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Adaptive Nonlinear Observer Design via a Polytopic Split of Signals ⋆

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Abstract: This paper provides a novel solution for adaptive observer design based on a polytopic representation of the error system. Thanks to a recently appeared factorisation, the proposal is able to deal with fully nonlinear system matrices and nonlinear outputs while dropping conditions on Lipschitz bounds and persistence of excitation. Examples are provided that illustrate the effectiveness and advantages of the new methodology over former approaches.

Keywords: State observers, Adaptive systems, Nonlinear systems, Lyapunov methods, Synchronization.

1. INTRODUCTION

The problem of observation in control systems has been studied for many decades as most control laws require access to the entire state vector, which usually is not fully available in real-time setups (Luenberger, 1964). As a dual problem with respect to control, observation can adopt a static, dynamic, or adaptive form; this work is concerned with the latter class. Besides stabilisation, adaptive observers have been used to carry out a variety of tasks such as unknown-input estimation (Dimassi and Loría, 2010), unknown-parameter identification (Cho and Rajamani, 1997), fault detection and isolation (Yang and Saif, 1995), fault-tolerant control (Ye and Yang, 2006), and synchronisation of chaotic systems (Feki, 2003), among others.

Some early works on adaptive observers are (Luders and Narendra, 1973; Carroll and Lindorff, 1973; Kreisselmeier, 1977), where the state-estimation and parameter-estimation problems were solved for linear time-invariant (LTI) single-input-single-output (SISO) systems. By taking into account these works, new approaches have been developed for the time-variant case: in (Bastin and Gevers, 1988) an adaptive observer has been proposed for SISO systems that can be put into the canonical observer form; in (Zhang, 2002; Zhao et al., 2012) adaptive observers were proposed to achieve the estimation task in multiple-input-multiple-output (MIMO) systems; these works show that state and parameter estimations converge to their real values if persistency of excitation (PE) is guaranteed. This condition can also be found in (Besançon, 2000; Farza et al., 2009; Loría et al., 2009; Farza et al., 2018).

Problem statement: Particularly, in (Loría et al., 2009) adaptive observers for synchronisation of chaotic systems of the form $\dot{x} = A(y)x + \Psi(x)\theta + B(t,x)$ have been proposed, where y is a measurable output, θ gathers constant parametric uncertainties, and x is the state to be estimated; $A(\cdot)$, $\Psi(\cdot)$, and $B(\cdot,\cdot)$ are allowed to be nonlinear. Nevertheless, the solution therein relies on Lipschitz bounds and conditions to guarantee PE; it does not allow the system matrix to depend on something else than the (known) linear output, thus limiting its applicability.

Contribution: Inspired by (Loría et al., 2009), this paper proposes a novel nonlinear adaptive observer for state estimation that overcomes some of the limitations of the referred work, namely: the system matrix and the output are now allowed to depend nonlinearly on any bounded signal, and conditions based on PE and Lipschitz constants can be avoided by a suitable factorisation of the error signal. For the sake of clarity, no parametric uncertainties are considered in this work, though extensions to that case are straightforward.

Methodology: The factorisation proposed in (Quintana et al., 2018) is used to construct an error system with explicitly known structure. By means of the sector nonlinearity approach (Taniguchi et al., 2001), this system is exactly rewritten as a convex sum of linear models where available signals are split from unmeasurable ones in a natural way. The direct Lyapunov method is combined with polytopic argumentations to guarantee convergence of the observation error to zero: an adaptive nonlinear observer gain is thus proposed.

Organisation: A first result on nonlinear adaptive observer design is provided in section 2, though strong limitations arise that are then considered and solved in section 3 via a polytopic representation of the error system. The effectiveness of the proposal is put at test and compared against former methodologies in section 4 via 3 nonlinear examples: 1 concerned with synchronisation, 2 others with nonlinear expressions for the system or the output. Finally, in section 5 conclusions and perspectives are discussed.

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Notation: Throughout this paper, given a matrix expression M, $M > 0$ ($M < 0$) stands for positive-definite (negative-definite), $M + (*) = M + M^T$. The convex hull of a set of vertices is denoted as **co**.

2. LYAPUNOV-BASED NONLINEAR ADAPTIVE OBSERVER DESIGN

Consider the following nonlinear system

$$\dot{x}(t) = A(x)x(t) + B(y)u(t), \quad y(t) = C(x)x(t), \tag{1}$$

where $x(t) \in \Omega \subset \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^o$ are the state, input, and output vectors, respectively, with $0 \in \Omega$; $A(\cdot)$, $B(\cdot)$, and $C(\cdot)$ are matrices of appropriate dimensions whose entries are bounded in Ω. Notice that matrix B is only allowed to depend on the system output.

Consider also the following nonlinear observer associated to (1)

$$\begin{aligned}\dot{\hat{x}}(t) &= A(\hat{x})\hat{x}(t) + B(y)u(t) + L(\hat{x},y,t)(y-\hat{y}),\\ \hat{y}(t) &= C(\hat{x})\hat{x}(t),\end{aligned} \tag{2}$$

where $\hat{x}(t) \in \hat{\Omega} \subset \mathbb{R}^n$ is the observer state and $L(\hat{x}, y, t) \in \mathbb{R}^{n\times o}$ is a possibly nonlinear observer gain; this gain should be designed such that the observation error $e(t) = x(t) - \hat{x}(t)$ goes to zero as time goes to infinity, i.e., $\lim_{t\to\infty} e(t) = 0$. Notice that asking the observer state $\hat{x}$ to lie in the compact set $\hat{\Omega}$ is somehow related with asking for Lipschitz bounds in classical observer design with unmeasured premise variables (Lendek et al., 2010) [1].

