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Bipartite Synchronization of coupled system with Markov switching topology

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Abstract: In this paper, we investigate the leaderless bipartite synchronization problem about nonlinear systems with time delay under Markov switching topologies. We first deal with the strongly connected part of the coupled system. Then, by utilizing the consensus trajectory of the strongly connected agents, we transform the consensus problem into a stable problem. A mild condition about switching topology is proposed which just require the union graph has a spanning tree. We also propose a novel method to deal with the error system. Then, sufficient conditions are presented to make all the systems achieve bipartite synchronization under ergodic Makrov switching topology. At last, an example is given to verify our theory.

Keywords: Bipartite synchronization, Leaderless, Time-delay, Makrov switching topology.

1. INTRODUCTION

The synchronization of multi-agent systems are inspired by many phenomena, e.g. distributed sensor, distributed computation, biology flocking Yu et al. (2011); Wu and Zheng (2009); Yang and Zheng (2019); Qin et al. (2013). This problem has attracted a large number of people from various fields, partially because of its application in formation control Qin et al. (2011a); Qin and Yu (2013); Qin et al. (2016). In recent years, the consensus problem is widely researched, offering control protocol that meet a wide range of requirements. To handle the doubled-integrator dynamics with communication delay under time-varying topology, the distributed sample date regulation method is proposed to solve the consensus of nonlinear systems. The cluster consensus problem of linear system is consider in Qin and Yu (2013). Then, the leader following framework of consensus is solved in Qin et al. (2013). In Yang and Zheng (2019), the discrete time fornasini-marchesini systems is considered.

However, in practical applications, to cope with the complex formation flying, we need the coupled system achieve bipartite consensus Liu et al. (2018); Yaghmaie et al. (2017). A great deal of literature focus on the bipartite consensus problem for dynamics modeled by linear of nonlinear system. In Li and Zheng (2019), the authors discuss the bipartite synchronization about neural networks with time delay. In Liu et al. (2018), the authors consider the bipartite leader-following synchronization in signed networks. Based on the upper boundary of time delay, a sufficient condition in form of LMI is given for reaching the synchronization. In Bian and Yao (2011), by using the Lyapunov theory, this paper designs a nonlinear derivative coupling to make the systems achieve bipartite synchronization with distributed delay. In Yaghmaie et al. (2016), heterogeneous linear systems are considered. Bipartite output leader-following synchronization can be achieved. In Yaghmaie et al. (2017), by proposing a new H criterion, the authors also presented a unified framework of output consensus and bipartite output consensus. Consensus problem of linear system with communication noise is considered in technical note Wang et al. (2015).

It is noted that the communication topology of coupled systems is time-varying as existing external disturbance. So far, there are lots of works about consensus problem under Makrov switching topology. A brief summary of related results in Markov switching topology are provided as follows. In Shang (2016), the stochastic consensus problem of linear system with time delay under Markov switching topology is considered. Under the appropriate time delay and Markov progress, the consensus can be achieved. In Todorov et al. (2018), the authors consider discrete-time coupled system. By designing a H_∞ controller with partial observation, in Miao et al. (2018), H consensus control of heterogeneous system over Makrov switching network is presented. Output consensus of first and second order system with time delay is also investigated. In Shang (2016), multi-agent with time-varying delay and topology uncertainty is considered. Synchronization of nonlinear Makov jump system is addressed in Dong et al. (2018). In Mo et al. (2017), leader-following mean square consensus of discrete system with persistent disturbances

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is considered. In Shen et al. (2019b), two-dimensional Markov jump systems is considered. The authors design an asynchronous fault detection filter, to produce signal. In Shen et al. (2019a), the model reduction of Makrov jump systems is considered, some new sufficient conditions are given to ensure the coupled systems achieving mean square stable with H performance. In practical application, the transition probability matrix may not be known exactly. Taking into account the uncertainty, the necessary and sufficient conditions are presented for mean square stability of discrete Makrov jump systems.

Finally, this paper is a continuation and improvement of previous papers Liu et al. (2018); Yaghmaie et al. (2017)Miao et al. (2018)Mo et al. (2017). In this paper, we consider the bipartite consensus problem of nonlinear system with time delay under Makrov switching topology. We just assume the union graph has a spanning tree. The system dynamics with time delay are more general. In Li and Zheng (2019)Liu et al. (2018), the authors consider the leader-following/pinning bipartite consensus. Different from these works, we consider the leaderless bipartite consensus problem. The bipartite synchronization can be achieved if the union graph has a spanning tree, the Markov progress is ergodic and further, the coupling strengths are sufficient strong.

