Event-Triggered Control for Extended Plants of Discrete-Time Linear Systems

Hiroyuki Ichihara [*] **Kenji Sawada** [**] **Koichi Kobayashi** [***]
Sophie Tarbouriech [****]

[*] *Meiji University, Kawasaki, Japan*
(e-mail: ichihara@messe.meiji.ac.jp)
[**] *The University of Electro-Communications, Chofu, Japan*
(e-mail: knj.sawada@uec.ac.jp)
[***] *Hokkaido University, Sapporo, Japan*
(e-mail: k-kobaya@ssi.ist.hokudai.ac.jp)
[****] *LAAS-CNRS, Université de Toulouse, CNRS, Toulouse, France*
(e-mail: tarbour@laas.fr)

Abstract: This paper deals with design methods of event-triggered control systems for discrete-time linear systems. An extended plant consisting of a given plant and a dynamical filter is considered and controlled by an event-triggered static output feedback. The triggering rule uses only the available signals and therefore is based on the difference between the triggered and non-triggered output signals. The paper deals with the co-design problem, that is the design of the triggering condition, the filter, and the controller simultaneously. Sufficient theoretical conditions are proposed in terms of linear matrix inequalities to ensure the asymptotic stability of the closed-loop system. Convex optimizations problems incorporate these conditions in order to optimize the closed-loop performance or to reduce the number of transmissions. Three numerical examples illustrate the design method of the triggering conditions as well as the simultaneous design method of the filter and controller.

Keywords: networked control system, event-triggered control, discrete-time control, convex optimization

1. INTRODUCTION

Traditional sampled-data based control, which adopts periodic sampling to transmit the output data of the plant to the controller, is convenient for analysis and synthesis of control systems. Since a wired or wireless network transmits the data, less energy consumption is preferable by reducing the number of transmissions. Indeed, there is still room to reduce the amount number of data to transfer in the sampled-data based control system without deteriorating too much the stability and performance of the system. In this viewpoint, several authors consider event-triggered control (ETC) that adopts periodic or aperiodic sampling (see, for example, Åström (2008), Heemels et al. (2012), Tallapragada and Chopra (2012), Postoyan et al. (2015), Tarbouriech et al. (2017), Wang and Lemmon (2011)). Continuous or time-regularized ETC monitors the system continuously and triggers the data transmission if an error between the current data and the previous one exceeds a certain amount of quantity to guarantee the stability and performance of the closed-loop system. In contrast, periodic ETC (PETC) monitors the system only at periodic sampling times to send or not the data. Although time-regularized ETC is capable of minimizing a waiting time between two sampling times to enhance the stability and performance of the system, PETC has several merits over time-regularized ETC (see, for example, Heemels et al. (2013), Aranda-Escolastico et al. (2015) and references

therein). One of the merits is that PETC is suitable to implement into digital devices that keep the sampling time to monitor the system.

Heemels et al. (2013) gave a general framework for PETC that includes three approaches: (1) The impulsive system approach allows the $\mathcal{L}_2$-gain from the error to the control signal; (2) The piecewise linear system approach provides conditions that are less conservativeness but computationally involved; (3) The perturbed linear system approach reduces the problems to numerically tractable conditions. Similar to the latter approach, Wu et al. (2015) proposed linear matrix inequality (LMI) conditions for given triggering rules that provide suboptimal event-triggered state feedback controllers. This approach derives the LMI conditions based on a lossless S-procedure relaxation with some approximation to avoid the conditions to be bilinear matrix inequalities (BMIs). Meng and Chen (2014) investigated co-design problems that design the parameters in the triggering rules and the matrices in static or dynamic output feedback controllers simultaneously. They use a non-lossless S-procedure relaxation to handle two different triggering rules in one event-triggered control system. The formulation of the problem of the static output feedback case is a BMI, and thus a linearization algorithm is used to solve the inequality. On the other hand, the formulation of the dynamic output feedback case is an LMI. Zhang and Han (2014) also investigated co-design problems with the

aid of Lyapunov-Krasovskii functional to design dynamic feedback controllers. However, the formulation is based on a BMI, in which the linearization algorithm is needed again or linearization of the non-linear terms results in an LMI with allowing some conservativeness.

This paper investigates event-triggered control systems for an extended plant of discrete-time linear systems with static output feedback controllers. The extended plant includes a given plant and a dynamic filter. The structure of the extended plant and static output feedback controller depends on the output error, that is the difference between the triggered and non-triggered output signals. Thanks to static output feedback controllers, it is not necessary to design or assume an event detector at the input side of the plant to solve the problems if there is a designed or assumed event detector at the output side. Then, we can concentrate on using the lossless S-procedure relaxation [1] to derive a co-design condition. The triggering conditions in this paper are generalizations from those in Wu et al. (2015), but are different from those in Meng and Chen (2014) and Zhang and Han (2014). Similar to the purpose in these papers, we investigate a convex structure for co-design of the parameters in the triggering rule, the filter, and controller to introduce dynamic elements in the system to generalize the approach in Wu et al. (2015) to the static output feedback case.

