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Sensor blending and Control allocation for non-square linear systems to achieve negative imaginary dynamics ⋆

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Abstract: This paper deals with the design of static pre- and post- compensators to transform stable, non-square LTI systems into the class of strongly strict negative imaginary systems. The pre-compensator plays the role of a control allocator while the post-compensator does sensor blending in order to make a non-square system square along with satisfying the strongly strict negative imaginary property. A specific structure of the post-compensator is also given that guarantees a feasible solution of the LMI conditions when applied to systems with number of outputs greater than or equal to number of inputs. The proposed pre- and post- compensators can also stabilize a non-square plant in closed-loop upon satisfying a particular DC-gain condition and furthermore, they can be utilized to develop a simple constant input tracking framework for non-square systems. The LMI-based design methodology offers a numerically tractable solution framework and hence the easy implementation of the proposed scheme in practical applications. Illustrative examples are provided throughout the paper to demonstrate the usefulness of the proposed results in widening the scope of the negative imaginary theory to non-square LTI systems (e.g. safety-critical systems having redundant sensors and actuators).

Keywords: Strongly strict negative imaginary systems; DC-gain; Non-square plants; Positive feedback; LMIs; Reference tracking.

1. INTRODUCTION

Negative Imaginary (NI) systems theory was introduced in (Lanzon and Petersen, 2008) and was inspired by the positive position feedback control of highly resonant flexible structures. NI theory has rapidly attracted the interest of the robust control community due to its simple internal stability condition for interconnected systems that depends only on the DC loop gain, and its wide applicability in different areas of control systems engineering. For example, such systems arise while considering the transfer functions from colocated force actuator to position sensor of a lightly-damped flexible structure (Lanzon and Petersen, 2008), from input voltage to output voltage in active electrical filters (Patra and Lanzon, 2011), from input voltage to shaft rotational velocity in DC servo motor (Song et al., 2012), etc. NI stability results find widespread applications in vibration control of cantilever beams (Bhikkaji et al., 2012), nano-positioning systems (Mabrok et al., 2014b), flexible structures with free-body motion (Mabrok et al., 2014a), etc. An NI system is a square, Lyapunov-stable system with real, rational and proper transfer function matrix $R(s)$ that satisfies the frequency-domain condition $j[R(j\omega) - R(j\omega)^*] \geq 0$ for all $\omega \in (0, \infty)$ such that $j\omega$ is not a pole of $R(s)$; while Strictly Negative Imaginary (SNI) property is defined for square and stable LTI systems satisfying the strict version of the aforementioned inequality. For SISO cases, it gives an appealing graphical interpretation – the Nyquist plot of an NI (SNI) transfer function lies below (strictly below) the real axis in the open positive frequency interval (Lanzon and Petersen, 2008). Another strict subset of SNI systems, called Strongly Strict Negative Imaginary (SSNI) systems, was introduced in (Lanzon et al., 2011) which satisfies two additional frequency-domain constraints at zero and infinite frequencies alongside the SNI properties. Since its inception in 2008, NI theory has witnessed rapid progress during the last ten years in both analysis and synthesis accompanied by practical applications. To study the most recent developments on the NI theory and the associated topics, the following papers (Dey et al., 2016; Bhowmick and Patra, 2017b,a; Khong et al., 2018; Dey et al., 2020; Dannatt and Petersen, 2019; Salcan-Reyes and Lanzon, 2019; Kurawa et al., 2019; Liu et al., 2019) can be referred.

Despite its strong theoretical background and potential applications in variety of control engineering problems, NI theory ceases to be applicable for non-square plants and for systems having relative degree more than two. This necessitates further research to develop some means by which NI theory can be applied to more general class

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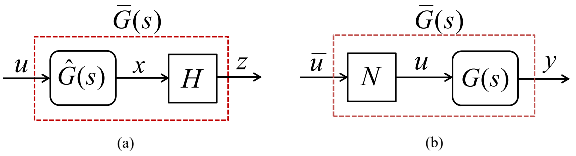