This paper is organized as follows. The preliminary knowledge of graph theory and some important lemmas are presented in Section 2. In Section 3, we introduce the problem considered in our paper. The main results in this paper are given in Section 4. Then, the illustration and conclusion are presented in Section 5 and 6, respectively.

Notation: Let I_n be the n-dimensional identity matrix. Sometimes, we use I to stand for identity matrix with compatible dimension. $sgn(\cdot)$ denotes the sign function. Denote $P > 0$ if P is a positive definite matrix. $\otimes$ denotes the Kronecker product. Let $a = [a_1, a_2, \cdots, a_n]^T \in \mathbb{R}^n$ be a vector, $diag(a) = diag(a_1, a_2, \cdots, a_n)$ is a diagonal matrix with ith element of diagonal is $a_i, i = 1, 2, \cdots, n$. $\lambda_{\min}(P)$, $\lambda_{\max}(P)$ are the minimum and maximum eigenvalues of matrix P, respectively.

2. PRELIMINARY

In this section, we present some preliminary knowledge about graph theory and some basic lemmas.

2.1 Graph Theory

Let $G = (V, E, \mathbb{A})$ be a weighted digraph of order N, where $V = \{1, 2, \cdots, N\}$ is the node set; $E \in V \times V$ is the edge set; $A = [a_{ij}]$ is adjacent matrix. $a_{ij} \neq 0$ if $(v_j, v_i) \in E$ and $a_{ij} = 0$ otherwise. Throughout this paper, we just consider the simple graph, which means $a_{ii} = 0$. The node set V can be divided into two disjoint node sets V_1 and V_2. Assume $a_{ij} > 0$ if $i, j \in V_1$ or $i, j \in V_2$ and $a_{ij} < 0$ if $i \in V_1$, $j \in V_2$ or $i \in V_2$, $j \in V_1$. The Laplacian matrix is defined as $L = diag\left(\sum_{j=1}^{N} |a_{1,j}|, \cdots, \sum_{j=1}^{N} |a_{N,j}|\right) - \mathbb{A}$ Qin et al. (2011b). The Laplacian matrix associated with G can be written in Frobenius form :

$$\begin{bmatrix} L_{11} & & & \\ L_{21} & L_{22} & & \\ \vdots & \vdots & \ddots & \\ L_{q1} & \cdots & \cdots & L_{qq} \end{bmatrix} \tag{1}$$

where $L_{ii} \in \mathbb{R}^{n_i \times n_i}$, $i = 1, \cdots, q$. A digraph is strongly connected is for any two different nodes of the graph are connected by a directed path. We say a directed graph has a spanning tree if there exists a node having a directed path to all other nodes.

In this paper, we consider the time-varying topology $G(t) = (V, E(t), \mathbb{A}(t))$. The Laplacian matrix of $G(t)$ is $L(t)$. Assume there is a series of digraph $G_1 = (V, E_1, \mathbb{A}_1)$, $G_2 = (V, E_2, \mathbb{A}_2)$, $\cdots$, $G_s = (V, E_s, \mathbb{A}_s)$, s is a positive integer, the union graph is defined as $G^{un} = \cup_{i=1}^{s} G_i = (V, \cup_{i=1}^{s} E_i, \cup_{i=1}^{s} A_i)$.

The switching topology is governed by a Markov progress $\theta(t)$, which taking value from a finite set $\{1, 2, \cdots, s\}$. That is mean, $G(t) = G_i$ if and only if $\theta(t) = i$.

Let $(\Omega, \mathbb{F}, \mathbb{P})$ stand for the probability space of Markov progress $\theta(t)$. Then, we have

$$P(\theta(t+\Delta t) = j | \theta(t) = i) = \begin{cases} \gamma_{ij}\Delta t + o(\Delta t), & \text{if} \quad i \neq j \\ 1 + \gamma_{ij}\Delta t + o(\Delta t), & \text{if} \quad i = j \end{cases} \tag{2}$$

where γ_{ij} stand for the transition rate from i to j and $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$. $o(\Delta t)$ denotes the higher order of Δt.

If the digraph G is strongly connected, one can find a positive vector $\beta \in \mathbb{R}^N$, such that $\beta^T L = 0$, $\beta^T \mathbf{1}_N = 1$, where $\mathbf{1}_N = [1, 1, \cdots, 1]^T$.