According to (Quintana et al., 2018), the error system can always be expressed as:

$$\dot{e}(t) = \left(\bar{A}(x,\hat{x}) - L(\hat{x},y,t)\bar{C}(x,\hat{x})\right) e(t), \tag{3}$$

where $\bar{A}(x, \hat{x})e(t) = A(x)x - A(\hat{x})\hat{x}$ and $\bar{C}(x, \hat{x})e(t) = C(x)x - C(\hat{x})\hat{x}$ are given explicitly. Let us illustrate this explicit rewriting by considering $p(x) = x_1 + x_1^2 x_2 + x_2^3$ as a polynomial expression appearing in some entry of $A(x)x$ or $C(x)x$; then, one of the possible factorisations of the error signals $e_1 = x_1 - \hat{x}_1$ and $e_2 = x_2 - \hat{x}_2$ from $p(x) - p(\hat{x})$ is

$$\begin{aligned}p(x) - p(\hat{x}) &= x_1 - \hat{x}_1 + x_1^2 x_2 - \hat{x}_1^2 \hat{x}_2 + x_2^3 - \hat{x}_2^3\\ &= e_1 + x_1^2 e_2 + \hat{x}_2\left(x_1^2 - \hat{x}_1^2\right) + \left(x_2^2 + x_2\hat{x}_2 + \hat{x}_2^2\right) e_2\\ &= e_1 + x_1^2 e_2 + \hat{x}_2\left(x_1 + \hat{x}_1\right) e_1 + \left(x_2^2 + x_2\hat{x}_2 + \hat{x}_2^2\right) e_2\\ &= \begin{bmatrix} 1 + \hat{x}_2(x_1 + \hat{x}_1) & x_1^2 + x_2^2 + x_2\hat{x}_2 + \hat{x}_2^2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.\end{aligned}$$

When $p(x)$ is non-polynomial, the Taylor series approach can be used to convert it to a Taylor polynomial of an arbitrary degree; then the procedure illustrated above to factorise $e(t)$ is the same.

Theorem 1. The origin of the nonlinear error system (3) is asymptotically stable if there exist matrices $P(t)$, $Q(t) \in \mathbb{R}^{n\times n}$ such that $0 < c_1 I \leq P(t) = P^T(t) \leq c_2 I$, $Q(t) = Q^T(t) > 0$, with $L(\hat{x}, y, t) = P^{-1}(t)\bar{C}^T(x, \hat{x})$ and

$$\dot{P}(t) = \bar{C}^T(x,\hat{x})\bar{C}(x,\hat{x}) - P(t)\bar{A}(x, \hat{x}) - \frac{1}{2}Q(t) + (*), \tag{4}$$

$\forall x \in \Omega$, $\forall \hat{x} \in \hat{\Omega}$, $\forall t \geq 0$. Moreover, any trajectory $e(t)$ starting within $\{e : e^T P(t)e \leq k, k > 0\} \subset \Omega_e$, with $\Omega_e = \{e : x \in \Omega, \hat{x} \in \hat{\Omega}\}$, goes to zero as time goes to infinity.

Proof. Since $P(t) = P^T(t) > 0$, then $V(t, e) = e^T P(t)e > 0$ is a Lyapunov function candidate; its time derivative is

$$\begin{aligned}\dot{V}(t, e) &= e^T \left(P(t)\bar{A}(x,\hat{x}) - P(t)L(\hat{x},y,t)\bar{C}(x,\hat{x}) + (*) + \dot{P}(t)\right) e\\ &= e^T \left(P(t)\bar{A}(x,\hat{x}) - \bar{C}^T(x,\hat{x})\bar{C}(x,\hat{x}) + (*) + \dot{P}(t)\right) e,\end{aligned}$$

where the error system (3) and $L(\hat{x}, y, t)$ have been substituted. Clearly, the condition $\dot{V}(t, e) < 0 \ \forall e \neq 0$ can be guaranteed if

$$P(t)\bar{A}(x,\hat{x}) - \bar{C}^T(x,\hat{x})\bar{C}(x,\hat{x}) + (*) + \dot{P}(t) = -Q(t),$$

which is equivalent to (4), thus proving that $V(t, e)$ is a valid Lyapunov function for the error system (3). It follows immediately that any trajectory beginning in the outermost Lyapunov level set within Ω_e goes asymptotically to zero as time goes to infinity; such Lyapunov level set has the form $\{e : e^T P(t)e \leq k, k > 0\}$, which concludes the proof.

Theorem 1 can be useful in the very particular case of having only available signals in the righthand side of (4) as illustrated in the next example.

Example 1. Consider the Lorenz oscillator given by equations (Loría et al., 2009):

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-16 & 16 & 0\\ 45.6 & -1 & -x_1\\ 0 & x_1 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}, \quad y = [1\ 0\ 0]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}, \tag{5}$$

where x_1, x_2, and x_3 are the states and $y = x_1$ is the (linear) output.

In this case, a nonlinear observer of the form (2) is:

$$\begin{bmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\\ \dot{\hat{x}}_3\end{bmatrix} = \begin{bmatrix}-16 & 16 & 0\\ 45.6 & -1 & -\hat{x}_1\\ 0 & \hat{x}_1 & -4\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\\ \hat{x}_3\end{bmatrix} + L(\hat{x}_1,\hat{x}_2,\hat{x}_3,y,t)(y - \hat{y})$$

where $\hat{x}_1$, $\hat{x}_2$, and $\hat{x}_3$ are the observer states and $\hat{y} = \hat{x}_1$ is the observer output.