Fig. 1. (a) Post-compensator (H) design for tall/square plants $G(s) \in \mathscr{RH}_\infty^{l\times r}$ ($r \leq l$) and (b) Pre-compensator (N) design for fat plants $G(s) \in \mathscr{RH}_\infty^{l\times r}$ ($r > l$), such that $\bar{G}(s)$ is SSNI. Here, $\hat{G}(s) \in \mathscr{RH}_\infty^{n\times r}$ is an auxiliary plant constructed from $G(s)$;

of LTI systems. In this paper, we propose LMI-based pre- and post- compensation schemes to transform stable, non-square/square LTI systems into the SSNI class. The post-compensator does judicious blending of the sensor signals, termed as 'sensor-blending', to make the number of inputs (u) and outputs (z) of the synthesized system equal, as depicted in Fig. 1(a). On the contrary, a pre-compensator executes the same task for systems having more inputs than outputs by distributing (or allocating) the control inputs, known as 'control-allocation', as shown in Fig. 1(b). The proposed pre- and post- compensators can also stabilize a non-square/square plant in closed-loop via unity positive feedback invoking the DC-gain condition of an NI-SNI interconnection (Lanzon and Petersen, 2008). In this study, we denote the set of non-square systems $G(s) \in \mathscr{RH}_\infty^{l\times r}$ with $l > r$ as 'tall' plants while the complementary set of systems having $r > l$ is recognised as 'fat' plants. Since many overactuated and underactuated systems (e.g. flexible structures with redundant actuators and sensors) belong to fat and tall categories respectively, the proposed theory may find potential applications in vibration control of large space structures. The pre- and post- compensators are also used to develop a simple constant-input tracking framework for non-square/square systems by exploiting integral controllability (IC) of SSNI systems (Bhowmick and Patra, 2018).

2. TECHNICAL BACKGROUND

We start by defining the classes of NI and SNI systems followed by a lemma to describe SSNI systems. After that we recall the internal stability result for a positive feedback interconnection of stable NI and SNI systems.

Definition 1. (**NI System**) (Mabrok et al., 2014a; Lanzon and Chen, 2017) Let $M(s)$ be the real, rational and proper transfer function matrix of a square system. $M(s)$ is said to be Negative Imaginary (NI) if the following conditions are satisfied:

(1) $M(s)$ has no poles in $\Re[s] > 0$;
(2) $j[M(j\omega) - M(j\omega)^*] \geq 0$ for all $\omega \in (0, \infty)$ except the values of ω where $j\omega$ is a pole of $M(s)$;
(3) If $s = j\omega_0$ with $\omega_0 \in (0, \infty)$ is a pole of $M(s)$, then it is at most a simple pole and the matrix $K_0 = \lim_{s\to j\omega_0} (s - j\omega_0)jM(s)$ is Hermitian and positive semidefinite;
(4) If $s = 0$ is a pole of $M(s)$, then $\lim_{s\to 0} s^k M(s) = 0$ for all $k \geq 3$ and $\lim_{s\to 0} s^2 M(s)$ is Hermitian and positive semidefinite.

Definition 2. (**SNI System**) (Lanzon and Petersen, 2008) Let $M(s)$ be the real, rational and proper transfer function matrix of a square system. $M(s)$ is said to be Strictly Negative Imaginary (SNI) if $M(s)$ has no poles in $\Re[s] \geq 0$ and $j[M(j\omega) - M(j\omega)^*] > 0$ for all $\omega \in (0, \infty)$.

SSNI systems (Lanzon et al., 2011) is a particular subset of the SNI class (Xiong et al., 2010) which satisfies two additional frequency-domain criteria in the neighbourhood of $\omega = 0$ and $\omega = \infty$. In a SISO setting, these two extra conditions indicates the patterns of departure from $\omega = 0$ and arrival at $\omega = \infty$ of the Nyquist plot of an SSNI transfer function.

