Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Stability Analysis of Multivariable Digital Control Systems with Uncertain Timing

Maximilian Gaukler * **Günter Roppenecker** * **Peter Ulbrich** *

* *Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU), Erlangen, Germany (e-mail: max.gaukler@fau.de, guenter.roppenecker@fau.de, peter.ulbrich@fau.de)*

Abstract: The ever increasing complexity of real-time control systems results in significant deviations in the timing of sensing and actuation, which may lead to degraded performance or even instability. In this paper we present a method to analyze stability under *mostly-periodic* timing with bounded uncertainty, a timing model typical for the implementation of controllers that were actually designed for strictly periodic execution. In contrast to existing work, we include the case of multiple sensors and actuators with *individual* timing uncertainty. Our approach is based on the discretization of a linear impulsive system. To avoid the curse of dimensionality, we apply a decomposition that breaks down the complex timing dependency into the effects of individual sensor-actuator pairs. Finally, we verify stability by norm bounding and a Common Quadratic Lyapunov Function. Experimental results substantiate the effectiveness of our approach for moderately complex systems.

Keywords: Control over Networks, Systems with Time-Delays, Sampled-Data Systems, Jitter, Common Quadratic Lyapunov Function, Linear Matrix Inequality, Linear Impulsive Systems

1. INTRODUCTION

The vast majority of control systems are implemented as discrete-time controllers executed on a real-time computing platform. In the design process, sampling the sensors and updating the actuators is generally assumed to be synchronous and strictly periodic. However, on modern computing platforms and due to the ever-growing overall system complexity, it is becoming increasingly difficult and often prohibitively costly to satisfy this assumption in the actual implementation: First, execution times are non-constant and hard to predict, especially when multiple applications share one processor. Second, contemporary digital sensors incorporate excessive signal pre-processing. Consequently, the sensor reading may be outdated by a small but varying duration, even if it is queried strictly periodically. Last but not least, the accuracy of time synchronization in distributed (i. e., ranging from multi-core to networked) systems is limited. All these factors jeopardize the controller's design assumptions and add to timing uncertainties in its input and output.

Therefore, the practical implementation of a controller with period T will in most cases result in a *mostly-periodic* system in which the sensor and actuator times do not lie on the intended periodic grid $t = kT$, $k \in \mathbb{N}$, but in a small *timing window* around these points. The resulting dynamics may be worse or even unstable. In practice, it is often assumed that the timing window is still small enough such that stability and convergence are not affected. This argument is problematic for two reasons: Firstly, without proper analysis, there is no guarantee that a certain timespan is "small enough". Secondly, larger timing windows relax and simplify the scheduling of real-time applications and are therefore even desirable from a (real-time) design point of view. Consequently, in this paper, we concentrate on the stability analysis of mostly-periodic digital control loops with given timing windows.

2. RELATED WORK

Providing a deterministic execution platform has always been a core aim in real-time scheduling and design. Here, the general approach to eliminate timing uncertainty is to rely on a time-triggered execution of the controller code at predetermined instants of time. Known representatives for this are the Cyclic Executive (Baker and Shaw, 1989) and Fixed-Priority Models (Sha and Goodenough, 1989) for periodic tasks. However, the focus is on deadline adherence rather than avoidance of jitter. Synchronous development models address the latter problem. For example, the logical execution time (LET) paradigm (Henzinger et al., 2003) suggests a decomposition of input and output: Sensors are sampled at fixed time instants (e. g., $t = kT$). Instead of updating the output immediately after the new value has been computed, the update is delayed until $t = kT + D_\mathrm{u}$ to eliminate jitter. In general, support for *exact* synchronization requires, however, tailored programming languages and hardware support and is thus inapplicable to a wide range of systems. Therefore, most practical implementations of LET resort to overapproximations and pessimism to match synchronicity within some uncertainty, which results in a timing window as considered in this work.

For the analysis of sampled-data systems with uncertain timing, a wide array of theoretical methods is available (cf. Hetel et al. (2017)). From a user's point of view, the existing results building upon these methods can be categorized by the employed timing model:

Copyright lies with the authors

Based on the small gain theorem, Cervin (2012) analyzes stability for a timing model similar to ours. The analysis is, however, restricted to the single-input-single-output (SISO) case, which is easier since there are only two scalar timing uncertainties, namely sensor and actuator delay. The same holds for multiple inputs and outputs if all sensors are jointly sampled and all actuators are jointly updated. This results in a system with SISO-like timing but vector-valued signals ("quasi-SISO"). However, the quasi-SISO assumption is invalid for systems with multiple sensors that are not exactly synchronized.

Quasi-SISO cases are analyzed in Kao and Rantzer (2007); Al Khatib et al. (2016); Bauer et al. (2012) and, with restriction to quantized output delays, in Fontanelli et al. (2013). To model network-controlled systems, Bauer et al. (2012) also offers the alternative model that exactly one sensor or actuator is updated in every control period, thereby transforming a multiple-input-multiple-output (MIMO) system to a switched quasi-SISO one. As this scenario is tailored to networked control with severely restricted communication resources, it does not match the common scenario of an embedded system that has enough resources to query all sensors in every period.

To the best of our knowledge, none of the existing publications address the actual MIMO case of multiple sensors and actuators with independent timing uncertainties. Filling this critical gap is the contribution of this work.

3. PROBLEM STATEMENT

System Model: A control loop that is exponentially stable for perfect timing is executed with uncertain timing. We employ the system model by Gaukler and Ulbrich (2019), restricted to the linear case without disturbance and measurement uncertainty:

The plant $\dot{x}_\mathrm{p}(t) = A_\mathrm{p} x_\mathrm{p}(t) + B_\mathrm{p} u(t)$ with state $x_\mathrm{p}(t) \in \mathbb{R}^{n_\mathrm{p}}$, output $y(t) = C_\mathrm{p} x_\mathrm{p}(t) \in \mathbb{R}^p$ and input $u(t) \in \mathbb{R}^m$ is controlled by a discrete-time controller with fixed period $T > 0$, state $x_\mathrm{d}(t) \in \mathbb{R}^{n_\mathrm{d}}$. The controller dynamics are

$$y_{\mathrm{d},k} = y_k, \quad x_{\mathrm{d},k+1} = A_\mathrm{d} x_{\mathrm{d},k} + B_\mathrm{d} y_{\mathrm{d},k}, \quad u_k = C_\mathrm{d} x_{\mathrm{d},k}, \quad (1)$$

where $y_\mathrm{d} \in \mathbb{R}^p$ is a measurement buffer introduced for formal reasons.