Lemma 1. (Qin et al. (2011b)) Let $e_i \in \mathbb{R}^n$ stand for a vector, where the ith element is 1 and all other elements are 0; $x \in \mathbb{R}^{nN \times nN}$ is a vector which satisfies $(\beta^T \otimes e_i^T)x = 0$. Then, for any positive semi-definite matrix P, one can get

$$x^T(\tilde{L} \otimes P)x \geq a(L) x^T (\Xi \otimes P)x$$

where $\Xi = diag(\beta)$, $a(L) = \min_{x^T\beta=0, x\neq 0} \frac{x^T \hat{L} x}{x^T \Xi x} > 0$, $\hat{L} = (\Xi L + L^T \Xi)/2$.

Lemma 2. (Qin and Yu (2013)). If digraph G has a spanning tree, then, for any $i = 2, \cdots, q$, we can find a vector β_i such that $\Xi_i L_{ii} + L_{ii}^T \Xi_i > 0$, where $\Xi_i = diag(\beta_i)$. The choosing of β_i can be found in Qin et al. (2011b); Qin and Yu (2013).

3. PROBLEM FORMULATION

Consider the coupled nonlinear multi-agent system consist of N agents. Every agent can be regarded as a node in the digraph G. The dynamics of each agent can be modeled as follows:

$$\dot{x}_i(t) = Ax_i(t) + Bf(x_i(t)) + cf(x_i(t - \tau(t))) + u_i \tag{3}$$

where $x_i(t) = [x_{i1}(t), \cdots, x_{in}(t)]^T \in \mathbb{R}^n$ is state of agent i, $A, B, C \in \mathbb{R}^{n\times n}$ are matrices, nonlinear function $f(x_i(t)) = [f(x_{i1}(t)), \cdots, f(x_{in}(t)))]^T \in \mathbb{R}^n$ is a continuous function; $\tau(t)$ is time delay and $\dot{\tau}(t) < 1$; $u_i(t)$ is control input of agent i.

We choose the following controller for agent i:

$$u_i = \gamma \sum_{j \in N} |a_{ij}(t)| (sgn(a_{ij}) x_j(t) - x_i(t)) \tag{4}$$

where $|a_{ij}(t)|$ denote the absolution of $a_{ij}(t)$, $\gamma > 0$ is the coupling strength.

Here, we define the gauge transformation S, where $S = diag\{s_1, s_2, \cdots, s_N\} \in \mathbb{R}^{N\times N}$, $s_i = 1$ when agent i belong to V_1, $s_i = -1$ when agent i belong to V_2.

Definition 1. The coupled agents with the partition V_1, V_2 is said to be achieve bipartite synchronization if there exits a controller u_i such that the state of agents satisfy

$$E[\|s_i x_i - s_j x_j\|] = 0, \quad \forall i, j \in \{1, 2, \cdots, N\}, i \neq j \quad (5)$$

for any initial value.

In the following, we introduce some assumptions.

Assumption 1. The union graph of $G_1, \cdots, G_s$ have a spanning tree.

Assumption 2. The function $f(x_i(t)$ satisfies the Lipschitz condition, that is, there exists a positive constant k_i, such that

$$|s_i f(x_i(t)) - s_j f(x_j(t))| \leq k_i |s_i x_i - s_j x_j|,$$

where k_i is a positive constant, $\forall i, j \in \{1, \cdots, N\}, i \neq j$.

Assumption 3. The Markov progress $\theta(t)$ is ergodic.

Remark 1. If the Markov progress $\theta(t)$ is ergodic, that is means, every state of the Markov progress can be reachable from other state. There also exists a invariant distribution $\pi = [\pi_1, \cdots, \pi_s]$, where $\pi_i > 0$, $i = 1, 2, \cdots, s$.

Remark 2. According to Assumption 1, the union graph G^{un} has a spanning tree. The matrix $L^{un} = \sum_{i=1}^{s} \pi_i L_i = \pi_1 L_1 + \pi_2 L_2 + \cdots + \pi_s L_s$ is also a Laplacian matrix. The Laplacian matrix L^{un} is also written in Frobenius form, like

$$\begin{bmatrix} L_{11}^{un} & 0 & \cdots & 0 \\ L_{21}^{un} & L_{22}^{un} & 0 & \vdots \\ \vdots & \vdots & \ddots & 0 \\ L_{q1}^{un} & \cdots & \cdots & L_{qq}^{un} \end{bmatrix} \triangleq \begin{bmatrix} L_{11}^{un} & 0 \ \cdots \ 0 \\ L_{21}^{un} & \\ \vdots & \widetilde{\boldsymbol{L}}^{un} \\ L_{q1}^{un} & \end{bmatrix} \quad (6)$$

Under Assumption 1, according to Lemma 1 and Lemma 2, one can choose appropriate β_1 and Ξ. Then there exists a β_i satisfies that $\beta_1^T L_{11}^{un} = 0$, where L_{11}^{un} is the top left submatrix of Laplacian matrix L^{un}.