Lemma 3. (**SSNI Lemma**) (Lanzon et al., 2011) Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a state-space realization of a real, rational and proper transfer function matrix $R(s) \in \mathscr{R}^{m\times m}$. Suppose, $[R(s) - R(-s)^T]$ has full normal rank m and the pair (A, C) is observable. Then, A is Hurwitz and $R(s)$ is SNI with

$$\begin{aligned} &\lim_{\omega\to\infty} j\omega\,[R(j\omega) - R(j\omega)^*] > 0 \quad \text{and} \\ &\lim_{\omega\to 0} j\frac{1}{\omega}\,[R(j\omega) - R(j\omega)^*] > 0 \end{aligned} \tag{1}$$

if and only if $D = D^T$ and there exists a real matrix $Y = Y^T > 0$ such that

$$AY + YA^T < 0 \quad \text{and} \quad B = -AYC^T. \tag{2}$$

The SSNI Lemma will be invoked later to prove the main results of this paper. Note that in case of SSNI systems the full normal rank constraint on $[R(s) - R(-s)^T]$ is implied by (2) when the B matrix has full column rank. This is proved in the following lemma.

Lemma 4. Let $R(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be an $(m\times m)$ SSNI system with $\text{rank}[B] = m$. Then, $[R(s) - R(-s)^T]$ has full normal rank m.

Proof. Since $AY + YA^T < 0$, there exists a square and non-singular matrix L such that $AY + YA^T = -L^T L$. For these L and Y, the transfer function matrix $N(s) = \left[\begin{array}{c|c} A & B \\ \hline LY^{-1}A^{-1} & 0 \end{array}\right]$ acquires full column rank at $s = j\omega$ for all $\omega \in \mathbb{R}$ since A is Hurwitz and $\text{rank}[B] = m$ via assumption and $\text{rank}[LY^{-1}A^{-1}] = n$. It implies from (Xiong et al., 2010, Corollary 1)

$$j\omega[R(j\omega) - R(j\omega)^*] = \omega^2 N(j\omega)^* N(j\omega) > 0 \tag{3}$$

for all $\omega \in \mathbb{R}\backslash\{0\}$ and $R(0) - R(0)^T = 0$ since $R(0) = CYC^T + D = R(0)^T$. This implies that there does not exist any continuous interval of $\omega \in \mathbb{R}$ for which $\det[R(j\omega) - R(j\omega)^*]$ remains zero. This in turn ensures that $[R(s) - R(-s)^T]$ must have full normal rank. Note minimality is not required. □

We now present an internal stability condition for a stable NI system interconnected with an SNI system via positive feedback. Please see (Lanzon and Chen, 2017) for updated internal stability results of NI-SNI interconnections without the restrictive suppositions.

Theorem 5. (Lanzon and Petersen, 2008; Lanzon and Chen, 2017) Let $M(s)$ be stable NI and $N(s)$ be SNI. Let either $M(\infty) = 0$, or else, let $M(\infty)N(\infty) = 0$ and $N(\infty) \geq 0$. Then, the positive feedback interconnection of $M(s)$ and $N(s)$ is internally stable if and only if $\lambda_{\max}[N(0)M(0)] < 1$.

3. MAIN RESULTS

This section presents the main theoretical contributions of the paper. The LMI-based static pre- and post- compensator design techniques are developed to transform stable, non-square/square, LTI systems into the SSNI class. A particular structure of the post-compensator is proposed in this paper which guarantees the SSNI property of any tall/square LTI plant with a full-rank B matrix. The pre-compensator, applicable for fat plants, requires to satisfy a set of sufficient-type LMI conditions and hence it may not always yield feasible solution. In order to overcome this limitation in the case of fat plants, a combined pre-post-compensation scheme is proposed in Subsection 3.3.

3.1 Post-compensator design for tall/square LTI plants

The post-compensation scheme is derived through Theorem 6 for stable, tall/square plants. Note that during the post-compensator design, it is assumed that all states of the underlying system are accessible for direct measurement. However, if some of the states are not measurable then a full-order observer can easily be included in the proposed scheme to estimate the states.

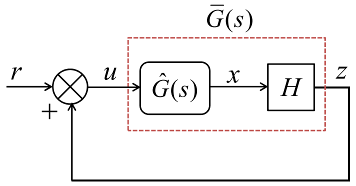

Fig. 2. A static post-compensation scheme for stable, tall/square LTI plants.