Notation: $\mathbb{R}$ is the set of real numbers. $\mathbb{R}^n$ is the set of the real vectors of dimension n. $\mathbb{R}^{n \times m}$ is the set of the real vectors of dimension $n \times m$. $\mathbb{S}^n$ is the set of the real symmetric matrices of dimension $n \times n$. I and 0 represent the identity and null matrices of appropriate dimensions, respectively. For $x \in \mathbb{R}^n$, $\|x\|$ is the Euclidean norm. For $A \in \mathbb{R}^{n \times n}$, $\mathrm{tr}(A)$ is the trace of A. $A \succ 0$ means that $A \in \mathbb{S}^n$ is a positive-definite matrix. For real symmetric matrices, the following notations are often used:

$$\begin{bmatrix} A & \star \\ B & C \end{bmatrix} = \begin{bmatrix} A & B^\top \\ B & C \end{bmatrix}, \ B^\top A [\star] = B^\top A B$$

2. PROBLEM FORMULATION

Consider the following discrete-time linear plant
$$x_p[k + 1] = A_p x_p[k] + B_p u[k], \tag{1}$$
$$y[k] = C_p x_p[k], \tag{2}$$
where $x_p \in \mathbb{R}^n$ is the state of the plant, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the output. A_p, B_p and C_p are constant known matrices of appropriate dimensions.

The event-triggered control system of interest is depicted in Fig. 1. We augment plant (1)-(2) with the following dynamical filter
$$x_f[k + 1] = A_f x_f[k] + v_1[k], \tag{3}$$
$$u[k] = C_f x_f[k] + v_2[k], \tag{4}$$
where $x_f \in \mathbb{R}^n$ is the state of the filter, A_f and C_f are constant matrices of appropriate dimensions. Finally, $v_1 \in \mathbb{R}^n$ and $v_2 \in \mathbb{R}^m$ are auxiliary inputs defined via a static output feedback controller as follows:
$$\begin{bmatrix} v_1[k] \\ v_2[k] \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \hat{y}[k], \tag{5}$$

---

[1] The lossless S-procedure is often used in control theory to unite two quadratic inequalities into one inequality without conservativeness (see, for example, Boyd et al. (1994)).
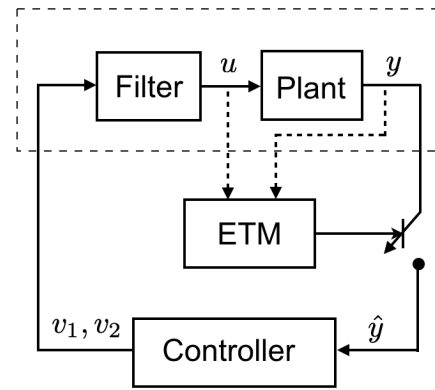

Fig. 1. Event-triggered control system for extended plant based on output error

where F_1 and F_2 are matrices of compatible dimensions. The signal $\hat{y}[k]$ is the signal issued from the event-triggering mechanism (ETM), which follows an event-triggering rule.

In the paper, we are interested in the following triggering rule:
$$\hat{y}[k] := \begin{cases} y[k], & g(\hat{y}[k-1], y[k], u[k]) > 0, \\ \hat{y}[k-1], & \text{otherwise.} \end{cases} \tag{6}$$

The function $g(\hat{y}, y, u)$ is defined by
$$g(\hat{y}[k-1], y[k], u[k]) := \|\hat{y}[k-1] - y[k]\|^2 - \sigma^2 \left(\mu \|y[k]\|^2 + (1-\mu)\|u[k]\|^2 \right), \tag{7}$$

where $\sigma > 0$ and $\mu \in [0, 1]$ are parameters to be tuned or chosen.

The first part of condition (7) represents the error between the triggered and non-triggered output. The meaning of (6) is the following: If (7) holds at step k, then signal $\hat{y}[k]$ is updated to be $y[k]$ because error $\|\hat{y}[k-1] - y[k]\|$ is too large to guarantee the stability and performance of the closed-loop system with some controller. On the other hand, if (7) does not hold at step k, then $\hat{y}[k]$ is not updated, that is, the previous one, $\hat{y}[k-1]$, is used as $\hat{y}[k]$ since the error is small enough to guarantee the stability and performance.

