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Robust Consensus and Connectivity-maintenance under Edge-agreement-based Protocols for Directed Spanning Tree Graphs

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Abstract: We address the consensus problem with connectivity maintenance for networks of multi-agent systems interconnected over directed spanning tree graphs in the edge-agreement space and, for the first time in the literature, we provide a strict Lyapunov function. The importance of this contribution is that it allows to establish uniform global asymptotic stability of the consensus manifold for a multi-agent system subject to proximity constraints. Moreover, robustness in the sense of input-to-state stability with respect to external disturbances is also established. These properties have not been established before when dealing with state-dependent constraints, even for a class of directed graphs, because most often in the literature only non-uniform convergence to the consensus manifold is established.

Keywords: Multi-agent systems, Directed graphs, Control under communication constraints, Lyapunov methods

1. INTRODUCTION

In recent years, distributed control of multi-agent systems has been extensively studied, since it offers considerable advantages over its centralized counterpart, such as versatility, robustness and less computational power. Of particular interest for multi-agent systems, the consensus problem, under a classical node-based graph representation, has been thoroughly studied for many years now. As it is well-known, for undirected graphs, consensus is achieved if and only if the graph is connected (there exists a path connecting every agent in the network); for directed graphs, the existence of a rooted directed spanning tree is necessary and sufficient (Ren et al. (2005)). Yet, although necessary for consensus, assuming connectivity may not be realistic in concrete applications. For instance, those involving autonomous multiagent cooperative systems, in which case the exchange of information between one system and its neighbours can be ensured only if the latter lie in "close" proximity. Therefore, connectivity is a constraint that must be established and not accounted for.

Several articles have addressed the connectivity maintenance problem. For undirected graphs, in Ji and Egerstedt (2007) an "edge-tension" function is proposed in order to guarantee consensus while preserving connectivity. Following the latter reference, in Dimarogonas and Kyriakopoulos (2008) and Boskos and Dimarogonas (2017) a general framework is developed using barrier functions, which encode the proximity constraints, over static and dynamic graphs; robustness with respect to additional bounded inputs is also established. For multiagent systems over directed graphs (digraphs), however, there are far fewer works addressing the problem of connectivity maintenance. In Sabattini et al. (2013), Poonawala and Spong (2017), Sabattini et al. (2015), and Cai et al. (2017) some consensus control laws derived as the gradient of barrier functions, are proposed, albeit for strongly connected digraphs. The methods proposed in these references, however, rely on the knowledge of the algebraic connectivity of the graph, which is a global parameter. Therefore, in order to apply a distributed control law it is required to estimate the algebraic connectivity. In Mukherjee et al. (2017), a control law is proposed to preserve local connectivity in a digraph, although only connectivity maintenance and not consensus is shown and estimation of the eigenvalues of the Laplacian matrix is also needed.

Now, from an analytic viewpoint, it is important to stress that the agreement problem has been mainly studied from the *node* perspective, according to which the properties of the resulting Laplacian matrix are fundamental to establish consensus relying, essentially, on linear algebra. This is in clear contrast with methods that focus on the study of consensus from a stability perspective. According to the latter, the system's states represent the difference among the states' values of the nodes, rather than the

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nodes themselves, so consensus is reformulated as the stabilisation of a set (Panteley and Loría (2017)) or, more particularly, of the origin. Not relying on methods tailored for the analysis of linear invariant systems, paves the way to important relaxations, such as considering time-varying interconnections in directed spanning trees (Maghenem and Loría (2017)) or nonlinear interconnections, as in the case of connectivity-preserving consensus algorithms (Dimarogonas and Kyriakopoulos (2008); Poonawala and Spong (2017); Sabattini et al. (2015)), as studied herein.

More specifically in this paper, in order to analyse the consensus problem from a stability perspective, we rely on the so-called *edge-agreement* representation Zelazo et al. (2007) —see also Alvarez-Jarquín and Loría (2014). This framework has been shown to offer some analysis advantages over the node perspective. In Zeng et al. (2014) and Zeng et al. (2017), respectively for nonlinear systems and systems with quantised information exchange, consensus over a digraph is guaranteed by establishing asymptotic stability of the agreement manifold by means of a strict Lyapunov function. The importance of the latter can hardly be overestimated. For instance, in Mukherjee and Zelazo (2018) it allows to assert the robustness of the consensus protocol for a network of second-order systems. In none of these works, however, the connectivity-maintenance problem is addressed; the interconnections are linear time-invariant (except for Alvarez-Jarquín and Loría (2014)), which greatly facilitates the construction of the Lyapunov function.