Theorem 6. Let $\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline I & 0 \end{array}\right] \in \mathscr{RH}_\infty^{n\times r}$ be constructed from the original plant $G(s) \in \mathscr{RH}_\infty^{l\times r}$ having a minimal state-space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $r \le l \le n$ and $\text{rank}[B] = r$. Suppose there exist real symmetric matrices $Y > 0$ and $Q_0 > 0$ such that

$$AY + YA^T < 0, \tag{4a}$$

$$\begin{bmatrix} I & (A^{-1}B)^T \\ A^{-1}B & Y \end{bmatrix} > 0, \tag{4b}$$

$$\begin{bmatrix} Q_0A + A^TQ_0 & I \\ I & -Y \end{bmatrix} \le 0. \tag{4c}$$

Let $z = Hx$ be defined as an auxiliary output of the system where $H = -B^TA^{-T}Y^{-1}$. Then the post-compensator H makes the combined system $\bar{G}(s) = H(sI - A)^{-1}B = \left[\begin{array}{c|c} A & B \\ \hline H & 0 \end{array}\right]$ from input u to the auxiliary output z SSNI and also stabilizes the closed-loop system with unity positive feedback as shown in Figure 2.

Proof. Since B has full column rank and $AY + YA^T < 0$ via (4a), $[\bar{G}(s) - \bar{G}(-s)^T]$ has full normal rank via Lemma 4. Note that the LMI constraint (4b) is equivalent to

$$Y - A^{-1}BB^TA^{-T} > 0 \tag{5}$$

via applying Schur-complement Lemma (Boyd et al., 1994). The new pair (A,H) is designed to be completely observable since (4c) implies the observability Gramian condition (Boyd et al., 1994) as shown below:

$$\begin{aligned} &\begin{bmatrix} Q_0A + A^TQ_0 & I \\ I & -Y \end{bmatrix} \le 0 \\ &\Leftrightarrow Q_0A + A^TQ_0 + Y^{-1} \le 0 \\ &\Leftrightarrow Y[Q_0A + A^TQ_0]Y + Y \le 0 \\ &\Rightarrow Y[Q_0A + A^TQ_0]Y + A^{-1}BB^TA^{-T} \le 0 \\ &\quad [\text{since } A^{-1}BB^TA^{-T} < Y \text{ via (5)}] \\ &\Leftrightarrow Q_0A + A^TQ_0 + Y^{-1}A^{-1}BB^TA^{-T}Y^{-1} \le 0 \\ &\Leftrightarrow Q_0A + A^TQ_0 + H^TH \le 0 \end{aligned}$$

on noting that $H = -B^TA^{-T}Y^{-1}$. The post-compensator H ensures $AYH^T = AY(-B^TA^{-T}Y^{-1})^T = -B$. Therefore, the compensated system $\bar{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline H & 0 \end{array}\right]$ satisfies all the required properties to be an SSNI system according to Lemma 3. Furthermore, the unity positive feedback interconnection of $\bar{G}(s)$ shown in Fig. 2 is closed-loop stable via exploiting the NI-SNI stability condition (Theorem 5) since $\bar{G}(0) < I \Leftrightarrow HYH^T < I \Leftrightarrow (A^{-1}B)^TY^{-1}(A^{-1}B) < I \Leftrightarrow$ (4b). This completes the proof. ■

Note that we may specify some upper and lower bounds for the matrices Y and Q_0 in Theorem 6 in order to avoid getting ill-conditioned Y and Q_0 due to numerical computational issues. The lower bound on Q_0 can be selected as the observability Gramian W_0 of the uncompensated plant $G(s)$. The upper bound on Y plays a crucial role in determining the DC-gain of the compensated system given by $\bar{G}(0) = (A^{-1}B)^TY^{-1}(A^{-1}B)$. A high upper bound on Y causes an overall reduction of the eigenvalues of Y^{-1} which in turn makes $\lambda_{\max}[\bar{G}(0)]$ very small. The other bounds can be selected depending on the plant. Note that $\lambda_{\max}[\bar{G}(0)]$ cannot be negative or zero since in Fig. 2 $\bar{G}(0) = HYH^T > 0$ and H has full rank.

