's inequality (1.2)

$$\begin{aligned} Y_i^T H_i Y_i &\leq \int_0^1 \varepsilon |f_i(\frac{t}{\varepsilon} - \theta)| \theta d\theta \\ &\quad \times \int_0^1 |f_i(\frac{t}{\varepsilon} - \theta)| \int_{t-\varepsilon\theta}^t \dot{x}^T(s) H_i \dot{x}(s) ds d\theta \\ &\leq f_i^* \int_0^1 \int_{t-\varepsilon\theta}^t \dot{x}^T(s) H_i \dot{x}(s) ds d\theta, \end{aligned}$$

we arrive at

$$\frac{d}{dt} V_{H_i} + 2\alpha V_{H_i} \leq \varepsilon \dot{x}^T(t) H_i \dot{x}(t) - \frac{2}{f_i^*} e^{-2\alpha\varepsilon} Y_i^T H_i Y_i. \quad (2.21)$$

Define a Lyapunov functional as

$$V = V(x(t), \dot{x}_t, \varepsilon) = V_P + V_G + V_H, \quad (2.22)$$

where $\dot{x}_t = \dot{x}(t + \theta)$, $\theta \in [-\varepsilon, 0]$. By Jensen's inequality (1.1), for all $\varepsilon \in (0, \varepsilon^*]$

$$V \geq V_p + V_G \geq \begin{bmatrix} x(t) \\ G(t) \end{bmatrix}^T \begin{bmatrix} P & -P \\ * & P + e^{-2\alpha\varepsilon} R \end{bmatrix} \begin{bmatrix} x(t) \\ G(t) \end{bmatrix} \geq c_1 |x(t)|^2 \quad (2.23)$$

with ε-independent $c_1 > 0$. Thus, V is positive-definite.

To compensate $\Delta A x$ in (2.15) we apply S-procedure: we add to $\dot{V}$ the left-hand part of

$$\lambda(\sigma^2 |x(t)|^2 - |\Delta A x|^2) \geq 0 \quad (2.24)$$

with some $\lambda > 0$. Then from (2.14)-(2.24), we have

$$\begin{aligned} \frac{d}{dt} V + 2\alpha V &\leq \frac{d}{dt} V + 2\alpha V + \lambda(\sigma^2 |x(t)|^2 - |\Delta A x|^2) \\ &\leq \xi^T \Phi \xi + \varepsilon^* \dot{x}^T(t) (R + \sum_{i=1}^N H_i) \dot{x}(t), \end{aligned} \quad (2.25)$$

where

$$\xi^T = [x^T(t), G^T, Y_1^T, \cdots, Y_N^T, x^T(t) \Delta A^T] \quad (2.26)$$

and Φ is given by (2.12). Substituting into (2.25) $\dot{x} = \sum_{i=1}^N f_i A_i x$ and applying Schur complements, we conclude that if

$$\begin{bmatrix} \Phi & \begin{matrix} \sqrt{\varepsilon^*}\sum_{i=1}^{N} f_i A_i^T (R+\sum_{j=1}^{N} H_j) \\ 0_{(N+2)n,n} \end{matrix} \\ \hline * \; * & -R-\sum_{j=1}^{N} H_j \end{bmatrix} < 0 \quad (2.27)$$

we have $\frac{d}{dt}V + 2\alpha V \le 0$, $\forall t \ge t_1$, implying

$$c_1|x(t)|^2 \le V(x_t) \le e^{-2\alpha(t-t_1)}V(t_1), \quad t \ge t_1. \quad (2.28)$$

LMIs (2.11) imply (2.27) since (2.27) is affine in $\sum_{i=1}^{N} f_i A_i^T$.

For all $\varepsilon \in (0, \varepsilon^*]$, V defined by (2.22) is upper bounded as

$$V(t_1) \le c_2\Big[|x(t_1)|^2 + \int_{t_1-\varepsilon}^{t_1} |\dot{x}(s)|^2 ds\Big]$$

with ε-independent $c_2 > 0$. For $t \in [0, t_1]$, $x(t)$ satisfies (2.1), where under **A2** we have $|A(\frac{t}{\varepsilon})| \le a$ for some $a > 0$ and all $t \ge 0$, $\varepsilon \in (0, \varepsilon^*]$. Hence, $\frac{d}{dt}|x(t)|^2 \le 2a|x(t)|^2$ for $t \in [0, t_1]$ yielding $|x(t)| \le e^{at}|x(0)|$ and $|\dot{x}(t)| \le a|x(t)| \le ae^{at}|x(0)|$ for $t \in [0, t_1]$. Therefore, $V(t_1)$ can be further upper bounded as