Under ideal timing, the measurement $y_{\mathrm{d},k}$ and actuation u would be updated at $t = kT$. Actually, updating the i-th actuator component $u^{(i)}$ is offset by the timing deviation $\Delta t_{\mathrm{u},i,k}$ and, respectively, sampling $y^{(i)}$ by $\Delta t_{\mathrm{y},i,k}$:

$$u^{(i)}(t) = u_k \quad \text{for} \quad kT + \Delta t_{\mathrm{u},i,k} \le t < (k+1)T + \Delta t_{\mathrm{u},i,k+1},$$
$$y^{(i)}_{\mathrm{d},k} = y^{(i)}(kT + \Delta t_{\mathrm{y},i,k}). \quad (2)$$

The timing deviations are unknown but bounded to

$$\underline{\Delta t}_{\{\mathrm{u,y}\},i} \le \Delta t_{\{\mathrm{u,y}\},i,k} \le \overline{\Delta t}_{\{\mathrm{u,y}\},i}, \quad (3)$$

where the bounds are less than half a period:

$$-T/2 < \underline{\Delta t}_{\{\mathrm{u,y}\},i} \le \overline{\Delta t}_{\{\mathrm{u,y}\},i} < T/2. \quad (4)$$

Formalization: To achieve a uniform formulation, the "discrete-time" variables u, x_d and y_d are treated as continuous-time signals that are updated at certain times and remain constant inbetween (zero-order hold). In this formulation, the k-th control period ($k \in \mathbb{N}$) is executed as follows: At $t_{\mathrm{y},i,k} = kT + \Delta t_{\mathrm{y},i,k}$, the i-th sensor, $i = 1, ..., p$, is sampled by setting the i-th component of $y_\mathrm{d}(t)$ to the i-th component of $y(t)$. Similarly, the j-th actuator, $j = 1, ..., m$, is updated at $t_{\mathrm{u},j,k} = kT + \Delta t_{\mathrm{u},j,k}$ by setting the j-th component of $u(t)$ to the j-th component of $C_\mathrm{d} x_\mathrm{d}(t)$. Finally, the discrete controller is updated at $t = (k + 1/2)T$ by setting $x_\mathrm{d}(t) = A_\mathrm{d} x_\mathrm{d}(t^-) + B_\mathrm{d} y_\mathrm{d}(t^-)$. As discussed later, fixing this time at $t = (k + 1/2)T$ is without loss of generality; it may be earlier or later as long as the order of events is maintained.

For readability, the startup behavior is defined such that the 0-th control period is skipped and the initial states are given at $t_0 = T/2$. The resulting system is linear but nondeterministic and time-variant. For a detailed discussion of this model, see Gaukler and Ulbrich (2019).

Goal: We want to prove exponential stability of the closed loop for moderate timing uncertainties. The focus is on an efficient solution that scales well to systems with a large number of inputs and outputs, even if this scalability makes the result more pessimistic and therefore the approach is only applicable to small timing uncertainties.

We define stability as the exponential decay of plant state x_p, controller state x_d, sampled measurement y_d and actuation u, which are combined in the state vector

$$x(t) = \begin{bmatrix} x_\mathrm{p}(t)^\top & x_\mathrm{d}(t)^\top & y_\mathrm{d}(t)^\top & u(t)^\top \end{bmatrix}^\top \in \mathbb{R}^n \quad (5)$$

of dimension $n = n_\mathrm{p} + n_\mathrm{d} + p + m$:

Definition 1. The closed loop with initial state $x(t_0)$ admits *Continuous-Time Globally Uniform Exponential Stability*, denoted as $\mathrm{CGES}(\lambda, D)$, iff there exist constants $D \in [1, \infty)$ and $\lambda < 0$ such that for all possible timings

$$|x(t)| \le D|x(t_0)|\mathrm{e}^{\lambda(t-t_0)} \quad \forall t \ge t_0, \ \forall x(t_0) \in \mathbb{R}^n. \quad (6)$$

4. NOTATION

Definitions are denoted with a colon, e. g., $a := b$ means that a is defined as b. We define $\mathbb{R}$ as the real numbers, $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z} := \{0, \pm 1, \dots\}$. For a set S, the number of elements is denoted $|S|$. Rounding down is $\lfloor x \rfloor := \max\{z \in \mathbb{Z} \mid z \le x\}$. The euclidean norm of $x \in \mathbb{R}^n$ is $|x| := \sqrt{x^\top x}$, where $^\top$ denotes transposition. If $A \in \mathbb{R}^{n \times n}$ has eigenvalues λ_i, it has spectral radius $\rho\{A\} := \max_i |\lambda_i|$. The spectral norm is $\|A\|_\sigma := \rho\{A^\top A\}$.

For $a, b \in \mathbb{Z}$, the reversed product $\widetilde{\Pi}$ is defined as

$$\widetilde{\prod_{i=a}^{b}} X_i := \prod_{i=-b}^{-a} X_{-i} = \begin{cases} X_b X_{b-1} \dots X_{a+1} X_a, & a \le b, \\ I, & a > b. \end{cases} \quad (7)$$

Positive definiteness of functions and matrices is denoted by $f(x) \succ 0$ and $P \succ 0$. The Cholesky Decomposition of $P \succ 0$ is $P =: P^{1/2}(P^{1/2})^\top$. We use common definitions and properties of matrix norms from Bernstein (2009); a detailed list is detached to the extended version of this paper (Gaukler et al., 2019, Section 4).

5. APPROACH

This section presents the high-level structure of our approach. Details are given in the subsequent sections.

Discretization: In Section 6, we apply a time discretization

$$x_k := x(t_k^+) := \lim_{\epsilon \to 0^+} x(kT + T/2 + \epsilon), \tag{8}$$

which leads to the linear discrete-time system $x_{k+1} = A_k x_k$, whose transition matrix $A_k = A(\Delta t_k)$ depends on the current timing vector

$$\Delta t_k := [\Delta t_{\mathrm{u},1,k} \ \dots \ \Delta t_{\mathrm{u},m,k} \ \Delta t_{\mathrm{y},1,k} \ \dots \ \Delta t_{\mathrm{y},p,k}]^\top . \tag{9}$$

The offset $+T/2$ was chosen such that the sensing and actuation events cannot move across the discretization times. This ensures that A_k depends only on Δt_k.

In the following, the subscript k of timing variables $\Delta t_{\dots}$ is often omitted. To further simplify the notation, the system dynamics are defined as right-side continuous, so that always $x(t^+) = x(t)$. Therefore, the discretization is simplified to $x_k := x(t_k)$ with $t_k := kT + T/2$.