Example 7. Let us consider the two-mode flexible structure taken in (Joshi and Kelkar, 2001) with two inputs and four outputs having non-colocated sensors and actuators where the inputs are generalized forces and the outputs are rates. The nominal plant model is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -0.1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & -0.1 \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 \\ 0.1 & -0.01 \\ 0 & 0 \\ -0.2 & 0.05 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0.8913 & 0 & 0.9218 \\ 0 & 0.7621 & 0 & 0.7382 \\ 0 & 0.4565 & 0 & 0.1763 \\ 0 & 0.0185 & 0 & 0.4057 \end{bmatrix} \text{ and } D = 0_{4\times 2} \text{ where } \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

is minimal and A is Hurwitz. Applying Theorem 6 to this system, the set of inequalities (4a)-(4c) yields

$$Y = \begin{bmatrix} 0.0340 & -0.0017 & 0.00 & 0.00 \\ -0.0017 & 0.0339 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.0108 & -0.0007 \\ 0.00 & 0.00 & -0.0007 & 0.0421 \end{bmatrix} > 0 \quad \text{and}$$

$$Q_0 = \begin{bmatrix} 677.1593 & 33.5965 & 0.00 & 0.00 \\ 33.5965 & 677.2235 & 0.00 & 0.00 \\ 0.00 & 0.00 & 999.7806 & 12.8268 \\ 0.00 & 0.00 & 12.8268 & 250.0677 \end{bmatrix} > 0$$

using the CVX toolbox (Grant and Boyd, 2014). It is easy to verify that $AY + YA^T < 0$, $B = -AYH^T$ and the new pair (A, H) is completely observable. The post-compensator $H = -B^T A^{-T} Y^{-1}$ is now obtained as

$$H = \begin{bmatrix} 2.9513 & 0.1440 & -4.6531 & -0.0785 \\ -0.2951 & -0.0144 & 1.1633 & 0.0196 \end{bmatrix}.$$

Therefore, the transformed system $\bar{G}(s) = H(sI - A)^{-1}B$ is SSNI according to Theorem 6. Now we check the DC-gain condition: $\lambda_{\max}[\bar{G}(0)] = \lambda_{\max} \begin{bmatrix} 0.5278 & -0.0877 \\ -0.0877 & 0.0175 \end{bmatrix} = 0.5424 < 1$. This guarantees that the post-compensator H makes $\bar{G}(s)$ SSNI and stabilizes the closed-loop system shown in Figure 2 via unity positive feedback.

3.2 Pre-compensator design for stable fat LTI plants

The pre-compensation scheme is derived via Theorem 8.

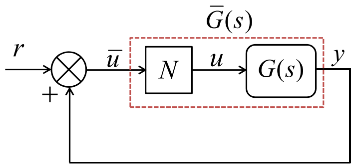

Fig. 3. A static pre- compensation scheme for fat plants.

Theorem 8. Let $\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ be a minimal state-space realization of a real, rational and proper transfer function matrix $G(s) \in \mathscr{RH}_\infty^{l\times r}$ with rank$[B] = r$ and $l < r \leq n$. Suppose there exist a full-rank matrix $N \in \mathbb{R}^{r\times l}$ and the real symmetric matrices $Y > 0$, $Q_c > 0$ such that

$$AY + YA^T < 0, \tag{6a}$$

$$BN = -AYC^T, \tag{6b}$$

$$CYC^T < I, \tag{6c}$$

$$\begin{bmatrix} AQ_c + Q_cA^T & BN \\ N^TB^T & -I \end{bmatrix} < 0. \tag{6d}$$

Then the pre-compensator N makes the combined system $\bar{G}(s) = G(s)N = \left[\begin{array}{c|c} A & BN \\ \hline C & 0 \end{array}\right]$ from $\bar{u}$ to y SSNI and also stabilizes the closed-loop system with unity positive feedback as shown in Figure 3.