Hence, in this paper we address the agreement control problem, with connectivity maintenance, for a network of first-order systems over directed spanning trees with proximity constraints. Our contribution is twofold; first, using a barrier function we construct a strict Lyapunov function. This allows us to establish uniform global asymptotic stability of the consensus manifold for the closed-loop system as well as connectivity maintenance. Secondly, we demonstrate via the proposed strict Lyapunov function that the system is input-to-state stable with respect to external disturbances. To the best of our knowledge, although strict Lyapunov functions have been proposed, even for edge-based graphs, this is done without addressing connectivity maintenance. As a matter of fact, the latter has been addressed mostly under the node-perspective and, even though in some works Lyapunov's method is used, this is done via non-strict Lyapunov functions, which lead to weak properties such as mere convergence.

The rest of the paper is organised as follows. In Section 2 we present some preliminaries on edge-based graph theory and state the control problem; our main results are presented in Section 3, illustrated through simulations in Section 4. We conclude some remarks in Section 5.

2. MODEL AND PROBLEM FORMULATION

2.1 Notations and preliminaries

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ be a weighted digraph defined by a vertex (node) set $\mathcal{V} = \{1, 2, \ldots, n\}$, corresponding to the nodes' states, and an edge set $\mathcal{E} \subseteq \mathcal{V}^2$ characterising the network communication topology. Throughout this paper it is assumed that the cardinality of $\mathcal{V}$ equals n and that of $\mathcal{E}$ equals m. $\mathcal{W} \in \mathbb{R}^{m\times m}$ is a diagonal matrix with strictly positive entries which represent the weights of the edges of the graph. A directed edge e_k is an ordered pair $(i, j) \in \mathcal{E}$ if and only if there exists a connection that starts at node i and ends at node j, where $i, j \in \mathcal{V}$. A directed path in a digraph is an ordered sequence of directed edges. A *directed tree* is a subgraph in which every node has exactly one parent except for the *root* node, which has no parent, but has a path to every other node. A *spanning tree* is a tree subgraph containing all nodes in $\mathcal{V}$.

The interconnection topology of a digraph may be described using the so-called incidence matrix, $E(\mathcal{G}) \in \mathbb{R}^{n\times m}$, which has n rows indexed according to the nodes and m columns indexed according to the edges. Its elements are defined as

$$[E]_{ik} := \begin{cases} -1 & \text{if } i \text{ is the terminal node of edge } e_k \\ 1 & \text{if } i \text{ is the initial node of edge } e_k \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for the purpose of analysis, it is useful to partition the incidence matrix as

$$E = E_\odot + E_\otimes \tag{1}$$

where $E_\odot(\mathcal{G}) \in \mathbb{R}^{n\times m}$ corresponds to the so-called *in-incidence* matrix, whose elements are defined as —*cf.* Zeng et al. (2014),

$$[E_\odot]_{ik} := \begin{cases} -1 & \text{if } i \text{ is the terminal node of edge } e_k \\ 0 & \text{otherwise} \end{cases}$$

and $E_\otimes(\mathcal{G}) \in \mathbb{R}^{n\times m}$ corresponds to the so-called *out-incidence* matrix, whose elements are defined as

$$[E_\otimes]_{ik} := \begin{cases} 1 & \text{if } i \text{ is the initial node of edge } e_k \\ 0 & \text{otherwise.} \end{cases}$$

Then, the weighted Laplacian matrix $L(\mathcal{G}) \in \mathbb{R}^{n\times n}$ of a digraph $\mathcal{G}$ can be defined in terms of the incidence and in-incidence matrices as

$$L(\mathcal{G}) = E_\odot(\mathcal{G})\mathcal{W}E(\mathcal{G})^\top. \tag{2}$$

For digraphs, $L(\mathcal{G})$ has a simple zero eigenvalue and all other non-zero eigenvalues are in the open left-half complex plane, if and only if the digraph contains a spanning tree (Ren et al. (2005)). In what follows, the argument $\mathcal{G}$ is dropped for brevity.