The closed-loop system in the paper can be written in a compact form as follows:
$$x[k+1] = \mathcal{A}x[k] + \mathcal{B}w[k], \tag{8}$$
$$y[k] = \mathcal{C}x[k], \tag{9}$$
$$u[k] = \mathcal{E}x[k] + \mathcal{F}w[k], \tag{10}$$

where $x = [x_p^\top \ x_f^\top]^\top \in \mathbb{R}^{2n}$ is the state, $w[k] = \hat{y}[k] - y[k] \in \mathbb{R}^p$ is the output error, and the matrices are

$$\mathcal{A} = \begin{bmatrix} A_p + B_p F_2 C_p & B_p C_f \\ F_1 C_p & A_f \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} B_p F_2 \\ F_1 \end{bmatrix},$$
$$\mathcal{C} = [C_p \ 0], \ \mathcal{E} = [F_2 C_p \ C_f], \ \mathcal{F} = F_2. \tag{11}$$

Consequently, we always have
$$\|w[k]\|^2 \leq \sigma^2 \left(\mu \|y[k]\|^2 + (1-\mu)\|u[k]\|^2 \right).$$

By rewriting the above equation with (9)-(10), we have that

$$\begin{bmatrix} x[k] \\ w[k] \end{bmatrix}^{\top} \left(\begin{bmatrix} \sigma^2 \mu \mathcal{C}^{\top} \mathcal{C} & 0 \\ 0 & -I \end{bmatrix} \right.$$
$$\left. + \sigma^2 (1-\mu) \begin{bmatrix} \mathcal{E}^{\top} \\ \mathcal{F}^{\top} \end{bmatrix} [\mathcal{E} \ \mathcal{F}] \right) \begin{bmatrix} x[k] \\ w[k] \end{bmatrix} \geq 0. \qquad (12)$$

The problem we intend to solve can be summarized as follows:

Problem 1. Design the filter (that is matrices A_f and C_f), the feedback gains F_1 and F_2 and the triggering condition (that is the parameters σ and μ) such that the closed-loop system (8)-(10) with triggering rule (6) is asymptotically stable.

3. MAIN RESULT

The results are based on the use of Lyapunov theory and more especially on the use of a quadratic Lyapunov function $\mathcal{V}(x) = x^{\top} \mathcal{P} x$, where $\mathcal{P} \in \mathbb{S}^{2n}$ is a positive definite matrix. Indeed, we are searching for a matrix $\mathcal{P} = \mathcal{P}^{\top} \succ 0$ such that for all $x[k] \neq 0$ and $w[k]$ satisfying (12), the following inequality holds

$$\begin{bmatrix} x[k] \\ w[k] \end{bmatrix}^{\top} \left(- \begin{bmatrix} \mathcal{A}^{\top} \\ \mathcal{B}^{\top} \end{bmatrix} \mathcal{P} [\mathcal{A} \ \mathcal{B}] + \begin{bmatrix} \mathcal{P} & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x[k] \\ w[k] \end{bmatrix} > 0. \quad (13)$$

On the basis of the above fact, we give a solution to the problem stated in Section 2.

3.1 Design of Triggering Rule

Consider the triggering condition (7). The following result can be stated to design the parameters σ and μ, when the filter and the controller are assumed given.

Lemma 1. Given filter (3)-(4) and controller (5), if there exist a positive definite matrix $\mathcal{P} \in \mathbb{S}^{2n}$ and positive scalars α, β_1 and β_2 satisfying

$$\begin{bmatrix} \mathcal{P} & \star & \star & \star & \star \\ 0 & \alpha I & \star & \star & \star \\ \mathcal{P}\mathcal{A} & \mathcal{P}\mathcal{B} & \mathcal{P} & \star & \star \\ \mathcal{C} & 0 & 0 & \beta_1 I & \star \\ \mathcal{E} & \mathcal{F} & 0 & 0 & \beta_2 I \end{bmatrix} \succ 0, \qquad (14)$$

then the closed-loop system (8)-(10) with the triggering rule (6) for which the parameters are defined by

$$\sigma = \sqrt{\frac{1}{\alpha}\left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right)}, \quad \mu = \frac{\beta_2}{\beta_1 + \beta_2} \qquad (15)$$

is asymptotically stable.

Remark 1. The parameters (σ, μ) of the triggering condition are characterized by the nonlinear functions in equation (15) of the triplet decision variables $(\alpha, \beta_1, \beta_2)$ of the LMI. The scalar α corresponds to the multiplier of S-procedure.

3.2 Co-Design

In this section, the co-design problem is addressed, that is in plus of designing the parameters μ and σ defined in the triggering rule, one wants to design the filter and the controller. At this aim, we consider that the matrix $\mathcal{P}$ is defined as follows:

$$\mathcal{P} = \begin{bmatrix} X & Z \\ Z & Z \end{bmatrix}, \qquad (16)$$

where $X \in \mathbb{S}^n$ and $Z \in \mathbb{S}^n$ are positive definite matrices so that $\mathcal{P}$ is a positive definite matrix. Note that the symmetric matrix Z in the non-diagonal blocks in $\mathcal{P}$ does not restrict the set of solutions to LMI (14). In fact, there is a variable change for a general form of the Lyapunov variable matrix whose non-diagonal block matrices are merely non-symmetric ones. Such a variable change preserves the characteristics of the controller and the Lyapunov variable matrix $\mathcal{P}$ takes the form in equation (16). The fact regarding the structure of the Lyapunov variable matrix is known in the literature (See, for example, Masubutch et al. (1998)).

Then, by expanding Lemma 1, the following result can be stated.