Proof. Note that rank$[BN] = l$ applying the matrix rank inequality rank$[B]$ + rank$[N] - r \leq$ rank$[BN] \leq \min\{$rank$[B]$, rank$[N]\}$ via the assumption rank$[B] = r$. Therefore, $[\bar{G}(s) - \bar{G}(-s)^T]$ has full normal rank via condition (6a) due to Lemma 4. Since $\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ is minimal, $\bar{G}(s) = \left[\begin{array}{c|c} A & BN \\ \hline C & 0 \end{array}\right]$ remains completely observable and via satisfying the controllability Gramian (Boyd et al., 1994) condition (6d), the new pair (A, BN) retains complete controllability. Hence, the pre-compensator N renders the compensated system $\bar{G}(s) = G(s)N$ SSNI via satisfying (6a) and (6b), and also ensures closed-loop stability of the scheme shown in Fig. 3 applying the DC-gain condition $\bar{G}(0) = CYC^T < I$. This completes the proof. ■

Example 9. Let us consider a (1×2) stable, LTI plant $G(s) = \left[\begin{array}{ccc|cc} -5 & -6.25 & 0 & 4 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 1 \\ \hline 6.25 & 1.563 & -1 & 0 & 0 \end{array}\right]$. Now, the set of conditions (6a)-(6d) is solved for this system and it gives $Y = \begin{bmatrix} 0.0827 & -0.0291 & 0.4711 \\ -0.0291 & 0.0293 & -0.1452 \\ 0.4711 & -0.1452 & 3.3186 \end{bmatrix} > 0$, $Q_c > 0$ and $N = \begin{bmatrix} 0.0138 \\ -3.0043 \end{bmatrix}$ by using the CVX toolbox (Grant and Boyd, 2014). We now compute the pre-compensated system $\bar{G}(s) = G(s)N = \dfrac{3.35s^2 + 17.1s + 76.84}{s^3 + 10s^2 + 50s + 125}$, the Nyquist plot of which is shown in Figure 4. It has also been checked that $\bar{G}(s)$ satisfies the SSNI properties: $AY + YA^T < 0$, $BN = -AYC^T$ and also retains minimality. Finally, we examine that the DC-gain condition is also satisfied, i.e., $\bar{G}(0) = 0.6147 < 1$. Therefore, the unity positive feedback interconnection of $\bar{G}(s)$ shown in Fig. 3 is closed-loop stable according to Theorem 8.

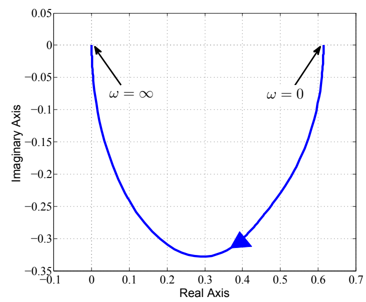

Fig. 4. Nyquist plot of the pre-compensated SSNI system $\bar{G}(s) = \frac{3.35s^2+17.1s+76.84}{s^3+10s^2+50s+125}$ in Example 9.

Remark 10. It is sometimes possible that the set of sufficient conditions (6a)-(6d), given in Theorem 8, may not be feasible. For instance, in Example 12, there does not exist any feasible pre-compensator for the system that renders the compensated system $\bar{G}(s)$ into the SSNI class. Moreover, a necessary condition for bi-proper fat plants $G(s)$ to be transformed into an NI system is that $DN = N^TD^T$, which is an excessively restrictive assumption in practice. To handle such cases, a post-compensator (H) with a specific structure can be used in addition to the pre-compensator (N) such that the condition $BN = -AYH^T$ becomes easier to satisfy. This requires us to discard the output equation $y = Cx+Du$ and work only with the state equation $\dot{x} = Ax+Bu$. Such a scheme with combined pre-post-compensator is discussed in the next subsection.

3.3 Combined pre-post-compensator design for fat plants

The theorem given below suggests a combined pre-post-compensator design technique to transform stable, fat LTI plants $G(s) \in \mathscr{RH}_\infty^{l\times r}$ where $r > l$ into the SSNI class leading to a closed-loop control scheme with unity positive feedback as shown in Fig. 5.