2.2 Directed Edge Laplacian and Reduced Order System

In Ren et al. (2005) it is shown that a multiagent system communicating through a directed graph achieves consensus if and only if the graph contains a directed spanning tree. This suggests that when addressing a consensus problem for a digraph, the effort may be concentrated on an existing rooted directed spanning tree that is a subgraph of $\mathcal{G}$. Actually, given an appropriate labelling of the edges (Mukherjee and Zelazo (2018)), it is possible to express the incidence matrix as

$$E = [\, E_t \quad E_c \,] \tag{3}$$

where $E_t \in \mathbb{R}^{n\times(n-1)}$ denotes the incidence matrix corresponding to an arbitrary spanning tree $\mathcal{G}_T \subset \mathcal{G}$ and $E_c \in \mathbb{R}^{n\times(m-n+1)}$ represents the incidence matrix corresponding to the remaining edges not contained in $\mathcal{G}_T$. The labelling is as follows: let the root node be labelled "1" and let the remaining nodes be labelled as follows. Any two nodes i and j belonging to a branch b_l of the tree are labelled such that if the path length from the root to i is shorter than the path length from the root to j, then $i < j$. Then, label the $n-1$ edges such that for any edge $e_k = (i, j)$, that is, with terminal node j, one has $j > k$.

Furthermore, in Mukherjee and Zelazo (2018), it is shown that there exists a linear transformation between the edges of the spanning tree $\mathcal{G}_T$ and the remaining cycles, that is

$$E_t T = E_c \tag{4}$$

where $T := \left(E_t^\top E_t\right)^{-1} E_t^\top E_c$. Thus, defining

$$R := [\, I \quad T \,] \tag{5}$$

with I denoting the identity matrix of adequate dimension, one obtains an alternative representation of the incidence matrix of the digraph that is given by

$$E = E_t R. \tag{6}$$

Remark 1. The in-incidence and out-incidence matrices also admit a decomposition as in (3), *i.e.*, defining $E_{\odot t}$ and $E_{\otimes t} \in \mathbb{R}^{n\times(n-1)}$ as the in-incidence and out-incidence matrices of a directed spanning tree $\mathcal{G}_T \subset \mathcal{G}$ and $E_{\odot c}$ and $E_{\otimes c} \in \mathbb{R}^{n\times(m-n+1)}$ as the in-incidence and out-incidence matrices of the remaining edges in $\mathcal{G}$, we have $E_\odot = [E_{\odot t}\ E_{\odot c}]$ and $E_\otimes = [E_{\otimes t}\ E_{\otimes c}]$. •

The identity (6) is useful to derive a dynamic model for the spanning-tree $\mathcal{G}_T \subset \mathcal{G}$, as we show next.

Let us consider the classic weighted consensus protocol (Ren et al. (2005)) for n first-order systems,

$$\dot{x}_i = u_i, \quad x_i \in \mathbb{R}^N,\ i \le n \tag{7}$$

where $u_i \in \mathbb{R}^N$ corresponds to the control input for each agent. In compact form, the systems' states are collected in the vector $x = \left[x_1^\top, \ldots, x_n^\top\right]^\top \in \mathbb{R}^{nN}$ and the control input is $u = \left[u_1^\top, \ldots, u_n^\top\right]^\top \in \mathbb{R}^{nN}$. Then, denoting by I_N the $N \times N$ identity matrix, in compact form, the networked systems' *nodes* dynamics is

$$\dot{x} = -[L \otimes I_N]x, \quad x \in \mathbb{R}^{nN},\ L \in \mathbb{R}^{n\times n}. \tag{8}$$

However, following Zelazo et al. (2007) we introduce the following coordinate transformation, mapping the nodes' space to that of the edges,

$$z := [E^\top \otimes I_N]x, \quad z := [z_1^\top \cdots z_k^\top \cdots z_m^\top]^\top. \tag{9}$$

That is, for each pair of nodes with states x_i and $x_j \in \mathcal{V}$, the state $z_k \in \mathbb{R}^N$, with $k \le m$, denotes the state of the k-th arc, interconnecting x_i and x_j. In other words, $z_k := x_i - x_j$ where $i, j \in \mathcal{V}$ and $k \le m$. Therefore, from (9), it is clear that the agreement condition, $\{x_i = x_j,\ \forall (i,j) \in \mathcal{V}^2\}$, is equivalent to $\{z = 0\}$. This is significant because in the edge-variables' representation, consensus may be reformulated as a stabilisation problem of the origin for the system

$$\dot{z} = -[E^\top E_\odot \mathcal{W} \otimes I_N]z, \tag{10}$$

which is obtained by differentiating both sides of (9) and using (2) and (8). This leads to the so-called edge Laplacian matrix $L_e(\mathcal{G}) \in \mathbb{R}^{m\times m}$,

$$L_e := E^\top E_\odot \mathcal{W} \tag{11}$$

which lies at the basis of the edge-representation framework. The matrix L_e is an edge-variant of the graph Laplacian and, as such, it has the property of having the same non-zero eigenvalues as the graph Laplacian L, and $\mathrm{rank}(L_e) = \mathrm{rank}(L) = n-1$ —see, *e.g.*, Zeng et al. (2014).