Theorem 1. If there exist positive definite matrices $X \in \mathbb{S}^n$ and $Y \in \mathbb{S}^n$, matrices $W \in \mathbb{R}^{n \times n}$, $U \in \mathbb{R}^{n \times p}$, $R \in \mathbb{R}^{m \times n}$ and $M \in \mathbb{R}^{m \times p}$, and scalars $\alpha > 0$, $\beta_1 > 0$ and $\beta_2 > 0$ satisfying

$$\begin{bmatrix} \Xi_{\mathcal{P}} & \star & \star & \star & \star \\ 0 & \alpha I & \star & \star & \star \\ \Xi_{\mathcal{A}} & \Xi_{\mathcal{B}} & \Xi_{\mathcal{P}} & \star & \star \\ \Xi_{\mathcal{C}} & 0 & 0 & \beta_1 I & \star \\ \Xi_{\mathcal{E}} & M & 0 & 0 & \beta_2 I \end{bmatrix} \succ 0, \qquad (17)$$

where

$$\Xi_{\mathcal{P}} = \begin{bmatrix} X & I \\ I & Y \end{bmatrix}, \quad \Xi_{\mathcal{A}} = \begin{bmatrix} XA_p + UC_p & W \\ A_p + B_p MC_p & A_p Y + B_p R \end{bmatrix},$$
$$\Xi_{\mathcal{B}} = \begin{bmatrix} U \\ B_p M \end{bmatrix}, \quad \Xi_{\mathcal{C}} = [C_p \ C_p Y], \quad \Xi_{\mathcal{E}} = [MC_p \ R], \qquad (18)$$

then the closed-loop system (8)-(10) with the triggering rule (6) with the parameters in equation (15) is asymptotically stable. Furthermore, the matrices in filter and controller (3)-(5) are

$$\begin{bmatrix} A_f & F_1 \\ C_f & F_2 \end{bmatrix} = \begin{bmatrix} Z & XB_p \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} W - XA_p Y & U \\ R & M \end{bmatrix} \begin{bmatrix} -Y & 0 \\ C_p Y & I \end{bmatrix}^{-1}, \qquad (19)$$

with $Z = X - Y^{-1}$.

Remark 2. Theorem 1 allows to design the parameters of the triggering condition, the matrices of the filter and the controller gains thanks to the satisfaction of LMI (17). Thus Theorem 1 gives a solution to Problem 1. In Meng and Chen (2014) a non-lossless S-procedure is implicitly applied to handle two different triggering rules, while Theorem 1 applies the lossless S-procedure. In addition, Meng and Chen (2014) fixes the multipliers in the non-lossless S-procedure while the multiplier α is a decision variable in Theorem 1.

3.3 Controller and Filter Design

By fixing σ and μ, we can design only the filter and controller gains stated in the following corollary derived from Theorem 1.

Corollary 1. For given scalars $\sigma > 0$ and $\mu \in [0,1]$ in condition (7), system (8)-(10) with triggering rule (6) is asymptotically stable if there exist positive definite matrices $X \in \mathbb{S}^n$ and $Y \in \mathbb{S}^n$, matrices $W \in \mathbb{R}^{n \times n}$, $U \in \mathbb{R}^{n \times p}$, $R \in \mathbb{R}^{m \times n}$ and $M \in \mathbb{R}^{m \times p}$, and a scalar $\alpha > 0$ satisfying

$$\begin{bmatrix} \Xi_{\mathcal{P}} & \star & \star & \star & \star \\ 0 & \alpha I & \star & \star & \star \\ \Xi_{\mathcal{A}} & \Xi_{\mathcal{B}} & \Xi_{\mathcal{P}} & \star & \star \\ \alpha\Xi_{\mathcal{C}} & 0 & 0 & \dfrac{\alpha}{\sigma^2\mu}I & \star \\ \alpha\Xi_{\mathcal{E}} & \alpha M & 0 & 0 & \dfrac{\alpha}{\sigma^2(1-\mu)}I \end{bmatrix} \succ 0, \qquad (20)$$

where matrices $\Xi_{\mathcal{P}}$, $\Xi_{\mathcal{A}}$, $\Xi_{\mathcal{B}}$, $\Xi_{\mathcal{C}}$ and $\Xi_{\mathcal{E}}$ are defined in (18). Then, matrices of the filter and controller (3)-(5) are obtained from (19) with $Z = X - Y^{-1}$.

Remark 4. Relation (20) is not an LMI due to the product of some decision variables with α, but if we fix α, condition (20) becomes an LMI condition. Thus in order to solve condition (20), we can use a line search algorithm in α, while finding the other decision variables.

Remark 5. Corollary 1 allows to expand the results of Wu et al. (2015), in which the closed-loop system does not include filter (3)-(4) and approximation to manage the inequality conditions leading to a certain conservatism are considered.

4. OPTIMIZATION ISSUES

In this section, we consider two optimization problems: the first one is related to the event-triggering rule and the number of data transmission; The second one is related to a certain level of performance for the closed-loop system.