Then, we first deal with the consensus problem of agents which are strongly connected. Denote $\bar{x} = s_i x_i$, we have

$$\begin{aligned} \dot{\bar{x}} =& s_i A x_i + s_B f(x_i) + s_i C f(x_i(t - \tau(t))) + s_i u_i \\ =& A\bar{x}_i + B s_i f(x_i(t)) + C s_i f(x_i(t - \tau(t))) \\ &+ s_i \sum_{j\in N_i} |a_{ij}| \Gamma(sgn(a_{ij}) x_j - x_i) \end{aligned}$$

Let $e_i = s_i x_i - \sum_{k=1}^{n_1} \beta_1^k s_k x_k$. We have

$$\begin{aligned} \dot{e}_i =& s_i A x_i + s_i B f(x_i(t)) + s_i C f(x_i(t - \tau(t))) + s_i u_i \\ &- \sum_{k=1}^{n_1} \beta_1^k s_k [A x_k + B f(x_k) + C f(x_k(t - \tau(t))) + u_k] \\ =& s_i A x_i + s_i B f(x_i(t)) + s_i C f(x_i(t - \tau(t))) + s_i u_i \\ &- \sum_{k=1}^{n_1} \beta_1^k s_k A x_k - \sum_{k=1}^{n_1} \beta_1^k s_k B f(x_k) \\ &- \sum_{k=1}^{n_1} \beta_1^k s_k C f(x_k(t - \tau(t))) - \sum_{k=1}^{n_1} \beta_1^k s_k u_k \\ =& A e_i + B \left(s_i f(x_i(t)) - \sum_{k=1}^{n_1} \beta_1^k s_k f(x_k) \right) \\ &+ C \left(s_i f(x_i(t - \tau(t))) - \sum_{k=1}^{n_1} \beta_1^k s_k f(x_k(t - \tau(t))) \right) \\ &+ s_i u_i - \sum_{k=1}^{n_1} \beta_1^k s_k u_k \end{aligned}$$

Let $e = [e_1^T, e_2^T, \cdots, e_{n_i}^T]^T$, we have

$$\begin{aligned} \dot{e} =& (I_{n_1} \otimes A) e + (I_{n_1} \otimes B) \Phi(t) + (I_{n_1} \otimes C) \Psi(t - \tau(t)) \\ &- (L_{11} \otimes \Gamma) e - (\beta_1^T L_{11} \otimes \Gamma) e \end{aligned}$$

where $\Phi(t) = \left[\phi_1^T \ \phi_2^T \ \cdots, \phi_{n_1}^T\right]$, $\phi_i = s_i f(x_i(t)) - \sum_{k=1}^{n_1} \beta_1^k s_k f(x_k)$; $\Psi(t - \tau(t)) = \left[\psi_1^T \ \psi_2^T, \ddots, \psi_{n_1}\right]$, $\psi_i = s_i f(x_i(t - \tau(t))) - \sum_{k=1}^{n_1} \beta_1^k s_k f(x_k(t - \tau(t)))$; The time varying Laplacian matrix $L(t)$ also is written in Frobenius form.

4. MAIN RESULTS

In this section, we will present a sufficient condition concerning the bipartite synchronization under Markov switching topology.

Theorem 1. Consider a group of systems (3) under Markov switching topology system. Suppose the Assumptions 1-3 hold, then there exists a

$$\gamma > \max\left\{ \frac{\lambda_{\min}(\Omega)}{a\left(L_{11}^{un}\right)\lambda_{\max}(P)}, \frac{\lambda_{\min}\left(\Omega'\right)\lambda_{\min}\left(\Xi'\right)}{\lambda_{\min}(H)\lambda_{\max}(P)} \right\}$$

such that the coupled systems (3) can achieve consensus in mean square by using control protocol (4), where $\Omega = PA + A^T P + PBB^T P + PCC^T P + \frac{\lambda_{\max}\left(K_{n_1}^T \Xi K_{n_1}\right)}{\lambda_{\min}(\Xi)} I_n + \frac{1}{1-t} \frac{\lambda_{\max}\left(K_{n_1}^T \Xi K_{n_1}\right)}{\lambda_{\min}(\Xi)} I_n$, $H = \Xi' \tilde{L}^{un} + \tilde{L}^{unT} \Xi'$, $\Omega' = PA + A^T P + PBB^T P + PCC^T P + \frac{\lambda_{\max}(K_{N-n_1}^T \Xi' K_{N-n_1})}{\lambda_{\min}(\Xi')}) I_n + \frac{1}{1-\bar{\tau}} \frac{\lambda_{\max}(K_{N-n_1}^T \Xi' K_{N-n_1})}{\lambda_{\min}(\Xi')}) I_n$, $K_{n_1} = \operatorname{diag}(k_1, \cdots, k_{n_1})$, $K_{N-n_1} = \operatorname{diag}\{k_{n_1+1}, \cdots, k_N\}$