Defining $e_i = x_i - \hat{x}_i$, $i \in \{1, 2, 3\}$, the error system (3) is

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\end{bmatrix} = \left(\begin{bmatrix}-16 & 16 & 0\\ 45.6 - \hat{x}_3 & -1 & -x_1\\ \hat{x}_2 & x_1 & -4\end{bmatrix} - L(\hat{x}_1,\hat{x}_2,\hat{x}_3,y,t)[1\ 0\ 0]\right)\begin{bmatrix}e_1\\ e_2\\ e_3\end{bmatrix},$$

where the fact that $x_1x_3 - \hat{x}_1\hat{x}_3 = x_1e_3 + \hat{x}_3e_1$ and $x_1x_2 - \hat{x}_1\hat{x}_2 = x_1e_2 + \hat{x}_2e_1$ has been used to achieve such factorisation.

The observation problem can be solved by means of Theorem 1 because all the signals in the nonlinear error are available, i.e., x_1 (the system output), $\hat{x}_2$, and $\hat{x}_3$. A simulation is performed by implementing the dynamic $P(t)$ in (4) along with the system and observer dynamics, from which the adaptive nonlinear observer gain $L(\hat{x}_1, \hat{x}_2, \hat{x}_3, y, t)$ is obtained on-line. Fig. 1 shows the behaviour of the error signals and the Lyapunov function along the time. Initial conditions $x(0) = [1\ 1\ 1]^T$, $\hat{x}(0) = [0\ 0\ 0]^T$, and

$$P(0) = \begin{bmatrix}5 & 1 & 2\\ 1 & 8 & 3\\ 2 & 3 & 9\end{bmatrix}, \quad \sigma\left(P(0)\right) = \{4.15, 5.65, 12.2\},$$

[1] Indeed, assumptions on the boundedness of the time-variant difference between the true and the estimated states in Bergsten and Driankov (2002), or for gradient expressions in Guerra et al. (2018), imply $\hat{x}$ belongs to some compact set.

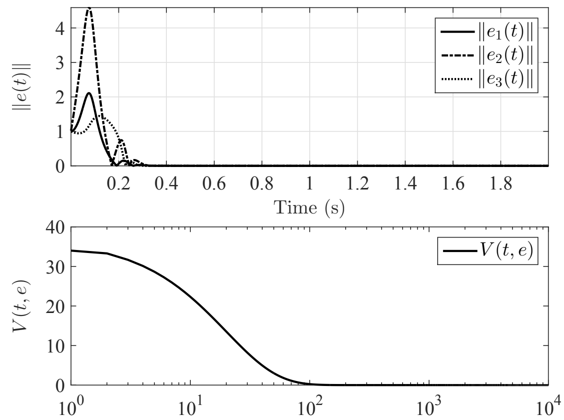

Fig. 1. Time evolution of the error norms and Lyapunov function $V(t,e)$ in example 1.

were employed. Matrix $Q(t)$ was chosen as a multiple of $P(t)$ to guarantee its definite-positiveness, more specifically, $Q(t) = 50P(t)$. Notice that the error signals go to zero and the Lyapunov function from Theorem 1 is monotonically decreasing, as expected.

Yet, as the following example shows, Theorem 1 cannot be used in most cases because $\dot{P}(t)$ usually depends on non-available signals:

Example 2. Consider the following nonlinear system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -x_1\\ -1 & -x_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ -1\end{bmatrix}u, \quad y = [x_1 \ 1]\begin{bmatrix}x_1\\ x_2\end{bmatrix}, \quad (6)$$

where x_1 and x_2 are the states and $y = x_1^2 + x_2$ is the nonlinear output.

By considering the following nonlinear observer

$$\begin{bmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\end{bmatrix} = \begin{bmatrix}0 & -\hat{x}_1\\ -1 & -\hat{x}_2\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix} + \begin{bmatrix}1\\ -1\end{bmatrix}u + L(\hat{x}_1,\hat{x}_2,y,t)(y-\hat{y}), \quad (7)$$

where the states are $\hat{x}_1$, $\hat{x}_2$, and $\hat{y} = \hat{x}_1^2 + \hat{x}_2$ is the observer output, the following error system is obtained

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\end{bmatrix} = \left(\begin{bmatrix}-\hat{x}_2 & -x_1\\ -1 & -x_2-\hat{x}_2\end{bmatrix} - L(\hat{x}_1,\hat{x}_2,t)\,[x_1+\hat{x}_1 \ 1]\right)\begin{bmatrix}e_1\\ e_2\end{bmatrix}. \quad (8)$$

Notice that unavailable states x_1 and x_2 appear in the error system above; therefore, Theorem 1 cannot be applied.

In order to overcome this problem, the fact that the dynamical equation (4) can be equivalently rewritten as a tensor product model by means of the sector nonlinearity approach (Taniguchi et al., 2001), can be used in the spirit of (Quintana et al., 2018) to split available signals ($\hat{x}$, y) from the remaining ones. The next section will show that convexity is useful to solve this problem under mild considerations.

3. NONLINEAR ADAPTIVE OBSERVERS AND TENSOR PRODUCT MODELS

Due to the bounds in the entries of the original system (1) and the nonlinear observer (2), those in (3) are also bounded. Bounded expressions can be rewritten as a convex sum of their bounds, i.e., given an expression $z \in [z^0, z^1]$, the following holds:

$$z = \underbrace{\frac{z^1 - z}{z^1 - z^0}}_{w_0(z)} z^0 + \underbrace{\frac{z - z^0}{z^1 - z^0}}_{w_1(z)} z^1,$$

where $w_0(z) + w_1(z) = 1$, $w_0, w_1 \in [0,1]$ for the referred interval in z: this is known as the convex sum property (Bertsekas et al., 2003).