4.1 Maximizing σ

To decrease the number of communication between the plant and controller, increasing the sensor inter-event interval is necessary. We can expect to increase the interval by maximizing σ, which allows a big difference between the triggered output and current output, in condition (7). From relation (15) it appears that σ can be written as a function of the triplet $(\alpha, \beta_1, \beta_2)$, which are decision variables in LMI (14). Then, we would like to maximize the following function

$$f(\alpha, \beta_1, \beta_2) = \frac{1}{\alpha}\left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right).$$

However, the above function is nonlinear in the decision variables α, β_1, β_2. A solution is to consider a first-order approximation around a triplet $(\alpha^*, \beta_1^*, \beta_2^*)$ by the truncated Taylor series expansion such that

$$f(\alpha^*, \beta_1^*, \beta_2^*) + \sum_{s \in \{\alpha, \beta_1, \beta_2\}} \left. \frac{\partial f(\alpha, \beta_1, \beta_2)}{\partial s} \right|_{(\alpha^*, \beta_1^*, \beta_2^*)} (s - s^*)$$
$$= g_0(\alpha^*, \beta_1^*, \beta_2^*) + g_1(\alpha, \beta_1, \beta_2),$$

where g_0 is constant and g_1 is a linear function of the triplet $(\alpha, \beta_1, \beta_2)$ such that

$$g_1(\alpha, \beta_1, \beta_2) = -\frac{1}{\alpha^*}\left\{\left(\frac{1}{\beta_1^*} + \frac{1}{\beta_2^*}\right)\frac{\alpha}{\alpha^*} + \frac{\beta_1}{(\beta_1^*)^2} + \frac{\beta_2}{(\beta_2^*)^2}\right\}.$$

Then, we can design a sub-optimal triggering condition that maximizes σ in the case of given the filter, the controller and an initial triplet $(\alpha^*, \beta_1^*, \beta_2^*)$. The proposed design is provided by Algorithm 1 through Lemma 1. To find a set of the filter and controller with a set of the initial parameters, for example, we can use Theorem 1.

Algorithm 1 Maximize σ for Lemma 1

Require: Matrices in (3)-(5)
Require: $(\alpha^0, \beta_1^0, \beta_2^0)$ that stabilizes (8)-(10) with (6)
  Construct σ^0 from $(\alpha^0, \beta_1^0, \beta_2^0)$ by (15)
  $(\alpha, \beta_1, \beta_2) \leftarrow (\alpha^0, \beta_1^0, \beta_2^0)$
  $\sigma \leftarrow \sigma^0$, $\sigma^* \leftarrow 0$
  $\varepsilon \leftarrow$ a small number
  while $\sigma - \sigma^* > \varepsilon$ **do**
    $(\alpha^*, \beta_1^*, \beta_2^*) \leftarrow (\alpha, \beta_1, \beta_2)$, $\sigma^* \leftarrow \sigma$
    Maximize $g_1(\alpha, \beta_1, \beta_2)$ subject to (14) to obtain $(\alpha, \beta_1, \beta_2)$
    Construct σ from $(\alpha, \beta_1, \beta_2)$
  end while

4.2 LQ Cost Optimization

Consider the quadratic cost

$$\mathcal{J}_\infty = \sum_{k=0}^{\infty} \Delta\mathcal{J}(k), \qquad (21)$$

where

$$\Delta\mathcal{J}(k) = \begin{bmatrix} x[k] \\ u[k] \end{bmatrix}^\top \mathcal{Q} \begin{bmatrix} x[k] \\ u[k] \end{bmatrix}$$

and $0 \preceq \mathcal{Q} \in \mathbb{S}^{2n+m}$ is the weighting matrix to be chosen. To evaluate control performance for system (8)-(10) with triggering rule (6), we consider the inequality

$$\mathcal{V}(x[k+1]) < \mathcal{V}(x[k]) - \Delta\mathcal{J}(k) \qquad (22)$$

that holds for all $x[k]$ and $w[k]$ satisfying equation (12). By summing up equation (22) from $k = 0$ to ∞, we have that $J_\infty < x[0]^\top \mathcal{P}x[0] < \text{tr}(\mathcal{P})\|x[0]\|^2$. If we assume $x_f[0] = 0$, then we have an upper bound of J_∞ as $\text{tr}(X)\|x_p[0]\|^2$. Thus we may solve the optimization problem to minimize cost (21) as follows:

$$\min_{X,Y,W,U,R,M,\alpha} \text{tr}(X) \text{ subject to} \qquad (23)$$

$$\left[\begin{array}{c|c} \mathcal{M}(X,Y,W,U,R,M,\alpha) & \star \\ \hline (\mathcal{Q})^{\frac{1}{2}}\begin{bmatrix}\Pi_{\mathcal{P}}^\top & 0 \\ \Xi_{\mathcal{E}} & M\end{bmatrix}\begin{array}{ccc}0 & 0 & 0 \\ 0 & 0 & 0\end{array} & \begin{array}{cc}I & 0 \\ 0 & I\end{array} \end{array}\right] \succ 0$$

with $\mathcal{M}(X,Y,W,U,R,M,\alpha)$ is the matrix of the left-hand side of relation (20) and

$$\Pi_{\mathcal{P}} = \begin{bmatrix} I & 0 \\ Y & -Y \end{bmatrix}. \qquad (24)$$

Remark 6. If α is fixed, then the last condition in problem (23) becomes an LMI. One can use the line search algorithm in α to solve this.