Stability of the discretized system is easier to analyze, but still *equivalent* to the desired continuous-time stability:

Definition 2. The discretized control loop

$$x_{k+1} = A_k x_k, \quad A_k \in \mathcal{A} \subset \mathbb{R}^{n \times n} \tag{10}$$

admits *Discrete-Time Globally Uniform Exponential Stability*, denoted as "$\mathcal{A}$ is DGES(ρ, C)", iff there exist constants $C \in [1, \infty)$ and $\rho \in (0, 1)$ such that

$$|x_k| \leq C |x_0| \rho^k \quad \forall k \geq 0, \forall x_0 \in \mathbb{R}^n, \forall A_0, A_1, ... \in \mathcal{A}. \tag{11}$$

Here, $\mathcal{A} = \{A(\Delta t_k) \mid \underline{\Delta t}_{\{\mathrm{u,y}\},i} < \Delta t_{\{\mathrm{u,y}\},i,k} < \overline{\Delta t}_{\{\mathrm{u,y}\},i}\}$ is the set of possible A_k for all possible timings $\Delta t_{\{\mathrm{u,y}\},i,k}$.

Theorem 3. For the given control loop, CGES $\Leftrightarrow$ DGES.

Proof. The proof given in Section 6.3 works by bounding the overshoot inbetween two discretization points. □

Next, we want to show DGES by a Common Quadratic Lyapunov Function (CQLF): Find $P \in \mathbb{R}^{n \times n}$ such that

$$V_P(x) := x^\top P x \succ 0 \quad \text{with} \quad V_P(A_k x) \prec V_P(x) \quad \forall A_k \in \mathcal{A}.$$

Difficulty: To the best of our knowledge, the straightforward extension of an existing method is not feasible:

A direct numerical approach based on a grid of possible Δt_k (e. g. grid-and-bound as in Heemels et al. (2010)) suffers from *exponential complexity* with regard to the number $m + p$ of sensors and actuators, which is also the dimension of the timing parameter space.

Similarly, an analytical approach which directly uses an explicit expression for $A(\Delta t_k)$ suffers from the prohibitively large number of case distinctions corresponding to the $(m + p)!$ possible orderings of sensor and actuator times.

Decomposition (Section 7): We avoid these difficulties by breaking up the dynamics into a sum:

Theorem 4. (Decomposition). The transition matrix A, which depends on $m + p$ scalar timing variables, can be split into a sum of functions of one scalar parameter each:

$$\begin{aligned} A(\Delta t) =& A(\Delta t = 0) + \sum_{i=1}^{m} \Delta A_{\mathrm{u},i}(\Delta t_{\mathrm{u},i}) \\ &+ \sum_{j=1}^{p} \Delta A_{\mathrm{y},j}(\Delta t_{\mathrm{y},j}) \\ &+ \sum_{i=1}^{m} \sum_{j=1}^{p} \Delta A_{\mathrm{uy},i,j}(\Delta t_{\mathrm{y},j} - \Delta t_{\mathrm{u},i}), \end{aligned} \tag{12}$$

where $A(\Delta t = 0)$ is the nominal case and $\Delta A_{\dots}$ are "deviations" that obey $\lim_{|\Delta t| \to 0} \Delta A_{\dots} = 0$.

Proof. See Section 7 for the proof and results. □

Loosely interpreted, $\Delta A_{\mathrm{u},i}$ is the deviation of A resulting from the timing of the i-th actuator, $\Delta A_{\mathrm{y},j}$ corresponds to the j-th sensor, and $\Delta A_{\mathrm{uy},i,j}$ to the influence of actuator i on sensor j. Explicit expressions are given in Section 7.

Stability by Norm Bounding (Sections 8 to 10): As assumed in the problem setting, the nominal case (perfect timing $\Delta t = 0$) is stable and therefore achieves DGES with $\rho < 1$. The resulting safety margin $1 - \rho > 0$ can be used to prove stability up to a certain amount of timing deviation. For this, we use a matrix norm corresponding to a CQLF:

Theorem 5. Let $V_P(x) = x^\top P x$, $P \in \mathbb{R}^{n \times n}$, be a positive definite function. Then the *P-ellipsoid norm*

$$\|A\|_P := \max_{x \neq 0} \sqrt{\frac{V_P(Ax)}{V_P(x)}} \tag{13}$$

is a submultiplicative matrix norm.

Proof. See Section 8. □

This norm $\|A\|_P$ represents the worst-case decay of $V_P(x)$ for the time-invariant system $x_{k+1} = A x_k$:

$$\|A\|_P \leq \rho \quad \Leftrightarrow \quad \big(V_P(x_{k+1}) \leq \rho^2 V_P(x_k) \quad \forall x_k\big). \tag{14}$$

In general, norm bounds can be highly pessimistic. However, this norm can *accurately* capture stability of the *nominal* case $x_{k+1} = A(\Delta t = 0) x_k$, for which $\rho\{A(\Delta t = 0)\} < 1$ is the minimal possible stability factor ρ for DGES.

Theorem 6. There exists P such that $\rho_\mathrm{n} := \|A(\Delta t = 0)\|_P$ is arbitrarily close to $\rho\{A(\Delta t = 0)\}$.

Proof. See Section 8, Theorem 13. □

To check stability for uncertain timing, choose any $P \succ 0$ for which $\rho_\mathrm{n} < 1$. This exists by the previous theorem; the implementation is discussed later. Then, stability under uncertain timing can be shown if the summands $\Delta A_{\dots}$ in (12), which represent timing deviation, are small enough:

Theorem 7. (Norm Bounding). The system is DGES if

$$\begin{aligned} \Bigg(\underbrace{\|A(\Delta t = 0)\|_P}_{=\rho_\mathrm{n}} &+ \sum_{i=1}^{m} \|\Delta A_{\mathrm{u},i}(\Delta t_{\mathrm{u},i})\|_P \\ &+ \sum_{j=1}^{p} \|\Delta A_{\mathrm{y},j}(\Delta t_{\mathrm{y},j})\|_P \\ + \sum_{i=1}^{m} \sum_{j=1}^{p} &\|\Delta A_{\mathrm{uy},i,j}(\Delta t_{\mathrm{y},j} - \Delta t_{\mathrm{u},i})\|_P \Bigg) < 1 \\ &\forall \Delta t_{\{\mathrm{u,y}\},i} \in \Big(\underline{\Delta t}_{\{\mathrm{u,y}\},i};\ \overline{\Delta t}_{\{\mathrm{u,y}\},i}\Big). \end{aligned} \tag{15}$$

Proof. Consider $\|A(\Delta t)\|_P$ and apply Theorem 4 and the triangle inequality to see that (15) implies $\|A(\Delta t)\|_P < 1$ for all possible Δt. This leads to DGES as detailed in Section 8, Theorem 14. □

For the practical implementation, upper bounds for $\|\Delta A_{\dots}\|_P$ are computed in Section 9 and P is determined by Linear Matrix Inequalities (LMIs) in Section 10.