Theorem 11. Let $\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline I & 0 \end{array}\right] \in \mathscr{RH}_\infty^{n\times r}$ be constructed from the original plant $G(s) \in \mathscr{RH}_\infty^{l\times r}$ having a

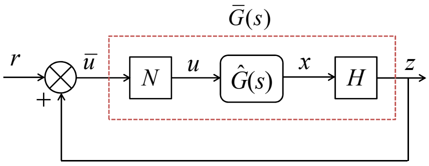

Fig. 5. A combined pre-post-compensation scheme for stable fat LTI plants.

minimal state-space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ where $l < r \leq n$ and $\text{rank}[B] = r$. Suppose there exist a full-rank matrix $N \in \mathbb{R}^{r \times l}$ and the real symmetric matrices $Y > 0$, $Q_0 > 0$ such that

$$AY + YA^T < 0, \tag{7a}$$

$$\begin{bmatrix} I & (A^{-1}BN)^T \\ A^{-1}BN & Y \end{bmatrix} > 0, \tag{7b}$$

$$\begin{bmatrix} Q_0 A + A^T Q_0 & I \\ I & -Y \end{bmatrix} \leq 0. \tag{7c}$$

Let $z = Hx$ be defined as an auxiliary output of the system where $H = -N^T B^T A^{-T} Y^{-1}$. Then the compensated system $\bar{G}(s) = H(sI - A)^{-1}BN = \left[\begin{array}{c|c} A & BN \\ \hline H & 0 \end{array}\right]$ from $\bar{u}$ to z is SSNI and the closed-loop system is stabilized with unity positive feedback as shown in Figure 5.

Proof. The proof follows directly by combining Theorems 6 and 8 on noting that $\text{rank}[BN] = l$, the pair $(A,\ H)$ is completely observable via (7c), $AYH^T = AY(-N^T B^T A^{-T} Y^{-1})^T = -BN$ and $\bar{G}(0) < I \Leftrightarrow HYH^T < I \Leftrightarrow Y - (A^{-1}BN)(A^{-1}BN)^T > 0 \Leftrightarrow$ (7b) applying Schur-complement Lemma. ■

Note that (7a) holds for any stable system $G(s)$ and $BN = -AYH^T$ automatically holds since $H = -N^T B^T A^{-T} Y^{-1}$. Henceforth, the combined pre-post-compensator guarantees the SSNI property of the compensated system $\bar{G}(s)$ when $G(s)$ is a stable, fat plant with a full-rank B matrix. Sufficiency of Theorem 11 arises due to satisfying other two conditions (7b) and (7c).

Example 12. Consider the transfer function matrix $G(s) \in \mathscr{RH}_\infty^{1\times 2} = \left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -3 & -1 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$ of a non-colocated spring-mass-damper system as shown in Figure 6. Discarding the output equation and applying Theorem 11, the set of inequalities (7a), (7b) and (7c) yields a set of solution matrices

$$Y = \begin{bmatrix} 0.6071 & -0.0777 & 0.3323 & -0.0137 \\ -0.0777 & 0.9565 & -0.0152 & -0.0525 \\ 0.3323 & -0.0152 & 0.5976 & -0.0759 \\ -0.0137 & -0.0525 & -0.0759 & 0.9532 \end{bmatrix} > 0,$$

$$Q_0 = \begin{bmatrix} 38.1796 & 5.3918 & -9.6688 & 3.2305 \\ 5.3918 & 15.3312 & 3.2276 & 5.5535 \\ -9.6688 & 3.2276 & 38.1888 & 5.3939 \\ 3.2305 & 5.5535 & 5.3939 & 15.3335 \end{bmatrix} > 0,$$

and $N = \begin{bmatrix} 1.00 \\ 0.00 \end{bmatrix}$. The post-compensator is then obtained as $H = -N^T B^T A^{-T} Y^{-1} = [\, 0.9047\ 0.0779\ 0.1722\ 0.0310 \,]$

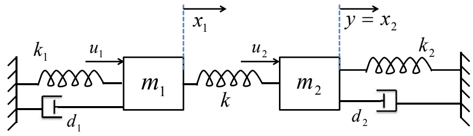

Fig. 6. A non-colocated spring-mass-damper system with two inputs and one output considered in Example 12.

and the compensated system is computed as

$$\bar{G}(s) = \frac{0.07791s^3 + 0.9826s^2 + 1.2s + 3.058}{s^4 + 2s^3 + 7s^2 + 6s + 5},$$

which satisfies all the criteria to be SSNI and retains both controllability and observability properties. The Nyquist plot of $\bar{G}(s)$ is given in Figure 7. Furthermore, the DC-gain condition is also satisfied: $\bar{G}(0) = 0.6117 < 1$. Therefore, the unity positive feedback interconnection is closed-loop stable via Theorem 11.

