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Sparse Representation of Feedback Filters in Delta-Sigma Modulators ⋆

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Abstract: In this paper, we propose sparse representation of FIR (Finite Impulse Response) feedback filters in delta-sigma modulators. The filter has a sparse structure, that is, only a few coefficients are non-zero, that stabilizes the feedback modulator, and minimizes the maximum magnitude of the noise transfer function at low frequencies. The optimization is described as an ℓ^1 minimization with linear matrix inequalities (LMIs), based on the generalized KYP (Kalman-Yakubovich-Popov) lemma. A design example is shown to illustrate the effectiveness of the proposed method.

Keywords: Delta-sigma modulators, signal processing, quantization, sparse representation, convex optimization, H^∞ optimization, generalized KYP lemma.

1. INTRODUCTION

Sparse representation is a recent technique of model reduction by sparsifying the system parameters with an allowable performance degradation (Mallat, 2008; Elad, 2010). In sparse representation, most parameters are set zero and only a few active parameters are used to represent the system characteristics. To measure the sparsity, we use the ℓ^0 norm, the number of non-zero elements in a vector. Sparse representation is done by finding a minimum ℓ^0-norm vector among admissible parameter vectors. This formulation is also known as compressed sensing (Donoho, 2006), or compressive sampling (Candes, 2006).

The ℓ^0 minimization in sparse representation (or compressed sensing) is known to be NP hard as shown in Natarajan (1995). To avoid this difficulty, the ℓ^1 norm is often adopted alternatively. Then, if the set of admissible vectors is convex, the optimization problem becomes a convex optimization problem, which can be efficiently solved by numerical softwares as SDPT3 (Toh et al., 1999; Tutuncu et al., 2003) or SeDuMi (Sturm, 1999). Sparse representation has wide applications in engineering and science, such as magnetic resonance imaging (MRI) (Lustig et al., 2008), black hole imaging (Honma et al., 2016), communications systems (Hayashi et al., 2013), optimal control (Nagahara et al., 2014, 2016; Ikeda et al., 2017), In particular, sparse representation has been applied to FIR (Finite Impulse Response) filter design Jiang et al. (2012); Chen et al. (2019) for sparse coefficients.

In this paper, we present a novel method for sparse representation of the feedback filter in a delta-sigma ($\Delta\Sigma$) modulator (Norsworthy et al., 1997; Schreier and Temes, 2005). A delta-sigma modulator is a feedback quantizer, which is widely used in oversampling analog-to-digital (AD) and digital-to-analog (DA) converters in, for example, digital audio (Janssen and Reefman, 2003) and digital communications (Gustavsson et al., 2010).

The key characteristic of delta-sigma modulation is quantization noise shaping. A delta-sigma modulator constitutes a pre-filter, a coarse uniform quantizer, and a feedback filter. The pre-filter defines the frequency domain characteristics of the input signals to be quantized. On the other hand, the feedback filter eliminates the in-band quantization noise by shaping the sensitivity function of the feedback system, called the noise transfer function (NTF). For low-frequency input signals, the feedback filter is designed to make the NTF a low-cut (or high-pass) filter, which well attenuates the quantization noise at low frequencies and push them out to the high-frequency band. From the viewpoint of control systems, the feedback filter can be considered as a controller for stabilization and noise reduction for the quantizer as a controlled plant. The stability criterion is given by the constraint on the H^∞ norm of the NTF, called the Lee criterion (Schreier and Temes, 2005; Chao et al., 1990).

The design problem is then to obtain a feedback filter that minimizes the frequency response gain of the NTF in a pre-specified low frequency range with the H^∞ norm constraint for stability. This has been considered in Nagahara and Yamamoto (2012) assuming that the feedback filter is assumed to be an FIR filter. The generalized KYP (Kalman-Yakubovich-Popov) lemma rewrites the optimization problem into LMI (Linear Matrix Inequality) optimization.