Since $x(t)$, $\hat{x}(t)$ are bounded in the compact sets Ω and $\hat{\Omega}$, respectively, the non-constant terms in the entries of matrices $\bar{A}(x,\hat{x})$ and $\bar{C}(x,\hat{x})$ can be rewritten as convex sums. Thus, by using the notation in (Quintana et al., 2018), we denote as $z_i(\hat{x},y) \in [z_i^0, z_i^1]$, $i \in \{1,2,\ldots,p\}$ the p non-constant bounded terms that depend exclusively on available signals and as $\zeta_j(x,\hat{x}) \in [\zeta_j^0, \zeta_j^1]$, $j \in \{1,2,\ldots,\rho\}$, the remaining ones. Due to convexity, the convex sums as well as the convex functions can be grouped at the leftmost side of any expression that contains them. In this way, the error system (3) can be expressed as the following algebraically equivalent tensor product model[2]:

$$\dot{e}(t) = \sum_{\mathbf{i}\in\mathbb{B}^p}\sum_{\mathbf{j}\in\mathbb{B}^\rho} \mathbf{w}_\mathbf{i}(\hat{x},y)\boldsymbol{\omega}_\mathbf{j}(x,\hat{x})\left(\bar{A}_\mathbf{ij} - L(\hat{x},y,t)\bar{C}_\mathbf{ij}\right)e(t), \quad (9)$$

where $\mathbb{B} = \{0,1\}$, $\mathbf{i} = (i_1, i_2, \ldots, i_p)$, $\mathbf{j} = (j_1, j_2, \ldots, j_\rho)$, $\mathbf{w}_\mathbf{i}(\hat{x},y) = w_{i_1}^1 w_{i_2}^2 \cdots w_{i_p}^p$, $w_0^i = (z_i^1 - z_i(\hat{x},y))/(z_i^1 - z_i^0)$, $w_1^i = 1 - w_0^i$, $i \in \{1,2,\ldots,p\}$, $\boldsymbol{\omega}_\mathbf{j}(x,\hat{x}) = \omega_{j_1}^1 \omega_{j_2}^2 \cdots \omega_{j_\rho}^\rho$, $\omega_0^j = (\zeta_j^1 - \zeta_j(x,\hat{x}))/(\zeta_j^1 - \zeta_j^0)$, $j \in \{1,2,\ldots,\rho\}$, $\bar{A}_\mathbf{ij} = \bar{A}(x,\hat{x})\big|_{\mathbf{w}_\mathbf{i}\boldsymbol{\omega}_\mathbf{j}=1}$, $\bar{C}_\mathbf{ij} = \bar{C}(x,\hat{x})\big|_{\mathbf{w}_\mathbf{i}\boldsymbol{\omega}_\mathbf{j}=1}$.

It is important to highlight that the tensor product model (9) is not an approximation of the nonlinear error system (3), but an exact representation.

Thus, based on the above convex rewriting, the following result can be stated.

Theorem 2. The origin of the nonlinear error system (3) with tensor product model (9), is asymptotically stable if there exist matrices $P(t)$, $Q(t) \in \mathbb{R}^{n\times n}$ such that $0 < c_1 I \le P(t) = P^T(t) \le c_2 I$, $Q(t) = Q^T(t) > 0$, and

$$\dot{P}(t) = \sum_{\mathbf{j}\in\mathbb{B}^\rho}\sum_{\mathbf{l}\in\mathbb{B}^\rho}\bar{\boldsymbol{\omega}}_\mathbf{j}\bar{\boldsymbol{\omega}}_\mathbf{l}\Bigg(\sum_{\mathbf{i}\in\mathbb{B}^p}\sum_{\mathbf{k}\in\mathbb{B}^p}\mathbf{w}_\mathbf{i}(\hat{x},y)\mathbf{w}_\mathbf{k}(\hat{x},y) \times\left(\bar{C}_\mathbf{kl}^T\bar{C}_\mathbf{ij} - P(t)\bar{A}_\mathbf{ij} - \frac{1}{2}Q(t) + (*)\right)\Bigg), \quad (10)$$

$\forall x \in \Omega$, $\forall \hat{x} \in \hat{\Omega}$, $\forall t \ge 0$, with $\bar{\boldsymbol{\omega}}_\mathbf{j} = \bar{\omega}_{j_1}^1 \bar{\omega}_{j_2}^2 \cdots \bar{\omega}_{j_\rho}^\rho$ and $\bar{\boldsymbol{\omega}}_\mathbf{l} = \bar{\omega}_{l_1}^1 \bar{\omega}_{l_2}^2 \cdots \bar{\omega}_{l_\rho}^\rho$ defined through arbitrary known functions $\bar{\omega}_0^i$, $\bar{\omega}_1^i$, $i \in \{1,2,\ldots,\rho\}$, such that $\bar{\omega}_0^i \in [0,1]$, $\bar{\omega}_1^i = 1 - \bar{\omega}_0^i$, and the nonlinear observer gain as

$$L(\hat{x},y,t) = P^{-1}(t)\sum_{\mathbf{l}\in\mathbb{B}^\rho}\bar{\boldsymbol{\omega}}_\mathbf{l}\sum_{\mathbf{k}\in\mathbb{B}^p}\mathbf{w}_\mathbf{k}C_\mathbf{kl}^T. \quad (11)$$

Moreover, any trajectory starting within $\{e : e^T P(t) e \le k, k > 0\} \subset \Omega_e$, with $\Omega_e = \{e : x \in \Omega, \hat{x} \in \hat{\Omega}\}$, goes to zero as time goes to infinity.