Proof: Step 1: We consider the candidate Lyapunov function $V(t) = E\left[\frac{1}{1-\bar{\tau}} \int_{t-\tau(t)}^{t} e^T(s) \left(K_{n_1}^T \Xi K_{n_1} \otimes I_n\right) e(s) ds\right] + E\left[e^T (\Xi \otimes P) e\right]$. Then, we have

$$\dot{V} = \dot{V}_1(t) + \dot{V}_2(t)$$

where $V_2 = E\left[\frac{1}{1-\boldsymbol{\tau}} \int_{t-\tau(t)}^{t} e^T(s) \left(K_{n_1}^T \Xi K_{n_1} \otimes I_n\right) e(s) ds\right]$, $V_1(t) = E\left[e^T (\Xi \otimes P) e\right]$, where $\Xi = \operatorname{diag}\left\{\beta_1^1, \cdots, \beta_1^{n_1}\right\}$.

Let $V_{1i}(t) = E\left[e^T(\Xi \otimes P)e1_{\theta(t)=i}\right]$, then, we have

$$dV_{1i}(t) = E\left[de^T(\Xi \otimes P)e + e^T(\Xi \otimes P)de\right] + \sum_{j=1}^{s} q_{ji}V_j dt + O(dt)$$

Then, we have

$$\begin{aligned}\frac{dV_1(t)}{dt} \leq & E\left[e^T\left(\Xi \otimes \left(PA + A^TP\right)\right)e + 2e^T(\Xi \otimes PB)\Phi(t)\right.\\ & + 2e^T(\Xi \otimes PC)\Psi(t-\tau(t))\\ & \left.-e^T\left(\gamma a\left(L_{11}^{un}\right)\Xi \otimes P\right)e\right]\\ \leq & E\left[e^T\left(\Xi \otimes \left(PA + A^TP\right)\right)e + e^T\left(\Xi \otimes PBB^TP\right)e\right.\\ & + e^T\left(K_{n1}^T\Xi K_{n_1} \otimes I_n\right)e + e^T\left(\Xi \otimes PCC^TP\right)e\\ & + e^T(t-\tau)\left(K_{n_1}^T\Xi K_{n_1} \otimes I_n\right)e(t-\tau)\\ & \left.-e^T\left(\gamma a\left(L_{11}^{un}\right)\Xi \otimes P\right)e\right]\\ \leq & E\left[e^T\left(\Xi \otimes \left(PA + A^TP + PBB^TP + PCC^TP\right.\right.\right.\\ & \left.+\frac{\lambda_{\max}\left(K_{n_1}^T\Xi K_{n_1}\right)}{\lambda_{\min}(\Xi)}I_n - \gamma a\left(L_{11}^{un}\right)P\right)e\\ & \left.+e^T(t-\tau)\left(K_{n_1}^T\Xi K_{n_1} \otimes I_n\right)e(t-\tau)\right]\end{aligned}$$

and

$$\begin{aligned}\frac{dV_2(t)}{dt} = & E\left[\frac{1}{1-\dot{\tau}}e^T\left(K_{n_1}^T\Xi K_{n_1} \otimes I_n\right)e\right.\\ & \left.-e^T(t-\tau)\left(K_{n_1}^T\Xi K_{n_1} \otimes I_n\right)e(t-\tau)\right]\end{aligned}$$

Then, according to Lemma 1, we can get the following results:

$$\begin{aligned}\dot{V}(t) \leq & E\left[e^T\left(\Xi \otimes \left(PA + A^TP + PBB^TP\right.\right.\right.\\ & + PCC^TP + \frac{\lambda_{\max}\left(K_{n_1}^T\Xi K_{n_1}\right)}{\lambda_{\min}(\Xi)}I_n\\ & \left.+ \frac{1}{1-\dot{\tau}}\frac{\lambda_{\max}\left(K_{n_1}^T\Xi K_{n_1}\right)}{\lambda_{\min}(\Xi)}I_n - \gamma\alpha\left(L_{11}^{un}\right)P))\right]\end{aligned}$$

Then, choosing $\gamma > \frac{\lambda_{\min}(\Omega)}{a\left(L_{11}^{un}\right)\lambda_{\max}(P)}$ we have

$$V(t) \leq e^{-\epsilon'(t-t_0)}V(t_0)$$

where $\epsilon' = \gamma a\left(L_{11}^{un}\right)\lambda_{\min}(P) - \lambda_{\max}(\Omega)$.