This paper combines the technique of sparse modeling with the feedback filter design in delta-sigma modulators. Sparse representation in delta-sigma modulation is very important in recent IoT (Internet of Things) systems or cyber-physical systems (Lee and Seshia, 2017), since

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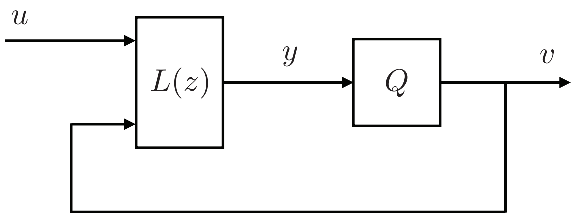

Fig. 1. Delta-sigma modulator with loop filter $L(z) = [L_0(z), L_1(z)]$ and uniform quantizer Q.

quantizers are implemented on edge sensors at the physical side, which should be compact due to the memory size limitation on the edge computers. To achieve sparse representation, we propose to find the sparsest FIR coefficients of the feedback filter that achieves stability and allowable performance at the same time. The problem itself is a semi-infinite programming, which we will show to be equivalently reduced to a finite-dimensional convex optimization that is efficiently solvable. We will also show a design example to illustrate merits of the proposed design method.

Notation

We denote by $\mathbb{R}^N$ the set of N-dimensional real vectors and $\mathbb{R}^{N\times N}$ the set of $N \times N$ real matrices. For a vector (or a matrix), we use the superscript $\top$ for the transpose. $\mathbb{S}_N$ is the set of $N \times N$ real symmetric matrices, that is,

$$\mathbb{S}_N \triangleq \{X \in \mathbb{R}^{N\times N} : X = X^\top\}. \tag{1}$$

For a symmetric matrix $X \in \mathbb{S}_N$, we write $X > 0$ (resp. $X < 0$) when X is positive definite (resp. negative definite). For a vector $c \in \mathbb{R}^N$, the ℓ^1 norm is defined by

$$\|c\|_1 \triangleq \sum_{k=1}^{N} |c_k|. \tag{2}$$

For a stable transfer function $H(z)$, the H^∞ norm is defined by

$$\|H\|_\infty \triangleq \max_{\omega\in[0,\pi]} |H(e^{j\omega})|. \tag{3}$$

For a vector $a = [a_1, \ldots, a_N]^\top$ or a sequence $a = \{a_1, a_2, \ldots\}$, the support set $\mathrm{supp}(a)$ is defined by

$$\mathrm{supp}(a) \triangleq \{k | a_k \neq 0\}. \tag{4}$$

Throughout this paper, we use the following abbreviations:

FIR: Finite Impulse Response
NTF: Noise Transfer Function
STF: Signal Transfer Function
LMI: Linear Matrix Inequality
KYP: Kalman-Yakubovich-Popov

2. DELTA-SIGMA MODULATION

In this section, we briefly review the delta-sigma ($\Delta\Sigma$) modulation. Figure 1 shows the general structure of a delta-sigma modulator. In this figure, $L(z)$ is a two-input/single-output linear time-invariant system, called the *loop filter*, with

$$L(z) \triangleq [L_0(z), L_1(z)], \tag{5}$$

and Q is a uniform quantizer that maps $\mathbb{R}$ to a finite alphabet $\{-M\delta, (-M+1)\delta, \ldots, M\delta\}$, where $\delta > 0$ is the step size and $M \in \mathbb{N}$ is the number of steps.

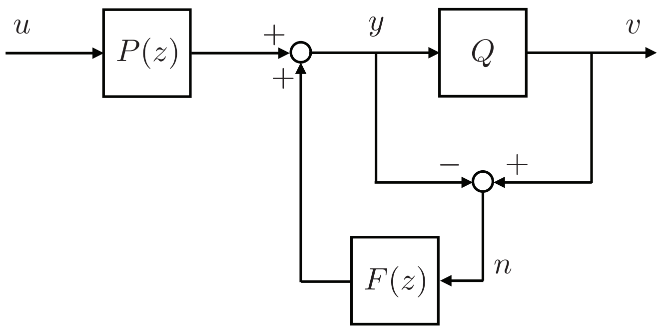

Fig. 2. Error feedback modulator

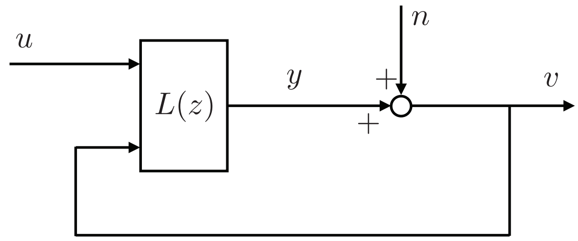

Fig. 3. Linearized model of the delta-sigma modulator

In this paper, we consider the error feedback modulator shown in Figure 2. In this block diagram, $P(z)$ is the pre-filter and $F(z)$ is the error feedback filter. For the stability and the well-posedness, we assume the following:

Assumption 1. $P(z)$ is stable and causal[1], and $F(z)$ is stable and strictly causal.