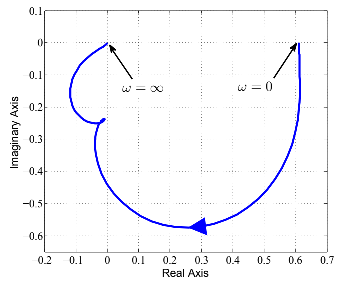

Fig. 7. Nyquist plot of the pre-post-compensated SSNI system $\bar{G}(s)$ designed in Example 12.

3.4 A constant input tracking scheme for non-square and square LTI plants using pre- and post- compensators

The static pre- and post- compensators can be utilised to develop a simple integral control framework with sensor/actuator failure tolerance for stable, non-square/square LTI systems by exploiting integral controllability (IC) and decentralized integral controllability (DIC) properties of SSNI systems having positive definite DC-gain matrix as discussed in (Bhowmick and Patra, 2018). DIC property facilitates constant-input tracking and preserves the closed-loop stability of an integral control scheme upon occurrence of sensor/actuator faults including the faulty channels while maintaining satisfactory tracking performance of the healthy channels. A sufficient condition for the DIC property when operated in a negative feedback loop is to check whether a stable and square system possesses positive definite DC-gain matrix. Fig. 8(a) shows the tracking scheme for a stable, fat plant $G(s)$, that requires a static pre-compensator N such that $\bar{G} = G(s)N$ is SSNI satisfying $\det[G(0)N] \neq 0$. The decentralized integral gain block has the structure $K_g = \text{diag}\{k_1, k_2, \ldots, k_l\}$ where $k_i \in [0, k^\star]$ for all $i \in \{1, \ldots, l\}$ with a finite upper bound $k^\star > 0$. Since $\bar{G}(0) = G(0)N = CYC^T > 0$, $\bar{G}(s)$ satisfies the DIC property and hence, facilitates constant input tracking with failure-tolerance. Similarly, for stable square plants, the tracking scheme is shown in Fig. 8(b) where the post-compensated plant $\bar{G}(s) = H(sI - A)^{-1}B$ satisfies DIC property due to being SSNI with $\bar{G}(0) = HYH^T > 0$. An input-shaping matrix $S = \bar{G}(0)G(0)^{-1}$ is designed so that the plant output y can track the actual reference

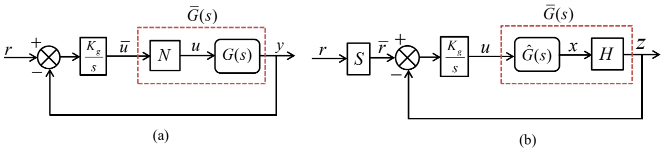

Fig. 8. A constant input tracking framework (a) for fat plants and (b) for square/tall plants, where K_g denotes the decentralized integral gain matrix.

r while the auxiliary output z automatically tracks the shaped reference $\bar{r}$. Note that the scheme shown in Fig. 8(b) can also be applied to tall plants by choosing the right set of outputs matching with the appropriate set of reference inputs.

4. CONCLUSION

This paper enables NI theory to be used on non-square LTI systems (e.g. safety-critical systems with redundant actuators and sensors) by designing static pre- and post- compensators. A specific structure of the post-compensator is also given such that any stable tall/square plant with a full-rank B matrix can be transformed into the SSNI class. Moreover, upon imposing the DC-gain condition on the compensated system, the pre- and post- compensators stabilize respectively fat and tall plants in closed-loop with unity positive feedback. Furthermore, the proposed compensators can be used to design a simple constant input tracking framework for stable non-square/square plants. The LMI-based design techniques offer numerically tractable solution framework and hence may find potential applications. In the future, we aim to develop observer-based post-compensation scheme to replace the state feedback by output feedback for applications where not all states are measurable. Furthermore, these compensation schemes can be modified to facilitate control effort allocation in case of large-scale flexible structure systems.

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