Benefits: The approach shows DGES and therefore CGES using norm bounds, which entails some conservatism. This will later be evaluated by experiments in Section 11. On the other hand, the chosen method is particularly well-suited for analyzing MIMO systems with moderate timing uncertainty:

Theorem 8. (Stability implies timing robustness). If the nominal case $\Delta t = 0$ is stable, then Theorem 7 can show stability for some nonzero (possibly small) timing bounds.

Proof. Consider the summands $\rho_\mathrm{n} + \sum \|\Delta A_{\dots}\|_P$ in (15). Assume a timing bound $|\Delta t| < \delta$ with sufficiently small $\delta > 0$. By Theorem 6, choose P such that $\rho_\mathrm{n} < 1$. By Theorem 4, $\Delta A_{\dots} \to 0$ for $|\Delta t| \to 0$, so choosing δ sufficiently small guarantees that $\sum \|\Delta A_{\dots}\|_P < 1 - \rho_n$. Then, (15) is true. For a detailed proof, see (Gaukler et al., 2019, Theorem 5.6). □

Remark 9. (Complexity). With increasing number of sensors and actuators, checking Theorem 7 requires only a *polynomially* increasing number of matrix norm computations. The approach therefore avoids the exponential growth suffered by gridding the parameter space. In detail, the computation consists of determining P, ρ_n, and then $p+m+mp$ bounds one-dimensional functions $\|\Delta A_{\dots}(\delta)\|_P$, where δ is a bounded scalar variable.

Remark 10. (Interpretability). Because each summand $\|\Delta A_{\dots}\|_P$ in Theorem 7 only refers to the timing of at most one sensor and one actuator, its maximum loosely corresponds to the amount of instability caused by the timing of one sensor, actuator or sensor-actuator-pair. This gives important hints on the timing sensitivity, which can be used to improve the design of the real-time system, e. g. to give priority to sensors with high sensitivity.

The following sections present the low-level details of every analysis step. Section 11 then shows experimental results.

6. DISCRETIZATION

6.1 Definition of a Linear Impulsive System (LIS)

A simple definition of a linear impulsive system is

$$\dot{x}(t) = A_\mathrm{cont} x(t), \quad t \neq \tau_i, \quad t > \tau_0, \tag{16a}$$

$$x(t) = E_i x(t^-), \quad t = \tau_i, \quad i \in \mathbb{N}, \tag{16b}$$

$$x(\tau_0) = x_0, \qquad \tau_0 < \tau_1 < \tau_2 < \dots. \tag{16c}$$

A_cont models continuous dynamics, which are interrupted by discrete events E_i at $t = \tau_i$. For ease of notation, this definition is chosen such that the resulting trajectory is right-continuous, i. e., $x(t^+) = x(t)$.

Extension to Concurrent Events This definition cannot handle concurrent events $\tau_i = \tau_{i+1}$, which is a problem for the basic case of perfect timing: In this case, all measurements and actuator updates occur at the same time $t = kT$. To solve this problem and allow the excluded case $\tau_i = \tau_{i+1}$, we directly define the trajectory as

$$x(t) := \mathrm{e}^{A_\mathrm{cont}(t-\tau_N)} E_N \mathrm{e}^{A_\mathrm{cont}(\tau_N - \tau_{N-1})} E_{N-1} \mathrm{e}^{A_\mathrm{cont}(\tau_{N-1}-\tau_{N-2})} \dots E_1 \mathrm{e}^{A_\mathrm{cont}(\tau_1 - \tau_0)} x_0 \tag{17}$$

$$= \mathrm{e}^{A_\mathrm{cont}(t-\tau_N)} \left(\widetilde{\prod}_{i=1}^N E_i \mathrm{e}^{A_\mathrm{cont}(\tau_i - \tau_{i-1})} \right) x(\tau_0) \tag{18}$$

with N such that $\tau_N \leq t < \tau_{N+1}$ and $\widetilde{\Pi}$ as defined in (7). For background, see (Gaukler et al., 2019, Section 6.1).

6.2 Model of Closed Loop as Linear Impulsive System

The closed loop defined in Section 3 can be rewritten in the framework of linear impulsive systems, similar to the derivations in Gaukler et al. (2018) and Rheinfels (2019). In the following, all block matrices are separated along the dimensions n_p, n_d, p, m of the four state components.

Continuous Dynamics The plant dynamics are continuous and all other variables are constant between the discrete events:

$$A_\mathrm{cont} = \begin{bmatrix} A_\mathrm{p} & 0 & 0 & B_\mathrm{p} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathrm{e}^{A_\mathrm{cont}\delta} = \begin{bmatrix} \mathrm{e}^{A_\mathrm{p}\delta} & 0 & 0 & \tilde{B}(\delta) \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

$$\forall \delta \in \mathbb{R}, \text{ with } \tilde{B}(\delta) := \textstyle\int_0^\delta \mathrm{e}^{A_\mathrm{p}\xi} \mathrm{d}\xi \, B_\mathrm{p}. \tag{19}$$

Discrete Events The k-th control period is defined as the time range $(k - 1/2)T < t \leq (k + 1/2)T$. Within this period, all sensors and actuators are updated near $t = kT$:

$$E_{\mathrm{u},i} = I + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & S_i C_\mathrm{d} & 0 & -S_i \end{bmatrix}, \quad t_{\mathrm{u},i,k} = kT + \Delta t_{\mathrm{u},i,k}, \tag{20}$$

$$E_{\mathrm{y},i} = I + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ S_i C_\mathrm{p} & 0 & -S_i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad t_{\mathrm{y},i,k} = kT + \Delta t_{\mathrm{y},i,k}. \tag{21}$$

$S_i := \mathrm{diag}(\underbrace{0, \dots, 0}_{i-1 \text{ times}}, 1, 0, \dots, 0)$ are selector matrices of appropriate dimension. The index "k" of the event times will later be omitted for better readability.

Just before the end of the control period, at $t = (k+1/2)T$, the new controller state is computed instantaneously from the recent measurements:

$$E_\mathrm{ctrl} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & A_\mathrm{d} & B_\mathrm{d} & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \quad t_{\mathrm{ctrl},k} = (k + 1/2)T \tag{22}$$

Note that the actual timing of the controller computation may deviate from this assumption by a bounded amount because updating the controller state has no physical impact. This can be proven by (18) and

$$E_\mathrm{ctrl} \mathrm{e}^{A_\mathrm{cont}\delta_1} \mathrm{e}^{A_\mathrm{cont}\delta_2} = \mathrm{e}^{A_\mathrm{cont}\delta_1} E_\mathrm{ctrl} \mathrm{e}^{A_\mathrm{cont}\delta_2} = \mathrm{e}^{A_\mathrm{cont}\delta_1} \mathrm{e}^{A_\mathrm{cont}\delta_2} E_\mathrm{ctrl} \quad \forall \delta_{1,2} \geq 0. \tag{23}$$

Therefore, the only timing requirements on the controller are its data dependencies: Computation may start as soon all measurements are available and may take until the first actuator is updated.