[2] In rewriting the tensor product model (9), it is not necessary to define a new convex function for expressions like z_i^r; e.g., $z_i^2 = z_i z_i$ can be rewritten as

$$z_1^2 = \left(\sum_{i_1=0}^{1} w_{i_1}^1 z_1^{i_1}\right)\left(\sum_{i_2=0}^{1} w_{i_2}^1 z_1^{i_2}\right) = \sum_{i_1=0}^{1}\sum_{i_2=0}^{1} w_{i_1}^1 w_{i_2}^1 z_1^{i_1} z_1^{i_2},$$

where only a new index was added (but not a nonlinearity).

Proof. Suppose that conditions in Theorem 2 hold. By expanding (10) we have

$$\begin{aligned}\dot{P}(t)=&\sum_{\mathbf{l}\in\mathbb{B}^\rho}\bar{\boldsymbol{\omega}}_{\mathbf{l}}\left(\sum_{\mathbf{k}\in\mathbb{B}^p}\mathbf{w}_{\mathbf{k}}\bar{C}_{\mathbf{kl}}^T\right)\sum_{\mathbf{j}\in\mathbb{B}^\rho}\bar{\boldsymbol{\omega}}_{\mathbf{j}}\left(\sum_{\mathbf{i}\in\mathbb{B}^p}\mathbf{w}_{\mathbf{i}}\bar{C}_{\mathbf{ij}}\right)\\ &-P(t)\sum_{\mathbf{j}\in\mathbb{B}^\rho}\bar{\boldsymbol{\omega}}_{\mathbf{j}}\left(\sum_{\mathbf{i}\in\mathbb{B}^p}\mathbf{w}_{\mathbf{i}}\bar{A}_{\mathbf{ij}}\right)-\frac{1}{2}Q(t)+(*),\\ =&\tilde{C}^T(\hat{x},y)\tilde{C}(\hat{x},y)-P(t)\tilde{A}(\hat{x},y)-\frac{1}{2}Q(t)+(*),\end{aligned}$$

where

$$\tilde{A}(\hat{x},y)=\mathbf{co}\left\{\sum_{\mathbf{i}\in\mathbb{B}^p}\mathbf{w}_{\mathbf{i}}\bar{A}_{\mathbf{ij}}\right\},\ \tilde{C}(\hat{x},y)=\mathbf{co}\left\{\sum_{\mathbf{i}\in\mathbb{B}^p}\mathbf{w}_{\mathbf{i}}\bar{C}_{\mathbf{ij}}\right\}.$$

Clearly, $\dot{P}(t)$ is a differential inclusion as $\bar{\boldsymbol{\omega}}_{\mathbf{j}}$ and $\bar{\boldsymbol{\omega}}_{\mathbf{l}}$ are *arbitrary* convex functions. Then, by invoking Theorem 1, we can prove that the origin of

$$\dot{e}(t)=\big(\tilde{A}(\hat{x},y)-\tilde{C}(\hat{x},y)L(\hat{x},y,t)\big)e(t), \tag{12}$$

is asymptotically stable, which ensures the same for the origin of (3), because system (3) belongs to the differential inclusion (12). The same Theorem guarantees the existence of the Lyapunov set $\{e : e^TP(t)e \le k, k > 0\} \subset \Omega_e$, thus concluding the proof.

It is worth noticing that 1) the observer gain (11) depends only on available signals and it should use all of them to get more flexibility, and 2) $P(t)$ and $L(\hat{x},y,t)$ are calculated through each iteration, as can be seen in the block diagram of Fig. 2.

4. EXAMPLES

Example 2. (continued). Recall that the error system associated to the nonlinear system (6) and the nonlinear observer (7) is given by (8). Thus, by choosing the non-constant terms as $z_1 = \hat{x}_1 \in [-1, 1]$, $z_2 = \hat{x}_2 \in [-2, 2]$ (available), $\zeta_1 = x_1 \in [-1, 1]$ and $\zeta_2 = x_2 \in [-2, 2]$ (unavailable), a convex rewriting of the error system (8) in the compact sets $\Omega = \{x : |x_1| \le 1, |x_2| \le 2\}$ and $\hat{\Omega} = \{\hat{x} : |\hat{x}_1| \le 1, |\hat{x}_2| \le 2\}$ is:

$$\dot{e}=\sum_{\mathbf{i}\in\mathbb{B}^2}\sum_{\mathbf{j}\in\mathbb{B}^2}\mathbf{w}_{\mathbf{i}}\boldsymbol{\omega}_{\mathbf{i}}\left(\begin{bmatrix}-z_2^{i_2} & -\zeta_1^{j_1}\\ -1 & -\zeta_2^{j_2}-z_2^{i_2}\end{bmatrix}-L(\hat{x},t)\begin{bmatrix}\zeta_1^{j_1}+z_1^{i_1} & 1\end{bmatrix}\right)e,$$

where $\mathbf{w}_{\mathbf{i}}(z) = w_{i_1}^1 w_{i_2}^2$, $w_0^1 = 0.5 - 0.5\hat{x}_1$, $w_1^1 = 1 - w_0^1$, $w_0^2 = 0.5 - 0.25\hat{x}_2$, $w_1^2 = 1 - w_0^2$, and $\boldsymbol{\omega}_{\mathbf{j}}(\zeta) = \omega_{j_1}^1\omega_{j_2}^2$, $\omega_0^1 = 0.5 - 0.5x_1$, $\omega_1^1 = 1 - \omega_0^1$, $\omega_0^2 = 0.5 - 0.25x_2$, $\omega_1^2 = 1 - \omega_0^2$.