Comparing Figure 2 with Figure 1, we obtain

$$L_0(z) = \frac{P(z)}{1+F(z)}, \quad L_1(z) = \frac{F(z)}{1+F(z)}. \tag{6}$$

As shown in Figure 2, the modulator feeds back the quantization noise n defined by

$$n \triangleq v - y = Q(y) - y. \tag{7}$$

To analyze the delta-sigma modulator, we introduce the linearized model shown in Figure 3, where the quantization noise n is taken as an independent disturbance added to the signal y to be quantized. This linear model is widely used in the literature of signal processing. For this linear model, we define a system called the noise transfer function (NTF) by

$$H(z) \triangleq \frac{1}{1-L_1(z)} = 1 + F(z), \tag{8}$$

which is the transfer function from the quantization noise n to the quantized output v in Figure 3.

Also, we define the *signal transfer function* (STF) by

$$T(z) \triangleq \frac{L_0(z)}{1-L_1(z)} = P(z), \tag{9}$$

which is the transfer function from the input u to the output v. The design of the delta-sigma modulator is to synthesize the filters $P(z)$ and $F(z)$ so that the noise transfer function $H(z)$ and the signal transfer function $T(z)$ have desired characteristics. For simplicity, we just

[1] A transfer function is said to be causal (resp. strictly causal) if the relative degree is greater than or equal to 0 (resp. strictly greater than 0).

consider the noise transfer function $H(z)$, and focus ourselves on the design of the feedback filter $F(z)$, assuming that the pre-filter $P(z)$ is the identity, that is,

$$P(z) \equiv 1. \tag{10}$$

In this case, the signal transfer function becomes $T(z) = 1$ from (9). This means that we do not explicitly assume the frequency-domain characteristic of the input signals. Note that the design of the pre-filter is considered in Okajima et al. (2015).

3. PROBLEM FORMULATION

Here we formulate the design problem of the feedback filter $F(z)$. First, for simple implementation, we assume that the feedback filter $F(z)$ is an FIR filter

$$F(z) = \sum_{k=0}^{N} c_k z^{-k}, \tag{11}$$

with

$$c_0 = 0, \tag{12}$$

since $F(z)$ is assumed to be strictly causal (see Assumption 1).

Then the problem is to find the FIR coefficients $c_1, \ldots, c_N$. The coefficients should be carefully chosen taking account of

(1) feedback stability,
(2) performance of quantization noise reduction,
(3) simple implementation.

In the following subsections, we formulate the design problem to satisfy the above requirements.

3.1 *Stability*

It is clear from (8) that the linearized feedback system is stable since $F(z)$ is stable. The problem is to guarantee the stability of the nonlinear feedback system in Figure 1 with nonlinear quantizer Q. A sufficient condition for the stability is given by

$$\begin{aligned} \|H\|_\infty &= \|1 + F\|_\infty \\ &\leq \frac{1}{(2N+1)\delta}(M + 1 + \delta - U_{\max}), \end{aligned} \tag{13}$$

where M and δ are respectively the number of steps and the step size of Q, N is the order of $F(z)$, and $U_{\max}$ is the maximum value of the input u. See Nagahara and Yamamoto (2012) for the proof of the above criterion. Also, the well-known Lee criterion is given by

$$\|H\|_\infty = \|1 + F\|_\infty < 1.5. \tag{14}$$

Note that this is neither sufficient nor necessary for the stability, but this is actually widely used in the real systems. See Schreier and Temes (2005); Chao et al. (1990) for details. In both cases, a stability condition is described by

$$\|H\|_\infty = \|1 + F\|_\infty < \gamma. \tag{15}$$

From (3) and (11), this condition can be rewritten as

$$\max_{\omega \in [0,\pi]} \left| 1 + \sum_{k=0}^{N} c_k e^{-j\omega k} \right| < \gamma. \tag{16}$$