To choose a reasonable weighting matrix $\mathcal{Q}$ of cost (21), we introduce a continuous-time setting of the plant such that

$$\dot{x}_p(t) = \mathfrak{A}_p x_p(t) + \mathfrak{B}_p u(t), \qquad (25)$$
$$y(t) = C_p x_p(t), \qquad (26)$$

where $x_p(t)$ and $u(t)$ represent the state and control input of the plant, respectively, that satisfy $x_p(kT) = x_p[k]$ and $u(kT) = u[k]$ for the sampling period T. Matrices $\mathfrak{A}_p$ and

$\mathfrak{B}_p$ have compatible dimensions. The quadratic cost for plant (25)-(26) can be written as follows:

$$\int_0^\infty \begin{bmatrix} x_p(s) \\ u(s) \end{bmatrix}^\top \begin{bmatrix} \mathfrak{Q}_p & 0 \\ 0 & \mathfrak{R} \end{bmatrix} \begin{bmatrix} x_p(s) \\ u(s) \end{bmatrix} ds \qquad (27)$$

$$= \sum_{k=0}^\infty \Delta\mathcal{J}(k),$$

where $0 \preceq \mathfrak{Q}_p \in \mathbb{S}^n$ and $0 \prec \mathfrak{R} \in \mathbb{S}^m$ are the weighting matrices, and $t_k = kT$. Then, we can choose $\mathcal{Q}$ as follows:

$$\mathcal{Q} = \int_0^T \begin{bmatrix} \hat{A}(\delta)^\top \mathfrak{Q}_p \hat{A}(\delta) & 0 & \hat{A}(\delta)^\top \mathfrak{Q}_p \hat{B}(\delta) \\ 0 & 0 & 0 \\ \hat{B}(\delta)^\top \mathfrak{Q}_p \hat{A}(\delta) & 0 & \hat{B}(\delta)^\top \mathfrak{Q}_p \hat{B}(\delta) + \mathfrak{R} \end{bmatrix} d\delta,$$

where $\hat{A}(\delta) = e^{\mathfrak{A}_p\delta}$ and $\hat{B}(\delta) = \int_0^\delta e^{\mathfrak{A}_p s} ds \mathfrak{B}_p$. Then, the matrices of discrete-time counterpart of plant (1) are $A_p = \hat{A}(T)$ and $B_p = \hat{B}(T)$. Consequently, we can obtain (A_p, B_p) and $\mathcal{Q}$ from $(\mathfrak{A}_p, \mathfrak{B}_p)$ and $(\mathfrak{Q}_p, \mathfrak{R})$ with T. One can cite a reference on the general framework on sampled-data systems as for example Chen and Francis (1994).

5. NUMERICAL EXAMPLES

5.1 Data

Consider the continuous-time system (25)-(26) where

$$\mathfrak{A}_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1.001 & -0.5117 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2.916 & 30.05 & 0 \end{bmatrix}, \quad \mathfrak{B}_p = \begin{bmatrix} 0 \\ 0.8455 \\ 0 \\ -2.461 \end{bmatrix}.$$

By using the sampling period $T = 0.01$, we obtain the discrete-time system (1) and matrices

$$A_p = \begin{bmatrix} 1 & 0.01 & 0 & 0 \\ 0 & 0.99 & -0.0051 & 0 \\ 0 & 0.0001 & 1.0015 & 0.01 \\ 0 & 0.0290 & 0.3006 & 1.0015 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 0.0084 \\ -0.0001 \\ -0.0245 \end{bmatrix}.$$

The parser and solver to solve the problems are YALMIP (Löfberg (2004)) and MOSEK 9.0 (MOSEK ApS (2019)), respectively. We commonly use the initial states of the plant and filter such that $x_p[0] = [0.3\ 0\ 0.3\ 0]^\top$ and $x_f[0] = [0\ 0\ 0\ 0]^\top$, respectively.

5.2 Example 1 (Co-Design):

Assume in (2) or (26) that

$$C_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

For the set of (A_p, B_p, C_p), a feasible solution to LMI (17) in Theorem 1 is as follows:

$(\alpha, \beta_1, \beta_2) = (221.3, 296.65, 636.0),$

$$\left[\begin{array}{c|c} A_f & F_1 \\ \hline C_f & F_2 \end{array}\right] = \left[\begin{array}{cccc|cc} 0.346 & 0.010 & -0.023 & 0.000 & -0.654 & -0.023 \\ -0.132 & 1.052 & -0.083 & 0.078 & -0.165 & -0.496 \\ -0.026 & -0.002 & 0.228 & 0.008 & -0.025 & -0.762 \\ -0.221 & -0.158 & -1.601 & 0.768 & -0.119 & -0.648 \\ \hline -1.304 & -7.431 & -8.298 & -9.315 & 2.710 & 41.826 \end{array}\right].$$

Then, from (15), one gets $(\sigma, \mu) = (0.0047, 0.6819)$.