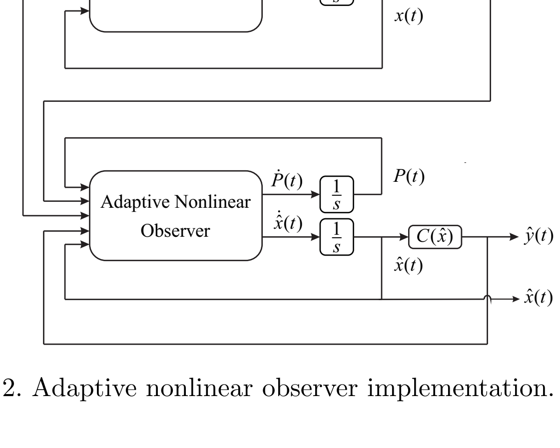

Fig. 2. Adaptive nonlinear observer implementation.

Once the error system is rewritten in a tensor product form, we can solve the observation problem by means of Theorem 2. Thus, replacing the convex functions $\boldsymbol{\omega}_{\mathbf{j}} = \omega_{j_1}^1\omega_{j_2}^2$ (depending on unavailable signals) by *known given* $\bar{\boldsymbol{\omega}}_{\mathbf{j}} = \bar{\omega}_{j_1}^1\bar{\omega}_{j_2}^2$, $\bar{\omega}_0^j = \bar{\omega}_1^j = 0.5$, $j \in \{1, 2\}$, the nonlinear observer gain (11) is

$$L(\hat{x}, t) = P^{-1}(t)\sum_{\mathbf{l}\in\mathbb{B}^2}\bar{\boldsymbol{\omega}}_{\mathbf{l}}\sum_{\mathbf{k}\in\mathbb{B}^2}\mathbf{w}_{\mathbf{k}}C_{\mathbf{kl}}^T.$$

Hence, following the scheme in Fig. 2, $P(t)$ and $L(\hat{x}, t)$ are computed on-line. Fig. 3 shows the behaviour of the nonlinear system states along with the nonlinear observer ones. Initial conditions are $x(0) = [-0.3\ 0.4]^T$, $\hat{x}(0) = [0\ 0]^T$, and $P(0) = I$, the input signal is $u(t) = 0.1\sin 5t + 0.2\sin 10t - 0.4\sin 20t$, and matrix $Q(t)$ is defined as $Q(t) = 20P(t)$. As it can be seen, the observation task takes place correctly.

Notice that, in contrast with results in Alessandri and Rossi (2015); Loría et al. (2009), the output is no longer limited to be linear; moreover, the system matrix is no longer required to depend exclusively on the output as in Loría et al. (2009). In this example, however, results in Loría et al. (2009) can still be used if the system is rewritten as $\dot{x} = Ax + B(t, x)$, but then Lipschitz conditions should be employed for $B(t, x)$. In Martínez-García et al. (2019), the systems under consideration are of the form $\dot{x}(t) = Ax + g(y, u)$ with a linear output; nevertheless, system (6) cannot be expressed in that form.

Example 3. Consider the following nonlinear system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-1-x_1 & \exp(x_1)\\ x_2 & -10x_2^2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0.5\\ 1\end{bmatrix}u,\quad y=x_1, \tag{13}$$

along with the nonlinear observer

$$\begin{bmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\end{bmatrix}=\begin{bmatrix}-1-\hat{x}_1 & \exp(\hat{x}_1)\\ \hat{x}_2 & -10\hat{x}_2^2\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix}+\begin{bmatrix}0.5\\ 1\end{bmatrix}u-L(\hat{x},y,t)(y-\hat{y}),$$

where $\hat{y} = \hat{x}_1$ is the observer output and, as before, $L(\hat{x},y,t)$ is the nonlinear gain.

Thus, the error system is given by

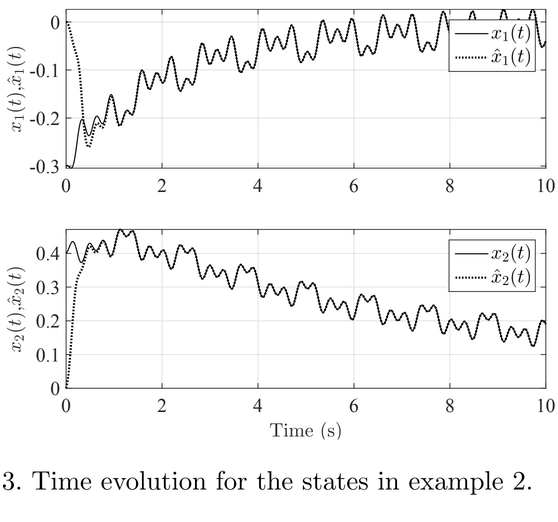

Fig. 3. Time evolution for the states in example 2.