Then, as $t \to \infty$, we have $V(t) \to 0$. Because $\Xi \otimes P$ and $I \otimes K_{n_1}^T K_{n_1}$ are positive definite matrices, we can get $e(t) \to 0$ in mean square.

Step 2: Denote the consensus of the systems $1, 2, \cdots, n_1$ as $x^*(t) = \sum_{j=1}^{n_1}\beta_1^j x_j$, where $x^*(t)$ satisfies the following dynamics:

$$\dot{x}^*(t) = Ax^*(t) + Bf\left(x^*(t)\right) + Cf\left(x^*(t-\tau(t))\right).$$

Now, we proceed to prove the consensus of agents $n_1 + 1, \cdots, N$. Let $e_i(t) = \bar{x}_i(t) - x^*(t) = \bar{x}_i - \sum_{j=1}^{n_1}\beta_1^j\bar{x}_j$, $i = n_1+1, n_1+2, \cdots, N$. Then, we have

$$\begin{aligned}\dot{e}_i = & A\bar{x}_i + Bs_if\left(x_i\right) + Cs_if\left(x_i(t-\tau)\right)\\ & + s_i\sum_{j\in N_i}\left|a_{ij}\right|\gamma\left(\operatorname{sgn}\left(a_{ij}\right)x_j - x_i\right)\\ & - \sum_{k=1}^{n_i}\left[A\bar{x}_k + Bs_kf\left(x_k\right) + Cs_kf\left(x_k(t-\tau)\right)\right.\\ & \left.+s_k\sum_{j\in N_k}\left|a_{kj}\right|\gamma\left(\operatorname{sgn}\left(a_{kj}\right)x_j - x_k\right)\right]\\ = & Ae_i + Bs_if\left(x_i\right) - \sum_{k=1}^{n_1}\beta_1^kBs_kf\left(x_k\right)\\ & + Cs_if\left(x_i(t-\tau)\right) - \sum_{k=1}^{n_1}\beta_1^kCs_kf\left(x_k(t-\tau)\right)\\ & + s_i\sum_{j\in N_i}\left|a_{ij}\right|\gamma\left(\operatorname{sgn}\left(a_{ij}\right)x_j - x_i\right)\\ & \left.- \sum_{k=1}^{n_i}s_k\sum_{j\in N_k}\left|a_{kj}\right|\gamma\left(\operatorname{sgn}\left(a_{kj}\right)x_j - x_k\right)\right]\end{aligned}$$

With an abuse of notation, we still use e to denote $e = \left[e_{n_1+1}^Te_{n_1+2}^T\cdots e_N^T\right]^T$. Then, we have

$$\begin{aligned}\dot{e} = & \left(I_{N-n_1} \otimes A\right)e + \left(I_{N-n_1} \otimes B\right)\Phi(t) - \tilde{L} \otimes \Gamma e\\ & + \left(I_{N-n_1} \otimes C\right)\Psi(t-\tau) - \mathbf{1}_{N-n_1}\left(\beta_1^TL_{11}(t) \otimes \Gamma\right)\bar{x}_{n_1}\end{aligned} \tag{7}$$

where $\tilde{L}(t) = \begin{bmatrix} L_{22}(t) & 0 & \cdots & 0\\ L_{32}(t) & L_{33}(t) & \ddots & \vdots\\ \vdots & \vdots & \ddots & 0\\ L_{q2}(t) & L_{q3}(t) & \cdots & L_{qq}(t)\end{bmatrix}$, and $\bar{x}_{n_1} = \left[s_1x_1, \cdots, s_{n_1}x_{n_1}\right]^T$.

We propose the candidate Lyapunov function

$$\begin{aligned}V(t) = & V_3(t) + V_4(t)\\ = & E\left[\sum_{i=2}^{q}\Delta_ie_i\left(\Xi_i \otimes P\right)e_i\right]\\ & + E\left[\frac{1}{1-\dot{\tau}}\int_t^{t-\tau}e^T\left(\Xi' \otimes K^TK\right)eds\right]\end{aligned}$$

where $V_3 = E\left[\sum_{i=n_1+1}^{N}\Delta_ie_i\left(\Xi_i \otimes P\right)e_i\right]$, Δ_i is positive constant, $V_4 = E\left[\frac{1}{1-\tau}\int_t^{t-\tau}e^T\left(K^T\Xi'K \otimes I_n\right)eds\right]$, where $\Xi_i = \operatorname{diag}\left\{\beta_i^1, \cdots, \beta_i^{n_i}\right\}$, $i = 2, 3, \cdots, q$.