Order of events With $\tau_0 := kT - T/2$, the set of events (E_i, τ_i) in the k-th control period is

$$EV_k := \{(E_i, \tau_i) | i = 1, \dots, N_e\} \tag{24}$$

$$= \{(E_{\mathrm{u},i}, t_{\mathrm{u},i,k}) | i = 0, \dots, m-1\} \cup \{(E_{\mathrm{y},i}, t_{\mathrm{y},i,k}) | i = 0, \dots, p-1\} \cup \{(E_\mathrm{ctrl}, t_{\mathrm{ctrl},k})\} \text{ with } \tau_i :\leq \tau_{i+1}, \tag{25}$$

$$|EV_k| := N_e := m + p + 1, \tag{26}$$

which means that events in each period are numbered as $i = 1, \ldots, N_e$ according to their temporal order and that all events occur exactly once. While the order of events with identical time τ_i is ambiguous, this is not a problem since, as detailed in (Gaukler et al., 2019, Theorem 6.1), all possible orders lead to the same trajectory.

6.3 CGES ⇔ DGES

In this section, the equivalence of DGES and CGES will be shown using the fact that the overshoot between two discrete samples is bounded.

Theorem 11. The growth rate of the closed control loop during one control period is bounded:

There exist constants $\bar{C} \geq 1, \bar{\lambda} \in \mathbb{R}$ such that $\forall k \geq 0$,

$$|x(t_k + \delta)| \leq \bar{C} e^{\bar{\lambda}\delta} |x(t_k)| \quad \forall \delta \in [0, T), \forall x(t_k) \in \mathbb{R}^n. \quad (27)$$

Note that this is not a stability result: Any discrete-time control effectively runs in open loop between the sampling instants, so $\bar{\lambda} > 0$ if the uncontrolled plant is unstable.

Proof. Assume $0 < \delta < T$ (the case $\delta = 0$ is trivially true). The event matrices from Section 6.2 are bounded:

$$C_{\mathrm{ev}} := \max_{M \in \{E_{\mathrm{ctrl}}, E_{\mathrm{u},1}, \ldots, E_{\mathrm{u},m}, E_{\mathrm{y},1}, \ldots, E_{\mathrm{y},p}\}} \|M\|_\sigma < \infty \quad (28)$$

exists because they are constant and finite.

Consider (18) with $N \in \{0, \ldots, m + p\}$ as the number of events in $(t_k, t_k + \delta]$. Note that by (24), the events are numbered such that the first event after $t = \tau_0 := t_k$ has the number $i = 1$. By the properties of the spectral norm,

$$\begin{aligned} |x(t_k+\delta)| &= \left| e^{A_{\mathrm{cont}}(t_k+\delta-\tau_N)} \left(\widetilde{\prod}_{i=1}^{N} E_i e^{A_{\mathrm{cont}}(\tau_i - \tau_{i-1})} \right) x(t_k) \right| \\ &\leq e^{\|A_{\mathrm{cont}}\|_\sigma (t_k+\delta-\tau_N)} \widetilde{\prod}_{i=1}^{N} \|E_i\|_\sigma e^{\|A_{\mathrm{cont}}\|_\sigma (\tau_i - \tau_{i-1})} |x(t_k)| \\ &\leq e^{\|A_{\mathrm{cont}}\|_\sigma t_k + \delta - \tau_0} C_{\mathrm{ev}}^N |x(t_k)| \\ &\leq \underbrace{e^{\|A_{\mathrm{cont}}\|_\sigma \delta}}_{e^{\bar{\lambda}\delta}} \underbrace{C_{\mathrm{ev}}^{m+p}}_{\bar{C}} |x(t_k)|. \quad \square \end{aligned} \quad (29)$$

Proof of Theorem 3 (CGES ⇔ DGES): The proof using Theorem 11 is similar to (Al Khatib et al., 2016, Prop. 2). Details are given in (Gaukler et al., 2019, Section 6.3).

7. DECOMPOSITION

In the following, we derive Theorem 4, a key result of our approach: The transition matrix A_k can be split into summands that depend on at most two timing variables.

Proof Sketch of Theorem 4: Consider the complete k-th control period from $x(t_{k-1})$, i. e., just after the controller state has been computed, until $x(t_k)$, i. e. just after the next controller computation. As discussed earlier, the period starts with the event counter $i = 0$ at $t = \tau_0 := t_{k-1} = kT - T/2$ and ends after event $i = N_e = m + p + 1$ at $t = \tau_{N_e} = t_k = kT + T/2$.

Equation (18) leads to $x(t_k) = A_{k-1} x(t_{k-1})$ with

$$A_{k-1} = E_{\mathrm{ctrl}} e^{A_{\mathrm{cont}}(\tau_{N_e} - \tau_{N_e - 1})} \underbrace{\widetilde{\prod}_{i=1}^{N_e - 1} E_i e^{A_{\mathrm{cont}}(\tau_i - \tau_{i-1})}}_{=:X}. \quad (30)$$

X only contains measurement and actuation events, i. e., all matrices E_i are either $E_i = E_{\mathrm{u},\ldots}$ or $E_i = E_{\mathrm{y},\ldots}$. The remainder of the proof, which can be found in (Gaukler et al., 2019, Section 7.3), then consists of expanding the product X and canceling most terms by exploiting the structure of $E_{\{\mathrm{y,u}\},i}$ and A_{cont}. This shows Theorem 4 with

$$\begin{aligned} &A(\Delta t = 0) \\ &= E_{\mathrm{ctrl}} e^{A_{\mathrm{cont}} T/2} \left(I + \sum_{i=1}^{m} (E_{\mathrm{u},i} - I) + \sum_{j=1}^{p} (E_{\mathrm{y},j} - I) \right) e^{A_{\mathrm{cont}} T/2}, \end{aligned} \quad (31)$$

$$\Delta A_{\mathrm{u},i}(\Delta t_{\mathrm{u},i}) = E_{\mathrm{ctrl}} e^{A_{\mathrm{cont}} T/2} (e^{-A_{\mathrm{cont}} \Delta t_{\mathrm{u},i}} - I)(E_{\mathrm{u},i} - I), \quad (32)$$

$$\Delta A_{\mathrm{y},j}(\Delta t_{\mathrm{y},j}) = E_{\mathrm{ctrl}} (E_{\mathrm{y},j} - I) e^{A_{\mathrm{cont}} T/2} (e^{A_{\mathrm{cont}} \Delta t_{\mathrm{y},j}} - I), \quad (33)$$

$$\begin{aligned} &\Delta A_{\mathrm{uy},i,j}(\Delta t_{\mathrm{y},j} - \Delta t_{\mathrm{u},i}) \\ &= \begin{cases} 0, & \Delta t_{\mathrm{y},j} - \Delta t_{\mathrm{u},i} \leq 0, \\ E_{\mathrm{ctrl}} (E_{\mathrm{y},j} - I) e^{A_{\mathrm{cont}}(\Delta t_{\mathrm{y},j} - \Delta t_{\mathrm{u},i})} (E_{\mathrm{u},i} - I), & \text{else.} \end{cases} \end{aligned} \quad (34)$$