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\end{bmatrix}=\begin{bmatrix}-x_1+\hat{x}_1-(x_1^2-\hat{x}_1^2)+x_2\exp(x_1)-\hat{x}_2\exp(\hat{x}_1)\\ x_1x_2-\hat{x}_1\hat{x}_2-10(x_2^3-\hat{x}_2^3)\end{bmatrix}-L(\hat{x},y,t)(x_1-\hat{x}_1).$$

The error signals $e_i = x_i - \hat{x}_i$, $i \in \{1, 2\}$, can be factorised as $x_1 - \hat{x}_1 = e_1$, $x_1^2 - \hat{x}_1^2 = (x_1 + \hat{x}_1)e_1$, $x_1x_2 - \hat{x}_1\hat{x}_2 = x_1e_2 + \hat{x}_2e_1$, $x_2^3 - \hat{x}_2^3 = (x_2^2 + x_2\hat{x}_2 + \hat{x}_2^2)e_1$, $x_2\exp(x_1) - \hat{x}_2\exp(\hat{x}_1) = \exp(x_1)(x_2 - \hat{x}_2) + \hat{x}_2(\exp(x_1) - \exp(\hat{x}_1))$, where the non-polynomial expression $\exp(x_1) - \exp(\hat{x}_1)$ is treated via its Taylor series. To do that, consider a Taylor polynomial of degree 3 around 0 for $\exp(x_1)$ and $\exp(\hat{x}_1)$; then, we have $x_2\exp(x_1)-\hat{x}_2\exp(\hat{x}_1)\approx\exp(x_1)e_2+\hat{x}_2\left(1+1/2\,(x_1+\hat{x}_1)+1/6\left(x_1^2+x_1\hat{x}_1+\hat{x}_1^2\right)\right)e_1$; from where the error signal arises. Thus, by choosing the available non-constant terms as $z_1 = x_1$, $z_2 = \hat{x}_1$, $z_3 = \hat{x}_2$, $z_4 = \exp(x_1)$, and the unavailable state as $\zeta_1 = x_2$ in the compact sets $\Omega = \{x : |x_1| \le 2, |x_2| \le 2\}$ and $\hat{\Omega} = \{\hat{x} : |\hat{x}_1| \le 2, |\hat{x}_2| \le 2\}$, a convex rewriting of the error system is

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\end{bmatrix}=\sum_{\mathbf{i}\in\mathbb{B}^7}\sum_{\mathbf{j}\in\mathbb{B}^2}\mathbf{w_i}(z)\boldsymbol{\omega}_\mathbf{j}(\zeta)\left(\begin{bmatrix}\bar{A}_{11} & z_4^{i_4}\\ z_3^{i_3} & \bar{A}_{22}\end{bmatrix}-L(\hat{x},y,t)\;[1\;\;0]\right)\begin{bmatrix}e_1\\ e_2\end{bmatrix},$$

where $\bar{A}_{11} = -1-(z_1^{i_1}+z_2^{i_2})+z_3^{i_3}(1+1/2(z_1^{i_1}+z_2^{i_2})+1/6(z_1^{i_1}z_1^{i_5}+z_1^{i_1}z_2^{i_2}+z_2^{i_2}z_2^{i_6}))$ and $\bar{A}_{22} = z_1^{i_1} - 10(\zeta_1^{j_1}\zeta_1^{j_2}+\zeta_1^{j_1}z_3^{i_3}+z_3^{i_3}z_3^{i_7})$, $\mathbf{w_i}(z) = w_{i_1}^1 w_{i_2}^2 w_{i_3}^3 w_{i_4}^4 w_{i_5}^1 w_{i_6}^2 w_{i_7}^3$, $w_0^1 = 0.5 - 0.25x_1$, $w_1^1 = 1-w_0^1$, $w_0^2 = 0.5-0.25\hat{x}_1$, $w_1^2 = 1-w_0^2$, $w_0^3 = 0.5-0.25\hat{x}_2$, $w_1^3 = 1 - w_0^3$, $w_0^4 = 1.02 - 0.138\exp(x_1)$, $w_1^4 = 1 - w_0^4$, $\boldsymbol{\omega}_\mathbf{j}(\zeta) = \omega_{j_1}^1\omega_{j_2}^1$, $\omega_0^1 = 0.5 - 0.25x_2$, $\omega_1^1 = 1 - \omega_0^1$.

By means of Theorem 2, the observer gain is

$$L(\hat{x},y,t)=P^{-1}(t)\sum_{\mathbf{l}\in\mathbb{B}^2}\bar{\boldsymbol{\omega}}_\mathbf{l}\sum_{\mathbf{k}\in\mathbb{B}^7}\mathbf{w_k}C^T=P^{-1}(t)C^T,$$

with $\bar{\omega}_0^1 = \bar{\omega}_1^1 = 0.5$, $C = [1\;0]$, and $Q(t) = 30P(t)$. Notice that a constant C makes the convex functions disappear from $L(\hat{x},y,t)$, but the dynamic equation of $P(t)$ still includes the known system nonlinearities via their convex rewriting.

A simulation has been run for initial conditions $P(0) = I$, $x(0) = [1.5\;1]^T$, and $\hat{x}(0) = [0\;0]^T$ under the input signal $u(t) = \sin 5t + \sin 20t - \sin 40t + \sin 60t$. In Fig. 4 the observer performance is shown, where $\hat{x}_2$ effectively estimates x_2.

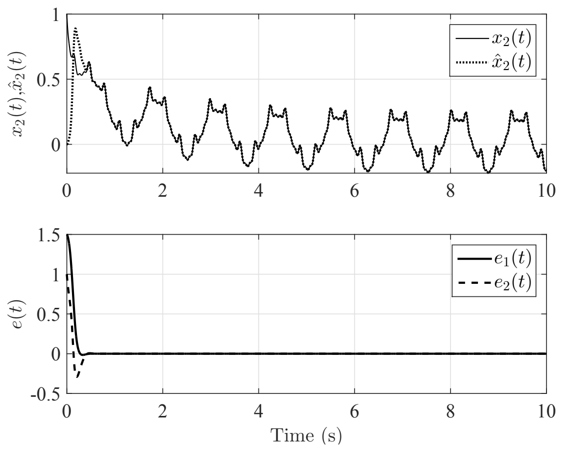

Fig. 4. Estimation of x_2 and time evolution of the nonlinear observer gain in example 3.