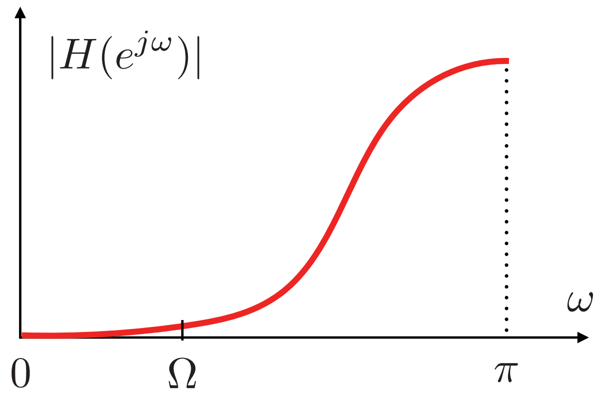

Fig. 4. A preferable low-cut (or high-pass) response of the NTF $H(z)$.

3.2 *Performance*

The noise transfer function from the quantization noise n to the quantized output v plays an important role in delta-sigma modulators. Let us assume that the highest frequency $\Omega \in (0, \pi)$ of the input signal u is known. That is, we assume that for any input u,

$$\mathrm{supp}(\hat{u}) \subset [0, \Omega], \tag{17}$$

holds where $\hat{u}$ is the Fourier transform of u. Then, we can shift the frequency distribution of the quantization noise in the quantizer output v to the high frequency domain $(\Omega, \pi]$, by making the NTF to be a low-cut (or high-pass) filter with cutoff frequency Ω (see Figure 4). We then introduce an optimization problem of the NTF to maximally attenuate the gain of the frequency response of the NTF in the low-frequency band $[0, \Omega]$:

$$\min_{c_1,\ldots,c_N} \max_{\omega \in [0,\Omega]} |H(e^{j\omega})|. \tag{18}$$

From (3) and (11), this condition can be rewritten as

$$\min_{c_1,\ldots,c_N} \max_{\omega \in [0,\Omega]} \left| 1 + \sum_{k=0}^{N} c_k e^{-j\omega k} \right|. \tag{19}$$

3.3 *Sparse representation*

The FIR filter $F(z)$ is often implemented on a digital device, which is often on the edge side (or the sensor side) of the IoT (Internet of Things) network. It is therefore preferable to represent the filter in the simplest form. For this purpose, we propose the sparse representation of the filter coefficients, by minimizing the ℓ^0 norm of the coefficient vector $c = [c_1, \ldots, c_N]^\top$, defined by

$$\|c\|_0 \triangleq |\mathrm{supp}(c)|, \tag{20}$$

where $|\mathrm{supp}(c)|$ is the number of the elements in the support set $\mathrm{supp}(c)$. That is, the ℓ^0 norm is the number of non-zero elements of c. Since the minimization of the ℓ^0 norm leads to an NP-hard problem as pointed out by Natarajan (1995), we often adopt a convex relaxation by minimizing the ℓ^1 norm

$$\|c\|_1 = \sum_{k=1}^{N} |c_k|. \tag{21}$$

It is well-known that ℓ^1 minimization often promotes the sparsity (see Elad (2010); Hayashi et al. (2013) for details), and hence by minimizing the ℓ^1 norm, we obtain a sparse representation of the FIR filter $F(z)$.

3.4 Optimization problem

Now, we summarize the design problem of the filter coefficients $c_1, \dots, c_N$ that satisfy the three requirements mentioned in the previous subsections. The design problem is described as an optimization problem:

$$\begin{aligned} &\underset{c_1,\dots,c_N \in \mathbb{R}}{\text{minimize}} && \max_{\omega\in[0,\Omega]} \left| 1 + \sum_{k=1}^{N} c_k e^{-j\omega k} \right| + \lambda \sum_{k=1}^{N} |c_k| \\ &\text{subject to} && \max_{\omega\in[0,\pi]} \left| 1 + \sum_{k=1}^{N} c_k e^{-j\omega k} \right| < \gamma, \end{aligned} \tag{22}$$

where $\lambda > 0$ is the regularization parameter that controls the tradeoff between performance and sparsity. This is a semi-infinite programming problem and hence hard to solve. In the next section, we show that this problem can be equivalently reduced to a finite-dimensional convex optimization problem.