All cases of the deviations $\Delta A_{\ldots}$ are of the form $M_1 (e^{A_{\mathrm{cont}} \delta(\Delta t)} - I) M_2$, where $M_{1,2} \in \mathbb{R}^{n \times n}$ depend on the event type and $\delta(\Delta t) \in \mathbb{R}$ on the timing such that $\lim_{|\Delta t| \to 0} \delta(\Delta t) = 0$. Consequently, we have $\lim_{|\Delta t| \to 0} \Delta A_{\ldots} = M_1 (I - I) M_2 = 0$. The results (31)–(34) are validated by numerical experiments.

8. P-ELLIPSOID NORM

This section presents connections between the Lyapunov candidate function $V_P(x) := x^\top P x$ and the P-ellipsoid matrix norm.

Proof of Theorem 5 ($\|\cdot\|_P$ is a submultiplicative norm): Since $P \succ 0$, $\sqrt{V_P(x)} = \sqrt{x^\top P x}$ is a vector norm (Bernstein, 2009, Fact 9.7.30). The P-ellipsoid norm $\|\cdot\|_P$ is its equi-induced matrix norm, therefore submultiplicative.

Theorem 12. $\|A\|_P = \|(P^{1/2})^\top A (P^{1/2})^{-\top}\|_\sigma$.

Proof. Rewrite the P-ellipsoid norm as

$$\|A\|_P = \max_{x \neq 0} |(P^{1/2})^\top A x| / |(P^{1/2})^\top x| \quad (35)$$

and change variables to z with $x = (P^{1/2})^{-\top} z$:

$$\begin{aligned} \|A\|_P &= \max_{z \neq 0} |(P^{1/2})^\top A (P^{1/2})^{-\top} z| / |z| \\ &= \|(P^{1/2})^\top A (P^{1/2})^{-\top}\|_\sigma. \quad \square \end{aligned} \quad (36)$$

Theorem 13. (Extreme Quadratic Lyapunov Function). If a time-invariant system $x_{k+1} = A x_k$ is stable, i. e., $\rho\{A\} < 1$, then there exists a quadratic Lyapunov function $V_P(x)$ that proves a stability factor $\bar{\rho}$ arbitrarily close to the spectral radius $\rho\{A\}$:

$$\begin{aligned} &\forall A \in \mathbb{R}^{n \times n} \text{ with } \rho\{A\} < 1 \quad \forall \bar{\rho} > \rho\{A\} \\ &\qquad \exists P \quad \|A\|_P = \max_{x \neq 0} \sqrt{V_P(Ax)/V_P(x)} \leq \bar{\rho}. \end{aligned} \quad (37)$$

Proof. See (Gaukler et al., 2019, Theorem 8.3). □

Theorem 14. (Robust stability from norm bounds). Let $A_k = \sum_{i=0}^{N} A_{k,i}$ with fixed N. Then, the system $x_{k+1} = A_k x_k$ is DGES($\bar{\rho}, C$) for some C if there are a submultiplicative matrix norm $\|\cdot\|$ and a bound $0 \leq \bar{\rho} < 1$ such that $\sum_i \|A_{k,i}\| \leq \bar{\rho} \ \forall k$.

Proof. Assume $\sum_i \|A_{k,i}\| \leq \bar{\rho} < 1 \ \forall k$. The triangle inequality leads to $\|A_k\| = \|\sum_{i=0}^{N} A_{k,i}\| \leq \sum_{i=0}^{N} \|A_{k,i}\| \leq \bar{\rho}$. Due to the equivalence of norms, there is a finite $C > 0$ such that $\|M\|_\sigma \leq C\|M\|$ for all $M \in \mathbb{R}^{n \times n}$. This leads to

$$|x_{k+1}| = \left| \left(\widetilde{\prod}_{j=0}^{k} A_j \right) x_0 \right| \leq \left\| \widetilde{\prod}_{j=0}^{k} A_j \right\|_\sigma |x_0|$$

$$\leq C \left\| \widetilde{\prod}_{j=0}^{k} A_j \right\| |x_0| \leq C\bar{\rho}^k |x_0| \quad \forall x_0 \in \mathbb{R}^n, \quad (38)$$

which proves DGES$(\bar{\rho}, C)$. □

9. NORM BOUNDING OF SUMMANDS

Theorem 7 provides a stability result based on the P-ellipsoid norm of the timing-dependent deviations $\Delta A_{\ldots}$. In this section, a bound for this norm is presented using the general form $\Delta A_{\ldots} = M_1(\mathrm{e}^{A\tau} - I)M_2$ shown in Section 7.

By (Gaukler et al., 2019, Section 9.1), a Taylor series of order r yields

$$0 \leq \|M_1(\mathrm{e}^{A\tau} - I)M_2\|_P \leq h(\delta) \quad \forall \tau \in [-\delta, \delta] \quad (39)$$

$$\text{with } h(\delta) = \|M_1\|_P \|M_2\|_P (\mathrm{e}^{\|A\|_P\, \delta} - 1) + \textstyle\sum_{i=1}^{r} \gamma_i \delta^i,$$
$$\gamma_i := \left(\|M_1 A^i M_2\|_P - \|M_1\|_P \|A\|_P^i \|M_2\|_P\right)/(i!). \quad (40)$$

As $\lim_{\delta \to 0^+} h(\delta) = 0$, this bound preserves the property

$$\|\Delta A_{\ldots}\|_P \to 0 \text{ for } \Delta t \to 0 \quad (41)$$

from Theorem 4, and therefore also the feasibility result from Theorem 8. In the implementation, $r = 10$ is used.

To ensure a safe overapproximation despite finite numerical precision, interval arithmetic is used to determine all norms and norm bounds. The interval computation of norms, based on Rump (2010), is explained in detail in (Gaukler et al., 2019, Section 9.2).