Consensus (hence stability of the origin for (10)) holds if and only if there exists a directed spanning tree $\mathcal{G}_T \subseteq \mathcal{G}$. This suggests that the control design and analysis problems for (10) may be simplified by making a distinction between the state variables corresponding to the edges in an arbitrary directed spanning tree $\mathcal{G}_T$, that we denote z_t and the remaining edges, that we denote z_c. That is, as in Mukherjee and Zelazo (2018), we split the edges' states as

$$z = \left[z_t^\top \ \ z_c^\top\right]^\top, \quad z_t \in \mathbb{R}^{n-1},\ z_c \in \mathbb{R}^{m-n+1}. \tag{12}$$

Then, we replace the latter expression of z in (the first equation in) (9) and compare it to (12) to obtain

$$z_t := \left[E_t^\top \otimes I_N\right]x. \tag{13}$$

Furthermore, after (4) it is readily seen that the states of the arcs not contained in the tree $\mathcal{G}_T$, z_c, satisfy

$$z_c = \left[T^\top \otimes I_N\right]z_t. \tag{14}$$

Other identity is obtained from (5) and (14); we have

$$z = \left[R^\top \otimes I_N\right]z_t \tag{15}$$

which, together with (6) and (10), implies that

$$\dot{z}_t = -\left[E_t^\top E_\odot \mathcal{W} R^\top \otimes I_N\right]z_t. \tag{16}$$

The latter equation has the advantage of having a reduced dimension with respect to (10) and, yet, in view of (14), it still captures the behaviour of the overall network. Thus, consensus is achieved if the origin for (10), or equivalently for (16), is asymptotically stable. The consensus problem for the system (16), with the weight matrix $\mathcal{W} = I_m$ and with linear interconnections, has been widely studied in the literature, including using Lyapunov's direct method —see Mukherjee and Zelazo (2018); Zeng et al. (2014).

In addition to consensus, the objective is to respect some proximity constraints that guarantee a reliable communication between any pair of nodes. This is defined as follows.

Definition 1. (Connectivity maintenance). For each $k \le m$, let $\Delta_k > 0$ denote the maximal distance between the nodes i and j such that the communication between them, through the arc $e_k = (i, j)$, is reliable. We say that the graph connectivity is maintained (hence, the proximity constraint holds) if the set

$$\mathcal{J} := \left\{z \in \mathbb{R}^{mN} : |z_k| < \Delta_k,\ \forall\, k \le m\right\}, \tag{17}$$

where $z_k = x_i - x_j$, is forward invariant. That is, if $|z_k(0)| < \Delta_k$ implies that $z(t) \in \mathcal{J}$ for all $t \ge 0$.

The design of the controller that we employ is based on so-called barrier Lyapunov functions; this leads to a *nonlinear* gradient-based control law. In other words, the closed-loop system is of the form (16), but with nonlinear interconnections that we refer to as a *connectivity potential.*

Definition 2. (Connectivity potential). Let $p_0 \in \mathbb{R}$ and, for each $k \le m$, let $\mathcal{B}_{\Delta_k} := \{z_k \in \mathbb{R}^N : |z_k| < \Delta_k\}$, $\alpha_k : \left[0, \Delta_k^2\right) \to \mathbb{R}_{\ge 0}$, $s \mapsto \alpha_k(s)$, be of class $\mathcal{C}^1$ on $\left[0, \Delta_k^2\right)$ and $p_k : \mathcal{B}_{\Delta_k} \to \mathbb{R}_{>0}$ be a continuous function such that

- $p_k(z_k) := \dfrac{\partial \alpha_k}{\partial s}(|z_k|^2) \ge p_0 > 0$, for all $|z_k| < \Delta_k$,
- $\alpha_k(s) \to \infty$ as $s \to \Delta_k^2$, so $p_k(z_k) \to \infty$ as $|z_k| \to \Delta_k$

Then, the connectivity potential $P(z)$ is a positive-definite matrix defined as $P(z) := \mathrm{diag}\left[p_k(z_k)\right] \in \mathbb{R}^{m\times m}$.