$$\begin{aligned} V(t_1) &\le c_2\Big[e^{2at_1}|x(0)|^2 + \int_{t_1-\varepsilon}^{t_1} a^2|x(s)|^2 ds\Big] \\ &\le c_2\Big[e^{2at_1}|x(0)|^2 + \varepsilon_1 a^2 e^{2at_1}|x(0)|^2\Big] \\ &\le c_3 e^{-2\alpha t_1}|x(0)|^2 \end{aligned} \quad (2.29)$$

for some ε-independent $c_3 > 0$. Then (2.13) follows from (2.28) and (2.29).

The feasibility of the strict LMIs (2.11) with $\alpha = 0$ implies the feasibility with the same decision variables and with a small enough positive $\alpha = \alpha_0$, and thus guarantees a small enough decay rate. □

Example 2.1. (Khalil (2002), Example 10.10): vibrational control. Consider the suspended pendulum with the suspension point that is subject to vertical vibrations of small amplitude and high frequency. The linearized at the upper equilibrium position model is given by

$$\dot{x}(t) = \begin{bmatrix} \cos\frac{t}{\varepsilon} & 1 \\ \gamma^2 - \cos^2\frac{t}{\varepsilon} & -\gamma\beta - \cos\frac{t}{\varepsilon} \end{bmatrix} x(t) \quad (2.30)$$

with $\gamma > 0$, $\beta > 0$. Note that we linearized f given above (10.32) on p. 410 of Khalil (2002) at $x_1 = \pi, x_2 = 0$ to derive (2.30). Similar to Remark 2.1, we change the time variable $t = 2\pi\bar{t}$ and define $\bar{x}(\bar{t}) = x(2\pi\bar{t}) = x(t)$, therefore,

$$\dot{\bar{x}}(\bar{t}) = 2\pi \begin{bmatrix} \cos\frac{2\pi\bar{t}}{\varepsilon} & 1 \\ \gamma^2 - \cos^2\frac{2\pi\bar{t}}{\varepsilon} & -\gamma\beta - \cos\frac{2\pi\bar{t}}{\varepsilon} \end{bmatrix} \bar{x}(\bar{t}) \quad (2.31)$$

Then we obtain

$$A_{av} = 2\pi \begin{bmatrix} 0 & 1 \\ \gamma^2 - 0.5 & -\gamma\beta \end{bmatrix}.$$

It follows from Theorem 10.4 of Khalil (2002) that for $\gamma^2 < 0.5$ and small enough ε, (2.30) is exponentially stable. We choose $\gamma = 0.2$ and $\beta = 1$.

Since A in (2.31) is ε-periodic, we have $\Delta A = 0$ and $\sigma = 0$. Note that $\cos \in [-1, 1]$ and $\cos^2 \in [0, 1]$. Therefore, (2.30) can be presented as a system with polytopic type uncertainty, where $A_1, \dots, A_4$ correspond to the four vertices:

$$\begin{aligned} A_1 &= 2\pi \begin{bmatrix} -1 & 1 \\ \gamma^2 & -\gamma\beta+1 \end{bmatrix}, \quad A_2 = 2\pi \begin{bmatrix} -1 & 1 \\ \gamma^2-1 & -\gamma\beta+1 \end{bmatrix}, \\ A_3 &= 2\pi \begin{bmatrix} 1 & 1 \\ \gamma^2 & -\gamma\beta-1 \end{bmatrix}, \quad A_4 = 2\pi \begin{bmatrix} 1 & 1 \\ \gamma^2-1 & -\gamma\beta-1 \end{bmatrix}. \end{aligned} \quad (2.32)$$

By verifying the feasibility of LMIs (2.11) in the four vertices, where for simplicity we take $\alpha = 0$ and $f_1^* = \dots = f_4^* = 0.5$, we find an upper bound $\varepsilon^* = 0.0031$ that preserves the stability of (2.30) for all $\varepsilon \in (0, \varepsilon^*]$. Numerical simulations under an arbitrary initial condition show that the system (2.30) is stable for a bigger upper bound $\varepsilon^* = 0.4755$, which may illustrate the conservatism of the proposed method.