10. SYNTHESIS OF P VIA LMIs

To show stability using Theorem 7, the CQLF matrix P must be determined such that the bound $\tilde{\rho}$ is less than 1:

$$\|A_k\|_P \leq \|A(\Delta t = 0)\|_P + \sum \|\Delta A_{\mathrm{u},\ldots}\|_P$$
$$+ \sum \|\Delta A_{\mathrm{y},\ldots}\|_P + \sum \|\Delta A_{\mathrm{uy},\ldots}\|_P \leq \tilde{\rho}. \quad (42)$$

Theorem 13 guarantees the existence of P with $\|A(\Delta t = 0)\|_P < 1$. Because the resulting bounds for $\|\Delta A_{\ldots}\|_P$ are often prohibitively large, remaining degrees of freedom in P must be used to minimize $\tilde{\rho}$ and show stability by $\tilde{\rho} < 1$. For this we employ an LMI-based approach.

10.1 Validity of Approximations

As shown in the following, determining P using LMIs entails finite numerical precision and approximations. It is important to note that the final stability result is valid no matter how P was determined, as long as $P \succ 0$: The underlying theorems are valid for any P-ellipsoidal norm $\|\cdot\|_P$ with $P \succ 0$. In the implementation, the numerical result P is checked for $P \succ 0$ and Theorem 7 using interval arithmetic and the results of Section 9. If these tests succeed, the system is stable. Otherwise, no conclusion can be drawn.

10.2 LMI Problem Formulation

To use the efficient framework of LMIs, the P-ellipsoid norms in (42) can be expressed using

$$\|M\|_P < c \quad \Leftrightarrow \quad M^\top P M \prec c^2 P \quad (43)$$

(Gaukler et al., 2019, Section 10.2) as

$$A^\top P\, A \prec \bar{\rho}^2 P \quad (\Leftrightarrow \|A\|_P < \bar{\rho}), \quad (44)$$

$$\Delta A_i^\top P\, \Delta A_i \prec \beta^2 P \quad (\Leftrightarrow \|\Delta A_i\|_P < \beta) \;\; \forall \Delta A_i \in \mathcal{D}, \quad (45)$$

where $A = A(\Delta t = 0)$ is the nominal-case dynamics and, for now, $\mathcal{D}$ the set of $\Delta A_{\ldots}$ in Theorem 4 for all possible Δt. Ignoring numerical errors, this leads to

$$\|A\|_P + \textstyle\sum_{\ldots} \|\Delta A_{\ldots}\|_P \overset{(44),\,(45)}{<} \bar{\rho} + \textstyle\sum_{\ldots} \beta \quad (46)$$

and the optimization goal

$$\min_{P \succ 0,\, \bar{\rho} > 0,\, \beta > 0} (\bar{\rho} + \textstyle\sum_{\ldots} \beta) \quad \text{subject to (44) and (45).} \quad (47)$$

However, this is not a valid LMI because (44) contains a product of the optimization variables P and $\bar{\rho}$. Additionally, to avoid numerically ill-conditioned P, the constraint

$$\gamma I \prec P \prec I \quad (\Leftrightarrow \lambda_{\min}(P) > \gamma \;\wedge\; \lambda_{\max}(P) < 1) \quad (48)$$

with $\gamma > 0$ is added. The optimization then becomes

$$\max_{P \in \mathbb{R}^{n \times n},\, \gamma > 0} \gamma \quad \text{subject to (44), (45) and (48),} \quad (49)$$

where the desired norm bounds $\bar{\rho}$ and β are constant within the LMI and instead optimized in an outer loop. Numerical robustness is further improved by preconditioning as detailed later in (Gaukler et al., 2019, Section 10.5).

While in theory, $\mathcal{D}$ should be the set of all $\Delta A_{\{\mathrm{u},\mathrm{y},\mathrm{uy}\},\ldots}$ for a representative set of timings, this would be prohibitively large for systems with many sensors and actuators. It is instead approximated as the set

$$\mathcal{D} = \big\{ A(\Delta t) - A(0) \;\big|\; \Delta t = [\Delta t_\mathrm{u}^\top \;\; \Delta t_\mathrm{y}^\top]^\top \in$$
$$\big(\{\underline{\Delta t}_\mathrm{u}, 0, \overline{\Delta t}_\mathrm{u}\} \times \{\underline{\Delta t}_\mathrm{y}, 0, \overline{\Delta t}_\mathrm{y}\}\big) \setminus \{0\} \big\} \quad (50)$$

representing eight extreme combinations of Δt_u and Δt_y. As noted in Section 10.1, this approximation does not restrict the validity of the final result.

10.3 Optimization of $\bar{\rho}$ and β

In the previous LMIs, the parameters $\bar{\rho}$ and β must be given, whereas the actual goal is to minimize the analysis result $\tilde{\rho}$. Mainly, $\bar{\rho}$ and β should be minimized because, by (42) and (46), neglecting the approximation of $\mathcal{D}$,

$$\tilde{\rho} = \bar{\rho} + \beta(m + p + mp) \quad (51)$$

is a worst-case bound for $\tilde{\rho}$. However, there are limits: Experiments show that smaller $\bar{\rho}$ increases $\|\Delta A_i\|_P$. Because $\beta > \|\Delta A_i\|_P$, $\bar{\rho}$ should not be too small. To show stability, $\bar{\rho} < 1$ is desirable. As $\bar{\rho} > \|A\|_P > \rho\{A\}$, we have $\rho\{A\} < \bar{\rho} < 1$. The implementation uses a fixed value $\bar{\rho} = 0.8 + 0.2\rho\{A\}$ in this range, and a heuristic search for β, as detailed in (Gaukler et al., 2019, Section 10.4).

11. EXPERIMENTAL RESULTS

The approach was prototypically implemented in Python using *CVXPY* for LMIs and *mpmath* for interval arithmetic. (Open-source code is available at https://github.com/qronos-project/timing-stability-lmi/.) Stability could successfully be proven for examples C2 and D2 from Gaukler and Ulbrich (2019), for which no previous stability result is known. These examples are the one- (C2) and three-axis (D2) angular rate control of a linearized quadcopter with a period of $T = 10\,\mathrm{ms}$ and a timing uncertainty of $\pm 1\,\%$. Example D2 is a multivariable system with $m = 4$, $p = 3$ and a total dimension of $n = 16$.