Then, the *Barrier* function $U_k : \mathcal{B}_{\Delta_k} \to \mathbb{R}_{\geq 0}$ is defined as

$$U_k(z_k) := \alpha_k(|z_k|^2). \tag{18}$$

Barrier functions are reminiscent of Lyapunov functions; as such, they have the property of being positive definite and radially unbounded, but $U_k(z_k)$ grows unboundedly as z_k approaches the border of an open set. Note that a connectivity potential may be regarded as the gradient of a barrier function, in the sense that

$$\frac{\partial U_k}{\partial z_k} = 2p_k(z_k)z_k, \quad \forall\, k \leq m. \tag{19}$$

Remark 2. For examples of Barrier functions satisfying the previous definition (albeit taking values in the *nodes* space), see the so-called "edge tension" function in Ji and Egerstedt (2007) and Boskos and Dimarogonas (2017). The Barrier Lyapunov Functions from Tang et al. (2013) and references therein constitute other examples. •

In the sequel, we present distributed control laws that solve the consensus-with-connectivity-maintenance problem for a directed spanning-tree, as well as the proposed strict Lyapunov functions.

3. MAIN RESULTS

3.1 Consensus with preserved connectivity

Consider a network of n dynamical systems (7) interconnected through a directed spanning tree graph $\mathcal{G}_T$. Because the graph is cycle-free, in the edge-coordinates representation, we have $z = z_t$, $E = E_t$, and

$$\dot{z}_t = \big[E_t^\top \otimes I_N\big]u. \tag{20}$$

Based on the barrier functions U_k the control law is defined as the gradient control law

$$u(z_t) = -c\big[E_{\odot t}P(z_t) \otimes I_N\big]z_t, \tag{21}$$

where $c > 0$ is the network's connectivity strength —*cf.* Panteley and Loría (2017), and the matrix $P(z_t) := \mathrm{diag}\big[p_k(z_k)\big]$ models the interconnections —see Definition 2. Hereafter, it is assumed without loss of generality that the weights matrix $\mathcal{W} = I_m$.

We emphasise that each component of u depends only on local information since $E_{\odot t}$ represents the incoming edges on each node, that is, the available information to each agent as defined by the digraph.

Proposition 1. Consider n systems as in (7) interconnected through a directed spanning tree $\mathcal{G}_T$ with proximity constraints. Then, the control law (21) guarantees that $z_k \to 0$ for all $k \leq m$ and the set $\mathcal{J}$, defined in (17), is forward invariant for the closed-loop trajectories. Furthermore the function

$$V(z_t) = \sum_{k \leq n-1} \gamma_k U_k(z_k), \tag{22}$$

where $\gamma_k > 0$ are design parameters and the functions U_k are defined in (18), is a strict Lyapunov function for system (20) with input (21).

Proof. First, we derive the closed-loop equation. To that end, we replace (21) into (20) and, akin to (11) albeit with an abuse of notation, we define the edge-Laplacian matrix $L_{et} \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ as $L_{et} := E_t^\top E_{\odot t} I_{n-1}$. We obtain

$$\dot{z}_t = -c\big[L_{e_t}P(z_t) \otimes I_N\big]z_t. \tag{23}$$

Next, we differentiate the function V along the trajectories of (23). We use (19) and define $\Gamma \in \mathbb{R}^{(n-1)\times(n-1)}$, $\Gamma := \mathrm{diag}\,[\gamma_k]$ with $\gamma_k > 0$ yet to be determined, to obtain

$$\frac{\partial V(z_t)}{\partial z_t} = \quad 2\big[P(z_t)\Gamma \otimes I_N\big]z_t \tag{24}$$

and, in turn,

$$\begin{aligned}\dot{V}(z_t) &= -2c\; z_t^\top\big[P(z_t)\Gamma L_{et}P(z_t) \otimes I_N\big]z_t\\ &= -c\; z_t^\top\big[P(z_t)\big(\Gamma L_{e_t} + L_{e_t}^\top\Gamma\big)P(z_t) \otimes I_N\big]z_t.\end{aligned}$$

On the other hand, the right hand side of the previous equality is negative definite if $E_\odot$ is constructed by applying the labelling approach described earlier, but on the in-incidence matrix. Indeed, in this case, we have

$$E_{\odot t} = \begin{bmatrix} 0_{1\times(n-1)} \\ -I_{n-1}\end{bmatrix}. \tag{25}$$