The ratio of events is 88.31% from $t = 0$ to 10 in the time evolution.

5.3 Example 2 (Design of Triggering Rule):

For the set of (A_p, B_p, C_p) in Example 1 with (A_f, C_f) and (F_1, F_2) that come from the solution to LMI (17) in Theorem 1, we apply Algorithm 1 with Lemma 1 and $\varepsilon = 10^{-6}$ to maximize σ. The initial triplet $(\alpha^0, \beta_1^0, \beta_2^0)$ also comes from the solution to (17) in Example 1. After 4 iterations, we have that $(\alpha, \beta_1, \beta_2) = (44.5485, 45.9587, 202.9852)$, that is, $(\sigma, \mu) = (0.0245, 0.8154)$. The time evolution of the closed-loop system for extended plant based on output error is shown in Fig. 2 where the bottom sub-figure shows the sensor inter-event interval. The figure illustrates that the closed-loop system is asymptotically stable while the maximum interval is 28 at $t = 0.34$. Then, the ratio of events is 32.77% from $t = 0$ to 10. Evolution with zoomed time scale of Fig. 2 from $t = 0.2$ to 0.8 is shown in Fig. 3. Since the output signal is not triggered from $t = 0.34$ to 0.62, the auxiliary input signal is constant while the input signal of the plant is not constant due to the filter.
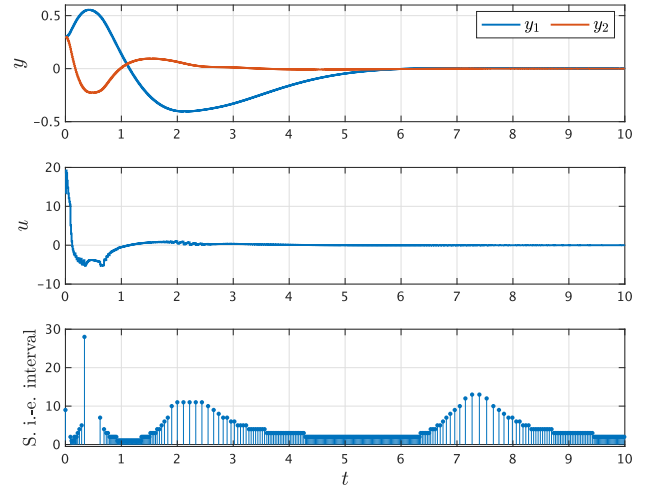

Fig. 2. Example 2 - Time evolution of event-triggered control system for extended plant based on output error

5.4 Example 3 (Controller and Filter Design):

Assume in (2) or (26) that C_p is the identity matrix, $C_p = I$. That is, we use state feedback control. For given pair (σ, μ) in condition (7), we design the filter and controller simultaneously based on problem (23), which minimizes the LQ cost. The weighting matrices $(\mathfrak{Q}_p, \mathfrak{R})$ of cost (27) is $\mathrm{diag}(1, 1, 0.5, 0.5)$ and 1, respectively. To compare this design method with Theorem 3.1 in Wu et al. (2015), we fix μ to be 1. The performance of the both methods are shown in Table 1 for several σ. In the table, $\underline{\mathcal{J}}_\infty$ is the lower bound of cost $\mathcal{J}_\infty$ that comes from the time evolution from $t = 0$ to 10, and $x_p[0]^\top X x_p[0]$ is the upper bound of $\mathcal{J}_\infty$ in problem (23) while $\underline{J}_\infty$ and $x_p[0]^\top P x_p[0]$ are the lower and upper bounds of $\bar{J}_\infty$, respectively, in Theorem 3.1 in Wu et al. (2015). R_{revent} denotes the ratio of events in the time evolution. The last row in Table 1, no ETC, represents that there is no triggering action, that is, standard LQ control, which indicates a limitation of control performance. For each σ, the solution to problem (23) has a better performance than that to the conventional problem. For larger σ, the conventional
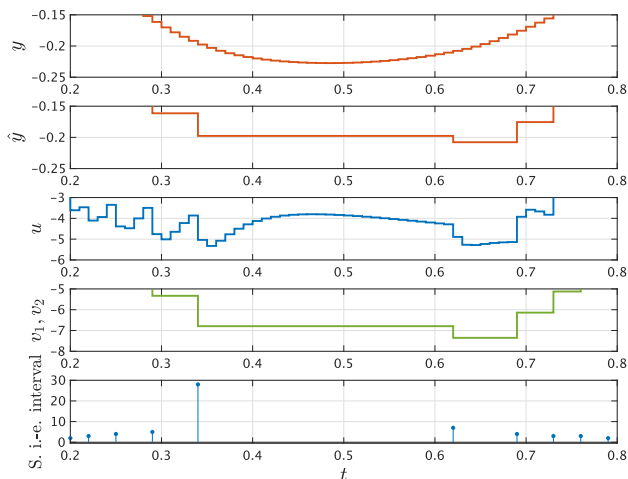
Fig. 3. Example 2 - Evolution with zoomed time scale of Fig. 2