We can write V_3 into following form:

$$V_3(t) = E\left[e^T(\Xi \otimes P)e\right] \tag{8}$$

where $\Xi' = \begin{bmatrix}\Delta_2\Xi_2 & & & \\ & \Delta_3\Xi_3 & & \\ & & \ddots & \\ & & & \Delta_q\Xi_q\end{bmatrix}$. Then, one can get

$$V_{3i}(t) = E\left[e^T\Xi' \otimes Pe1_{\{\theta(t)=i\}}\right] \tag{9}$$

Differenting the function $V_{3i}(t)$, we can get the following results:

$$
\begin{aligned}
dV_{3i}(t) =& E\left[de^T\Xi'\otimes Pe1_{\{\theta(t)=i\}}\right] \\
&+ E\left[e^T\Xi'\otimes Pde1_{\{\theta(t)=i\}}\right] \\
&+ \sum_{j=1}^{s}\gamma_{ij}V_j(t)dt + o(dt) \\
=& E\left[e^T\Xi'\otimes P\left[\left(I_{N-n_1}\otimes A\right)e + \left(I_{N-n_1}\otimes B\right)\Phi\right.\right. \\
&+ \left(I_{N-n_1}\otimes C\right)\Psi - \gamma\left(\tilde{L}\otimes I_n\right)e \\
&\left.\left.-\gamma\mathbf{1}_{N-n_i}\left(\beta_1^T L_{11}(t)\otimes I_n\right)\bar{x}_{n_1}\right]\right. \\
=& E\left[e^T\Xi'\otimes\left(PA + A^TP\right)e + 2e^T\left(\Xi'\otimes PB\right)\Phi(t)\right. \\
&+ 2e^T\left(\Xi'\otimes PC\right)\Psi(t-\tau) \\
&- \gamma e^T\left(\Xi'\tilde{L}(t) + \tilde{L}^T(t)\Xi'\otimes P\right)e \\
&\left.-\gamma e^T\left(\Xi'\mathbf{1}\beta_1 L_{11}\otimes P\right)\bar{x}_{n_1}\right] \\
&+ \sum_{j=1}^{s}\gamma_{ij}V_j(t)dt + o(dt)
\end{aligned}
$$

where $\mathbf{1}$ is vector of all ones with compatible dimension. Following the trajectory of (7), we have

$$
\begin{aligned}
\dot{V}_3(t) =& E\left[e^T\left(\Xi'\otimes\left(PA + A^TP\right)\right)e + 2e^T\left(\Xi'\otimes PB\right)\Phi(t)\right. \\
&+ 2e^T\left(\Xi'\otimes PC\right)\Psi(t-\tau) \\
&- \gamma e^T\left(\Xi'\tilde{L}(t) + \tilde{L}^T(t)\Xi'\otimes P\right)e \\
&\left.-\gamma e^T\left(\Xi'\mathbf{1}\beta_1 L_{11}\otimes P\right)\bar{x}_{n_1}\right] \\
\leq& E\left[e^T\left[\Xi'\otimes\left(PA + A^TP + PBB^TP + PCC^TP\right.\right.\right. \\
&\left.\left.+\frac{\lambda_{\max}\left(K_{N-n_1}^T\Xi'K_{N-n_1}\right)}{\lambda_{\min}\left(\Xi'\right)}\right)I_n\right]e \\
&\left.+e^T(t-\tau)\left(K_{N-n_1}^T\Xi'K_{N-n_1}\otimes I_n\right)e(t-\tau)un\right] \\
&- \gamma e^T(H\otimes P)e
\end{aligned}
$$

According to Lemma 2 and the proof of Qin et al. (2011b), one can choose appropriate Δ_i such that $H > 0$.

Then, we have

$$
\begin{aligned}
\dot{V}_4 =& E\left[\frac{1}{1-\dot{\tau}}e\left(K_{N-n_1}^T\Xi'K_{N-n_1}\otimes I_n\right)e\right. \\
&\left.-e^T(t-\tau)\left(K_{N-n_1}^T\Xi'K_{N-n_1}\otimes I_n\right)e(t-\tau)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{V} =& E\left[e^T\left(\Xi'\otimes PA + A^TP + PBB^TP + PCC^TP\right.\right. \\
&+\frac{\lambda_{\max}\left(K_{N-n_1}^T\Xi'K_{N-n_1}\right)}{\lambda_{\min}\left(\Xi'\right)}\Big)I_n \\
&+\frac{1}{1-\dot{\tau}}\frac{\lambda_{\max}\left(K_{N-n_1}^T\Xi'K_{N-n_1}\right)}{\lambda_{\min}\left(\Xi'\right)}\Big)I_n \\
&\left.\left.-\gamma\frac{\lambda_{\min}(H)}{\lambda_{\min}\left(\Xi'\right)}P\right)e\right]
\end{aligned}
$$