Now, using the fact that $E_t = E_{\odot t} + E_{\otimes t}$, the directed spanning-tree edge Laplacian satisfies

$$L_{et} = E_t^\top E_{\odot t} = E_{\odot t}^\top E_{\odot t} + E_{\otimes t}^\top E_{\odot t} =: I - B \tag{26}$$

where we defined $B := -E_{\otimes t}^\top E_{\odot t}$ and we used (25) to show that $E_{\odot t}^\top E_{\odot t} = I$. Furthermore, since having $[E_{\otimes t}]_{ij} = 1$ implies $[E_{\odot t}]_{ij} = 0$ and, in view of the previous labelling, $[E_{\otimes t}^\top]_{ij} = 0$ for $i < j$, it follows that B is lower triangular matrix with zero diagonal and all other elements either equal to 0 or 1. Moreover, for a directed spanning tree, all the eigenvalues of L_{e_t} lie on the open left-hand complex plane and $\mathrm{rank}(L_{e_t}) = n - 1$; indeed, they coincide with the eigenvalues of the graph's Laplacian L. Thus, from the latter and (26), we conclude that L_{e_t} is a non-singular M-matrix (Plemmons (1977)), that is, a real matrix with non-positive off-diagonal elements and eigenvalues with strictly positive real parts. Now, after Plemmons (1977), every non-singular M-matrix is diagonally stable, that is, for any $Q = Q^\top > 0$, L_{et} admits a diagonal solution $\Gamma :=\mathrm{diag}\big[\gamma_k\big]$, to the Lyapunov inequality

$$\Gamma L_{e_t} + L_{e_t}^\top \Gamma \geq Q > 0. \tag{27}$$

Choosing γ_k in (22) so that (27) holds, and since $P(z_t) > 0$, we have

$$\begin{aligned}\dot{V}(z_t) \;&\leq\; -c\, z_t^\top\big[P(z_t)QP(z_t) \otimes I_N\big]z_t, \quad \forall\, z_t \in \mathbb{R}^{(n-1)N}\\ &\leq\; -c\,\lambda_{min}(Q)\big|\big[P(z_t) \otimes I_N\big]z_t\big|^2 < 0,\ \forall z_t \neq 0,\end{aligned} \tag{28}$$

where $\lambda_{min}(Q)$ is the smallest eigenvalue of Q. Thus, V in (22) is a strict Lyapunov function for (23).

Now we show that $\mathcal{J}$ is forward invariant along closed-loop solutions. We proceed by contradiction. Suppose that there exists $T > 0$ such that for all $t \in [0, T)$, $z_t(t) \in \mathcal{J}$ and $z_t(T) \notin \mathcal{J}$. That is, we have $|z_k(t)| \to \Delta_k$ as $t \to T$ for at least one $k \leq n - 1$. Then, by definition, we have $V(z_t(t)) \to \infty$ as $t \to T$. This, however, is in contradiction with (28), which implies that $V(z_t(t)) \leq V(z_t(0)) < \infty$ for all $t \geq 0$. Connectivity maintenance follows.

Now, define the set $\mathcal{J}_\varepsilon := \{z_t \in \mathbb{R}^{(n-1)N} : |z_k| < \Delta_k - \varepsilon, \forall\, k \leq n-1\}$ and its closure $\bar{\mathcal{J}}_\varepsilon$, where $\varepsilon > 0$ is an arbitrarily small constant. We see that $V(z_t)$ is positive definite for all $z_t \in \bar{\mathcal{J}}_\varepsilon$ and can be bounded as

$$\beta|z_t|^2 \leq V(z_t) \leq h(|z_t|) \tag{29}$$

where β is a positive constant and $h(\,\cdot\,)$ is a positive strictly increasing function defined everywhere in $\bar{\mathcal{J}}_\varepsilon$ and $h(0) = 0$.

This means that $V(z_t) \to 0$ as $z_t \to 0$. Therefore, from (28) and standard Lyapunov theory it follows that for all trajectories of the closed-loop system starting in $\mathcal{J}_\varepsilon$, the origin is attractive for all $z_k(0) \in \mathcal{J}_\varepsilon$ (and all $k \leq n-1$). Taking the limit $\varepsilon \to 0$, asymptotic stability of the origin for all trajectories starting in $\mathcal{J}$ follows. Thus, consensus is achieved and the proximity constraints are satisfied. ∎

3.2 Robustness of the spanning-tree topology

Consider now systems with an additive bounded disturbance, that is,

$$\dot{x}_i = u_i + d_i, \quad x_i \in \mathbb{R}^N. \tag{30}$$

For this system we have the following.

Proposition 2. The multiagent system (30), with a communication topology defined by a directed spanning tree $\mathcal{G}_T$ and under proximity constraints, in closed loop with the controller (21) is input-to-state stable with respect to disturbance $d := \left[d_1^\top \cdots d_n^\top\right]^\top \in \mathbb{R}^{nN}$. Furthermore, the digraph remains connected in the presence of bounded d.