Table 1 compares the results and computation times obtained using interval arithmetic ($\tilde{\rho}$, t) with those from

name	n	$\tilde{\rho}_{\text{approx}}$	$\lvert\tilde{\rho} - \tilde{\rho}_{\text{approx}}\rvert$	t_{approx}	t
C2	5	0.914	$9.2 \cdot 10^{-8}$	1.0	1.6
D2	16	0.926	$9.3 \cdot 10^{-8}$	17.5	98.1
D2_b: $2\Delta t$	16	1.073	—	12.8	—
D2_c: $2n$	32	1.021	—	312.1	—
D2_d: $2n$, $\frac{\overline{\Delta t_\text{y}}}{10}$	32	0.979	$9.8 \cdot 10^{-8}$	308.1	2196.3

All values are rounded up to the last shown digit. Times are wall-times in seconds on an Intel i7-8750H CPU with 16GB RAM.
$n = n_\text{p} + n_\text{d} + m + p$: Total state dimension
$\tilde{\rho}$: Upper bound on stability factor with interval arithmetic
$\tilde{\rho}_{\text{approx}}$: Fast approximation of $\tilde{\rho}$
t_{approx}, t: Time for computing $\tilde{\rho}_{\text{approx}}$, ρ.
Modified system parameters are indicated as $2n$ (dimension doubled by repetition) and $K\Delta t$ (timing variable(s) increased by factor K)

Table 1. Experimental results

a simplified approximation ($\tilde{\rho}_{\text{approx}}$, t_{approx}), in which the norm bounds from Section 9 are replaced by the floating-point maximum $\max_\tau \|\Delta A_{\dots}(\tau)\|$ over 100 samples of τ. While this approximation is not guaranteed to be correct, it is about eight times faster. The small deviations $|\tilde{\rho}_{\text{approx}} - \tilde{\rho}|$ show that the norm bounds are accurate.

While stability ($\tilde{\rho} < 1$) can be shown for example D2, this does not hold for doubled timing uncertainty (D2_b), which may be due to conservatism or due to actual instability. To analyze the scalability, the dimension of D2 was doubled by block-diagonal repetition. By construction, the resulting system D2_c of dimension $n = 32$ has the same stability properties as D2. It can still be analyzed approximately within six minutes and verified within one hour, however at the cost of increased conservatism: Stability can only be shown for reduced timing uncertainty (D2_d, $\overline{\Delta t_\text{y}}$ reduced to 1/10th). This conservatism relates to the fact that the summands of Theorem 4 are norm-bounded individually, while their total effect is generally less severe.

12. CONCLUSION

We presented a stability verification approach for control systems with multiple inputs and outputs under uncertain timing for sensing and actuating. Here, the challenge is that the system dynamics depends on the combination of all individual timing variables, that is, varying jitter for each sensor and actuator. To avoid the resulting curse of dimensionality, we exploit the system model's structural properties: A decomposition of the discrete-time dynamics leads to summands with at most two timing variables. Subsequently, we can bound these summands in terms of a norm that corresponds to a Common Quadratic Lyapunov Function. The experimental results show that our approach facilitates the stability analysis for moderately complex systems for which, to the best of our knowledge, previously no analysis methods were known.

Future research will be concerned with extending the approach to the nonlinear case and improving the scalability by a more efficient implementation.

REFERENCES

Al Khatib, M., Girard, A., and Dang, T. (2016). Stability verification and timing contract synthesis for linear impulsive systems using reachability analysis. *Nonlinear Analysis: Hybrid Systems*. doi:10.1016/j.nahs.2016.08.007.

Baker, T.P. and Shaw, A. (1989). The Cyclic Executive Model and ADA. *Real-Time Systems*, 1(1), 7–25.

Bauer, N., van Loon, S., Donkers, M., van de Wouw, N., and Heemels, W. (2012). Networked control systems toolbox: Robust stability analysis made easy. *IFAC Proceedings Volumes*, 45(26), 55–60. doi:10.3182/20120914-2-us-4030.00014.

Bernstein, D.S. (2009). *Matrix Mathematics : Theory, Facts, and Formulas*. Princeton Univ. Pr., 2nd edition.

Cervin, A. (2012). Stability and worst-case performance analysis of sampled-data control systems with input and output jitter. In *2012 American Control Conference (ACC)*, 3760–3765. doi:10.1109/ACC.2012.6315304.

Fontanelli, D., Greco, L., and Palopoli, L. (2013). Soft real-time scheduling for embedded control systems. *Automatica*, 49(8), 2330–2338. doi:10.1016/j.automatica.2013.04.036.

Gaukler, M., Michalka, A., Ulbrich, P., and Klaus, T. (2018). A new perspective on quality evaluation for control systems with stochastic timing. In *Proceedings of the 21st International Conference on Hybrid Systems: Computation and Control (part of CPS Week) - HSCC '18*. ACM Press. doi:10.1145/3178126.3178134.

Gaukler, M., Roppenecker, G., and Ulbrich, P. (2019). Details and proofs for: Stability analysis of multivariable digital control systems with uncertain timing. URL https://arxiv.org/abs/1911.02537. Extended version of this paper with proofs and additional details.

Gaukler, M. and Ulbrich, P. (2019). Worst-case analysis of digital control loops with uncertain input/output timing. In *ARCH19. Int. Workshop on Applied veRification for Continuous and Hybrid Systems*. doi:10.29007/c4zl.

Heemels, W.P., van de Wouw, N., Gielen, R.H., Donkers, M.C., Hetel, L., Olaru, S., Lazar, M., Daafouz, J., and Niculescu, S. (2010). Comparison of overapproximation methods for stability analysis of networked control systems. In *Proceedings of the 13th ACM international conference on Hybrid systems: computation and control - HSCC '10*. ACM Press. doi:10.1145/1755952.1755979.

Henzinger, T.A., Kirsch, C.M., Sanvido, M.A.A., and Pree, W. (2003). From control models to real-time code using Giotto. *IEEE Control Systems Magazine*, 23(1), 50–64. doi:10.1109/MCS.2003.1172829.

Hetel, L., Fiter, C., Omran, H., Seuret, A., Fridman, E., Richard, J.P., and Niculescu, S.I. (2017). Recent developments on the stability of systems with aperiodic sampling: An overview. *Automatica*, 76, 309–335. doi:10.1016/j.automatica.2016.10.023.

Kao, C.Y. and Rantzer, A. (2007). Stability analysis of systems with uncertain time-varying delays. *Automatica*, 43(6), 959–970. doi:10.1016/j.automatica.2006.12.006.

Rheinfels, T. (2019). *Leveraging Machine-Learning Techniques for Quality-Aware Real-Time Scheduling*. Master's thesis, Friedrich-Alexander-Universität Erlangen-Nürnberg.

Rump, S.M. (2010). Verified bounds for singular values, in particular for the spectral norm of a matrix and its inverse. *BIT Numerical Mathematics*, 51(2), 367–384. doi:10.1007/s10543-010-0294-0.

Sha, L. and Goodenough, J.B. (1989). Real-time Scheduling Theory and ADA. Technical report, Carnegie-Mellon University, Pittsburgh PA, Software Engineering Institute.