Example 2.2. (Hetel and Fridman, 2013): stabilization by fast switching. Consider a switched system

$$\dot{x}(t) = \begin{cases} A_1 x(t), & t \in [k\varepsilon, k\varepsilon + \beta\varepsilon), \\ A_2 x(t), & t \in [k\varepsilon + \beta\varepsilon, (k+1)\varepsilon), \end{cases} \quad (2.33)$$

where $\varepsilon > 0$, $k = 0, 1, \dots$ and $\beta \in (0, 1)$, with unstable modes

$$A_1 = \begin{bmatrix} 0.1 & 0.3 \\ 0.6 & -0.2 \end{bmatrix}, \; A_2 = \begin{bmatrix} -0.13 & -0.16 \\ -0.33 & 0.03 \end{bmatrix}. \quad (2.34)$$

Then (2.33) can be presented as (2.1) with

$$A(\tau) = f_1(\tau)A_1 + f_2(\tau)A_2, \; \tau = \frac{t}{\varepsilon} \in [k, k+1), \; k = 0, 1, \dots$$

where $f_1(\tau) = \chi_{[k,k+\beta)}(\tau)$ is the indicator function of $[k, k+\beta)$, $f_2(\tau) = 1 - f_1(\tau)$. Choose $\beta = 0.4$ that leads to Hurwitz

$$A_{av} = \beta A_1 + (1-\beta)A_2.$$

Here $A(\tau)$ is periodic implying $\Delta A = 0$ and $\sigma = 0$.

The bounds (2.10) in this example can be found as follows:

$$\begin{aligned} \int_0^1 \varepsilon f_1(\frac{t}{\varepsilon} - \theta)\theta d\theta &= \int_0^\beta \varepsilon\theta d\theta \le 0.5\varepsilon^*\beta^2 \triangleq f_1^*, \\ \int_0^1 \varepsilon f_2(\frac{t}{\varepsilon} - \theta)\theta d\theta &= \int_\beta^1 \varepsilon\theta d\theta \le 0.5\varepsilon^*(1-\beta^2) \triangleq f_2^*. \end{aligned}$$

By verifying the feasibility of LMIs (2.11) with $\alpha = 0$ in the two vertices, we find an upper bound $\varepsilon^* = 0.1871$ that preserves the stability of (2.33) for all $\varepsilon \in (0, \varepsilon^*]$. Compared with $\varepsilon^* = 0.1871$ that is obtained in the theory, numerical simulations show that the system (2.33) with $\beta = 0.4$ is stable for a much bigger upper bound $\varepsilon^* = 37.8$.

Remark 2.2. The presented approach can be applied to persistently excited systems:

$$\dot{\bar{x}}(\bar{t}) = -\varepsilon p(\bar{t})p^T(\bar{t})\bar{x}(\bar{t}), \; \bar{t} \ge 0, \quad (2.35)$$

where $\bar{x}(\bar{t}) \in \mathbb{R}^n$, $p : [0, \infty) \to \mathbb{R}^n$ is measurable and $\varepsilon > 0$ is a small parameter. Here, similar to Pogromsky and Matveev (2017), it is assumed that function p has the following properties:

Boundedness: there exists a constant M such that for almost all $\tau \ge 0$

$$p(\tau)p^T(\tau) \le M^2 I_n.$$

Persistency of excitation: there is a constant $\rho > 0$ such that

$$\int_0^1 p(\tau-\theta)p^T(\tau-\theta)d\theta \geq \rho I_n, \ \forall \tau \geq 1.$$

The system (2.35) has been studied in Pogromsky and Matveev (2017); Zhang et al. (2019), where sufficient conditions are provided to guarantee the stability. In Pogromsky and Matveev (2017), a bound on the decay rate has been derived by introducing a novel non-quadratic Lyapunov functional. Time-varying Lyapunov functions for PE were considered in Efimov and Fradkov (2015); Verrelli and Tomei (2019). Note that our time-delay approach to averaging should lead to a time-independent quadratic Lyapunov functional and simple conditions in terms of LMIs.

3. CONCLUSION

The presented time-delay approach to the averaging allows, for the first time, to derive efficient constructive conditions on the upper bound of the small parameter that preserves the stability. This method provides a direct Lyapunov approach to linear fast-varying systems. It can be extended to input-to-state stability, to linear fast-varying systems with state-delay and to persistently excited systems.

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