Proof. Applying the edge transformation (13) and the control (21), we obtain the closed-loop equation $-cf.$ (23)

$$\dot{z}_t = -c\left[E_t^\top E_{\odot t} P(z_t) \otimes I_N\right] z_t + \left[E_t^\top \otimes I_N\right] d. \tag{31}$$

Consider again the Lyapunov function $z_t \mapsto V(z_t)$ as in (22); using (28), we obtain that its total derivative along the trajectories of (31) satisfies

$$\dot{V}(z_t) \leq -c'\left|\left[P(z_t) \otimes I_N\right] z_t\right|^2 + 2z_t^\top\left[P(z_t)\Gamma E_t^\top \otimes I_N\right] d$$

where $c' := c\lambda_{min}(Q)$. Now, given c, Q, and Γ, let $\delta > 0$ be such that

$$c' := c\,\lambda_{min}(Q) - \frac{1}{\delta}\lambda_{max}(\Gamma E_t^\top E_t \Gamma) > 0.$$

It follows from Young's inequality that

$$\dot{V}(z_t) \leq -\,c'\left|\left[P(z_t) \otimes I_N\right] z_t\right|^2 + \delta|d|^2 \tag{32}$$

and from the latter and $P(z_t) \geq p_0$ —see Definition 2, we have

$$\dot{V}(z_t) \leq -c' p_0^2 |z_t|^2 + \delta|d|^2, \tag{33}$$

so the system (31) is input-to-state stable.

To assert connectivity maintenance in presence of additive disturbances it suffices to show that in the proximity of the limits of the connectivity region, that is, as $|z_k| \to \Delta_k$ for any $k \leq m$, the first term on the right-hand side of (32) dominates over the second term. Let $\varepsilon > 0$ be arbitrarily fixed and let $z_t \in \mathbb{R}^{n-1}$ be such that, for some $k \leq m$ we have $|z_k| \geq (\Delta_k - \varepsilon)$. Then, $|z_t| \geq (\Delta_k - \varepsilon)$ and it follows from (32) that

$$\dot{V}(z_t) \leq -c'|P(z_t)|^2(\Delta_k - \varepsilon)^2 + \delta|d|^2, \tag{34}$$

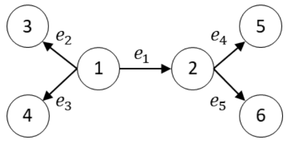
which, from $|P(z_t)| \geq p_k((\Delta_k - \varepsilon)^2)$, implies that

$$\dot{V}(z_t) \leq -c' p_k((\Delta_k - \varepsilon)^2)^2(\Delta_k - \varepsilon)^2 + \delta|d|^2.$$

The previous reasoning holds for any z_k arbitrarily close to the boundary Δ_k, that is, for arbitrarily small ε. Furthermore, since $p_k(s) \to \infty$ as $s \to \Delta_k^2$, then $|P(z_t)| \to \infty$ as $\varepsilon \to 0$. It follows that $\dot{V}(z_t) \leq 0$ for sufficiently small $\varepsilon > 0$. The proof of connectivity maintenance follows similar arguments as in the proof of Proposition 1. ∎

4. SIMULATION RESULTS

In this section, we present simulation results that demonstrate the performance of the connectivity-preserving consensus algorithm analysed above. We considered a system composed of six agents, whose communication topology is given by the spanning tree digraph in Figure 1.



$$E_{\odot t} = \begin{bmatrix} 0_{1\times(n-1)} \\ -I_{n-1} \end{bmatrix}$$

Fig. 1. Directed spanning tree for 6 agents

For the simulations, we considered each agent to be described by a single integrator system subject to a vanishing perturbation —see (30). The perturbations were modelled as $d_i(t) = \bar{d}_i(t)\,[1\ 1]^\top$, with

$$\bar{d}_i(t) = \begin{cases} -2.4(\tanh(2(t-15)) - 1) + \dfrac{1}{(t+10)}, & i = \{3, 5\} \\ 2.4(\tanh(2(t-15)) - 1) + \dfrac{1}{(t+10)}, & i = \{2\} \\ 0, \ i = \{1, 4, 6\}. \end{cases} \tag{35}$$

The input perturbation is set to take its maximal value at $t = 0$, since, for the considered simulation, the agents are closest to the boundaries of the proximity regions at $t = 0$.