Notice that unlike the approaches proposed in (Ekramian et al., 2013; Na et al., 2017; Dimassi et al., 2019), no Lipschitz conditions and/or extended output are required to solved the observation problem for system (13).

A key aspect of this work has been the fact that unavailable signals, while captured in convex structures, allow choosing them in an arbitrary way if convexity holds for the proposed functions. So far, examples have chosen constant values due to simplicity; nevertheless, as the following example shows, the given known functions $\bar{\omega}$ can be time-varying as long as the convex-sum property hold:

Example 4. Consider the equations of the chaotic oscillator of Lü (Loría et al., 2009):

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-2.2587 & -x_3 & 0\\ 0 & -10 & x_1\\ 0 & x_1 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix},\quad y=\underbrace{\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}_{C}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix},$$

along with a nonlinear observer in the form (2)

$$\begin{bmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\\ \dot{\hat{x}}_3\end{bmatrix}=\begin{bmatrix}-2.2587 & -\hat{x}_3 & 0\\ 0 & -10 & \hat{x}_1\\ 0 & \hat{x}_1 & -4\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\\ \hat{x}_3\end{bmatrix}-L(\hat{x},y,t)(y-\hat{y}),$$

where $L(\hat{x}, y, t)$ is the nonlinear observer gain and $\hat{y} = [\hat{x}_2\;\hat{x}_3]^T$ is the observer output.

Thus, the error system is

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\end{bmatrix}=\left(\begin{bmatrix}-2.2587 & -x_3 & -\hat{x}_2\\ \hat{x}_3 & -10 & x_1\\ \hat{x}_2 & x_1 & -4\end{bmatrix}-L(\hat{x},y,t)C\right)\begin{bmatrix}e_1\\ e_2\\ e_3\end{bmatrix},$$

which can be rewritten in convex form by considering non-constant terms $z_1 = x_3 \in [-40, 20]$, $z_2 = \hat{x}_2 \in [-40, 30]$, $z_3 = \hat{x}_3 \in [-40, 20]$, and $\zeta_1 = x_1 \in [-40, 40]$, where the first three are known and the fourth one is not available:

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\end{bmatrix}=\sum_{\mathbf{i}\in\mathbb{B}^3}\sum_{\mathbf{j}\in\mathbb{B}^1}\mathbf{w_i}\boldsymbol{\omega}_\mathbf{j}\left(\begin{bmatrix}\bar{A}_{11} & -z_1^{i_1} & -z_2^{i_2}\\ z_3^{i_3} & -10 & \zeta_1^{j_1}\\ z_2^{i_2} & \zeta_1^{j_1} & -4\end{bmatrix}-L(\hat{x},y,t)C\right)\begin{bmatrix}e_1\\ e_2\\ e_3\end{bmatrix},$$

where $\bar{A}_{11} = -2.2587$, $w_0^1 = 1/3 - x_3/60$, $w_1^1 = 1 - w_0^1$, $w_0^2 = 3/7 - \hat{x}_2/70$, $w_1^2 = 1 - w_0^2$, $w_0^3 = 1/3 - \hat{x}_3/60$, $w_1^3 = 1 - w_0^3$, and $\omega_0^1 = 1/2 - x_1/80$, $\omega_1^1 = 1 - \omega_0^1$.

By using Theorem 2, we can consider $\bar{\omega}_0^1 = 0.5 - 0.4\sin 20t$ and $\bar{\omega}_1^1 = 1 - \bar{\omega}_0^1$ in order to implement the dynamic

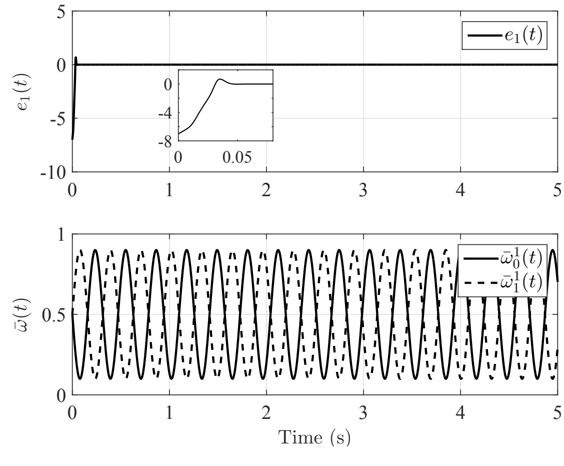

Fig. 5. Time evolution of the error signal e_1 and the time-varying convex function $\bar{\omega}$ in example 4.

$P(t)$ in (10); the nonlinear observer gain is $L(\hat{x}, y, t) = P^{-1}C^T$. A simulation is performed by implementing (10) for initial conditions $P(0) = I$, $x(0) = [3\ -4\ 2]^T$, and $x(0) = [10\ 10\ 10]$. In Fig. 5 error signal e_1 and the time-varying convex functions $\bar{\omega}_0^1$ and $\bar{\omega}_1^1$ are shown.

5. CONCLUSIONS

A novel adaptive nonlinear observer has been presented. The design is based on a factorisation of the error signal via a recently appeared explicit methodology along with a polytopic representation that naturally separates available from non-available signals. Thanks to the polytopic split, a dynamic implementation of the Lyapunov matrix has been made possible that allows the adaptive nonlinear observer gain to be obtained on line. In contrast with recent works, it has been shown that the proposed design avoids conditions on persistency of excitation and Lipschitz bounds. Illustrative examples have been provided. Adaptive output feedback and fault-tolerant control under this observation scheme are left for future work.

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