If $\gamma > \frac{\lambda_{\min}(\Omega')\lambda_{\min}(\Xi')}{\lambda_{\min}(H)\lambda_{\max}(P)}$, we can get $V(t) \leq e^{-\epsilon''(t-t_0)}V(t_0)$, where $\epsilon'' = \gamma\lambda_{\min}(H)\lambda_{\max}(P) - \lambda_{\max}(\Omega')\lambda_{\min}(\Xi')$.

Then, we have $V(t) \leq e^{-\epsilon''(t-t_0)}V(t_0)$, where $\epsilon = \min\{\epsilon', \epsilon''\}$. Finally, one can get the $e(t) \to 0$ in mean square. Now, we can give the conclusion the leaderless coupled systems under Makrov switching topology can achieve consensus in mean square.

Remark 3. Note that our results are more general which including the results in Liu et al. (2018),Yaghmaie et al. (2017) Bian and Yao (2011)Li and Zheng (2019), which just consider the leader-following or pinning bipartite problem.

5. EXAMPLE

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & -0.1 \\ -5.0 & 1.5 \end{bmatrix}, C = \begin{bmatrix} -1.5 & 0.1 \\ -0.2 & -1 \end{bmatrix}$, $f(x) = (|x+1| - |x-1|)/2$, $\tau(t) = 0.5 + 0.1\sin(t)$. The switching topologies are shown as follows. We assume $V_1 = \{1, 4\}$ and $V_2 = \{2, 3\}$, see Figure 1. The invariant distribution is $\pi = [0.4\ 0.2\ 0.4]$. When $(v_j, v_i) \in E$, we choose $a_{ij} = 1$ if $i, j \in V_1$, otherwise $a_{ij} = -1$. According to the Theorem 1, we choose the $\gamma = 6$.

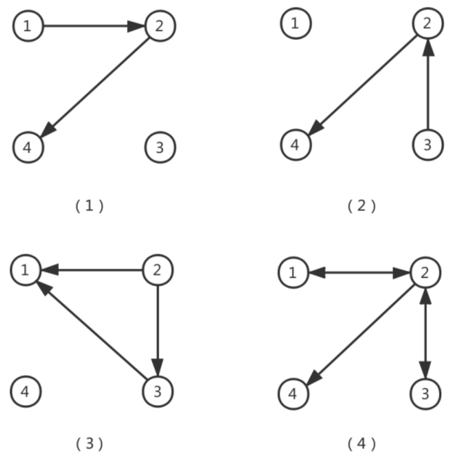

Fig. 1. The topology of the networks (1): G_1, (2): G_2 , (3): G_3 and (4) is the union graph of G_1, G_2, G_3.

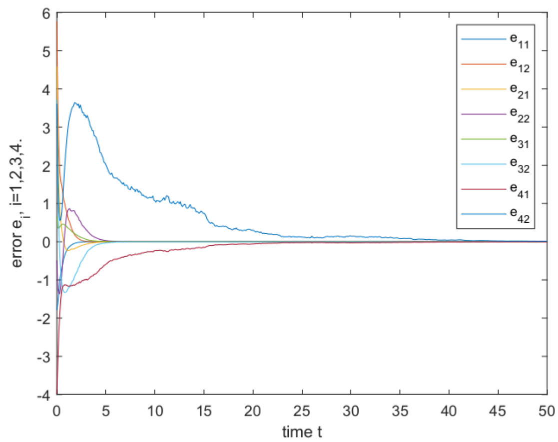

Fig. 2. Consensus errors e i with Makrov switching topologies, $i = 1, 2, 3, 4$. From the Figure, one can find e_1, e_2, e_3 synchronize to zero more fast than e_4.

We choose the initial state of each agent randomly form $[-10,\ 10]$. Then, the consensus error $e_i(t)$ is shown in Figure 2, $i = 1, \cdots, 4$. From Figure 2, one can find e_1, e_2, e_3 synchronization to zero quickly, while e_4 synchronization to zero slowly, because the union graph L_{un}^{11} is strongly connected.

6. CONCLUSION

In this paper, we consider the bipartite consensus of nonlinear system with time-delay under Markov switching topology. We have shown the leaderless bipartite can be achieved if the union graph has a spanning tree, the Markov progress is ergodic and coupling strength strong enough.

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