Furthermore, two scenarii were considered. For the first scenario, we used the gradient control law proposed in (21), where the barrier function was defined as

$$U_k(z_k) = |z_k|^2 + \ln\left(\frac{\Delta_k^2}{\Delta_k^2 - |z_k|^2}\right) \tag{36}$$

and we set $c = 2$ and $\gamma_k = 1$, for all $k \leq 5$. Hence, the expression for each u_i is as follows:

$$u_i = -2\sum_{k\leq m}[E_\odot]_{ik}\left(1 + \frac{1}{\Delta_k^2 - |z_k|^2}\right) z_k. \tag{37}$$

For the second scenario, we used the edge-based consensus algorithm without connectivity maintenance proposed in Mukherjee and Zelazo (2018), of the form

$$u_i = -2\sum_{k\leq m}[E_\odot]_{ik} z_k. \tag{38}$$

For both scenarios, the initial conditions for the agents were set to $x_1(0) = [2.4,\ 0]$, $x_2(0) = [-0.58,\ -0.9]$, $x_3(0) = [4.5,\ 2]$, $x_4(0) = [5,\ -2]$, $x_5(0) = [-4.2,\ -0.45]$, and $x_6(0) = [-2,\ -4.2]$ and the radii of the connectivity regions were assumed to be $\Delta = [2.5,\ 3.2,\ 3.3,\ 3.9,\ 3.7,\ 4]$. In Figure 2 we show the evolution of the edge states for the system with the proposed controller (37). It is clear from the Figure that once the disturbance vanishes, the edge states converge to the origin, which implies that consensus is achieved. Moreover, the distance constraints (dashed lines) are always respected, even in the presence of disturbance d. On the contrary, it can be seen from Figure 3 that the consensus algorithm (38) does not guarantee connectivity maintenance, hence consensus is not achieved.

It is worth mentioning that the disturbance d could be considered as a bounded additional control input aiming

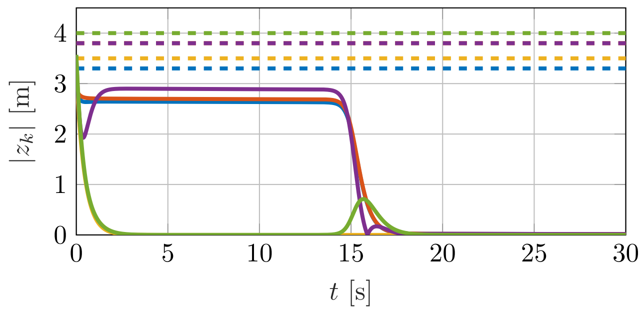


Fig. 2. Trajectories of the norm of the edges' states under control law (37). Dashed lines: distance constraints.

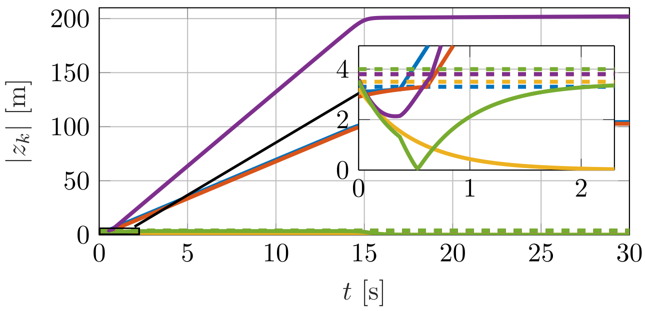


Fig. 3. Trajectories of the norm of the edges' states under control law (38). Dashed lines: distance constraints.

to achieve a secondary task. Therefore, as can be seen from the simulation results, the proposed controller is able to guarantee the respect of the distance constraints even in the presence of other multi-robot tasks more challenging from a connectivity maintenance point of view.

5. CONCLUSIONS

Using an edge-based representation, we constructed a strict Lyapunov function for the consensus problem with connectivity maintenance of a first-order system communicating over a directed spanning tree with proximity constraints. The proposed strict Lyapunov function allows to directly conclude uniform asymptotic stability of the agreement subspace as well as connectivity maintenance. Moreover, a strict Lyapunov function naturally leads to asserting strong robustness properties such as input-to-state stability with respect to additive disturbances. Such results are without precedent in the literature for directed graphs, even for first-order systems.

We are confident that our main results pave the way for significant extensions. Future and currently developed work addresses the consensus problem for more general digraphs as well as higher-order nonlinear systems. For instance, the results provided in this paper may be used as a starting block in considering additional inter-agent or information constraints such as collision/obstacle avoidance and quantised information.

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