Table 1. Example 3 - LQ performance for state error case

Problem	σ	$\overline{\mathcal{J}}_\infty$	$x_p[0]^\top X x_p[0]$	R_{event}
(23)	0.005	10.49	10.92	93.61
(23)	0.010	10.50	11.17	53.25
(23)	0.020	10.58	12.17	36.76
(23)	0.050	11.15	19.29	20.98
(23)	0.100	13.26	40.32	17.68
Problem	σ	$\underline{J}_\infty$	$x_p[0]^\top P x_p[0]$	R_{event}
Th. 3-1	0.005	10.51	11.08	95.25
Th. 3-1	0.010	10.62	12.18	63.86
Th. 3-1	0.020	11.48	17.56	47.46
Th. 3-1	0.050	—	—	—
Th. 3-1	0.100	—	—	—
no ETC	—	10.51	10.51	100

method does not have feasible solutions while problem (23) have feasible solutions. However, for too small σ, problem (23) tends to be numerically unstable to solve.

6. CONCLUSION

This paper clarified the structure of an extended plant (that is the plant + the filter) in order to address the event-triggered control problem of discrete-time systems. Theoretical conditions in emulation and co-design contexts were described by LMI conditions, allowing to obtain the matrices of the dynamic filter and static output feedback controller as well as the parameters of the triggering rule. The triggering rule considered was based on the output signal of the plant. The effectiveness of our approach was illustrated to compare with the conventional approach from an optimization viewpoint.

The results open the door for future works. In particular, it could be interesting to generalize the class of controller to be designed together with the kind of triggering rule, which could depend on the input.

ACKNOWLEDGEMENTS

REFERENCES

Aranda-Escolastico, E., Guinaldo, M., and Dormido, S. (2015). A novel approach for periodic event-triggering based on general quadratic functions. In *2015 International Conference on Event-based Control, Communication, and Signal Processing (EBCCSP)*, 1–6. Krakow, Poland.

Boyd, S., Ghaoui, L., Feron, E., and Balakrishnan, V. (1994). *Linear Matrix Inequalities in System and Control Theory*. SIAM, Philadephia.

Chen, T. and Francis, B. (1994). *Optimal Sampled-Data Control Systems*. Springer.

Heemels, W., Donkers, M., and Teel, A. (2013). Periodic event-triggered control for linear systems. *IEEE Transactions on Automatic Control*, 58(4), 847–861.

Heemels, W., Johansson, K., and Tabuada, P. (2012). An introduction to event-triggered and self-triggered control. *Proc. of the 51st IEEE Conference on Decision and Control*, 3270–3285.

Löfberg, J. (2004). YALMIP : A toolbox for modeling and optimization in MATLAB. In *Proc. of the CACSD Conf.* Taipei, Taiwan. Available from http://control.ee.ethz.ch/˜joloef/yalmip.php.

Masubutch, I., Ohara, A., and Suda, N. (1998). LMI-based controller synthesis: a unified formulation and solution. *International Journal of Robust and Nonlinear Control*, 8(8), 669–686.

Meng, X. and Chen, T. (2014). Event detection and control co-design of sampled-data systems. *International Journal of Control*, 87(4), 777–786.

MOSEK ApS (2019). *The MOSEK optimization toolbox for MATLAB manual. Version 9.0.* URL http://docs.mosek.com/9.0/toolbox/index.html.

Postoyan, R., Tabuada, P., Nešić, D., and Anta, A. (2015). A framework for the event-triggered stabilization of nonlinear systems. *IEEE Trans. Autom. Control*, 60(4), 982–996.

Åström, K. (2008). *Analysis and Design of Nonlinear Control Systems: In Honor of Alberto Isidori*, chapter Event Based Control, 127–147. Springer.

Tallapragada, P. and Chopra, N. (2012). Event-triggered dynamic output feedback control for LTI systems. In *51st IEEE Conference on Decision and Control*, 6597–6602. Maui, HI, USA.

Tarbouriech, S., Seuret, A., Moreira, L.G., and Gomes da Silva Jr., J.M. (2017). Observer-based event-triggered control for linear systems subject to cone-bounded nonlinearities. *IFAC-PapersOnLine*, 50(1), 7893–7898. 20th IFAC World Congress.

Wang, X. and Lemmon, M.D. (2011). Event-triggering in distributed networked control systems. *IEEE Trans. Autom. Control*, 56(3), 586–601.

Wu, W., Reimann, S., Görges, D., and Liu, S. (2015). Suboptimal event-triggered control for time-delayed linear systems. *IEEE Transactions on Automatic Control*, 60(5), 1386–1391.

Zhang, X.M. and Han, Q.L. (2014). Event-triggered dynamic output feedback control for networked control systems. *IET Control Theory & Applications*, 8(4), 226–234.