4. SPARSE FILTER DESIGN VIA ℓ^1 OPTIMIZATION

Here we show how to efficiently solve the semi-infinite programming problem in (22). Let us define state-space matrices A, b, and c of the transfer function $H(z) = 1 + F(z)$ as

$$A := \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}, \quad b := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{23}$$
$$c^\top := [c_N, c_{N-1}, \dots, c_1],$$

that is,

$$H(z) = 1 + c^\top (zI - A)^{-1} b. \tag{24}$$

First, we show two lemmas that are the key to relaxing the problem in (22) to a finite-dimensional convex optimization.

Lemma 1. (KYP lemma). The inequality (15), or (16) holds if and only if there exists a real symmetric matrix $X \in \mathbb{S}_N$ such that

$$\Phi_\gamma(X, c) \triangleq \begin{bmatrix} A^\top X A - X & A^\top X b & c \\ b^\top X A & b^\top X b - \gamma^2 & 1 \\ c^\top & 1 & -1 \end{bmatrix} < 0, \quad X > 0. \tag{25}$$

For the proof of KYP lemma, see Rantzer (1996); Nagahara (2011).

Lemma 2. (Generalized KYP lemma). The following inequality

$$\max_{\omega\in[0,\Omega]} \left| 1 + \sum_{k=1}^{N} c_k e^{-j\omega k} \right| < \rho \tag{26}$$

holds if and only if there exist real symmetric matrices $Y, Z \in \mathbb{S}_N$ such that

$$\Psi_\rho(Y, Z, c) \triangleq \begin{bmatrix} M_1(Y, Z) & M_2(Y, Z) & c \\ M_2(Y, Z)^\top & M_3(Y, \rho^2) & 1 \\ c^\top & 1 & -1 \end{bmatrix} < 0, \quad Z > 0, \tag{27}$$

where

$$\begin{aligned} M_1(Y, Z) &= A^\top Y A + Z A + A^\top Z - Y - 2Z \cos\Omega, \\ M_2(Y, Z) &= A^\top Y b + Z b, \\ M_3(Y, \rho^2) &= b^\top Y b - \rho^2. \end{aligned}$$

The proof is found in Iwasaki and Hara (2005).

Next, we equivalently rewrite the optimization problem in (22) as

$$\begin{aligned} &\underset{\rho, c_1,\dots,c_N \in \mathbb{R}}{\text{minimize}} && \rho + \lambda \sum_{k=1}^{N} |c_k| \\ &\text{subject to} && \max_{\omega\in[0,\pi]} \left| 1 + \sum_{k=1}^{N} c_k e^{-j\omega k} \right| < \gamma, \\ & && \max_{\omega\in[0,\Omega]} \left| 1 + \sum_{k=1}^{N} c_k e^{-j\omega k} \right| < \rho. \end{aligned} \tag{28}$$

Finally, from Lemmas 1 and 2, we can equivalently relax the optimization problem (28) into

$$\begin{aligned} &\underset{\substack{\rho\in\mathbb{R}, c\in\mathbb{R}^N \\ X,Y,Z\in\mathbb{S}_N}}{\text{minimize}} && \rho + \lambda \|c\|_1 \\ &\text{subject to} && \Phi_\gamma(X, c) < 0, \\ & && \Psi_\rho(Y, Z, c) < 0, \\ & && X > 0, \\ & && Z > 0. \end{aligned} \tag{29}$$

This is a convex optimization problem and can be efficiently solved by e.g. SDPT3 (Toh et al., 1999; Tutuncu et al., 2003) or SeDuMi (Sturm, 1999).

Remark 1. (NTF zero). For ensuring perfect reconstruction of the DC input level, and reducing low-frequency tones, it is preferable for the NTF to have zeros at $z = 1$, or $\omega = 0$ (Schreier and Temes, 2005). The zeros of the NTF, $H(z)$, can be assigned by linear constraints of $c_1, \dots, c_N$. If we define

$$G(z) \triangleq z^N H(z) = z^N + \sum_{k=1}^{N} c_k z^{N-k}. \tag{30}$$

Then, $H(z)$ has m zeros at $z = 1$ if and only if

$$\left. \frac{d^l G(z)}{dz^l} \right|_{z=1} = 0, \quad l = 0, 1, \dots, m-1, \tag{31}$$

where

$$\frac{d^0 G(z)}{dz^0} = G(z).$$

We can combine these linear constraints with the convex optimization (29).

Remark 2. We can use a non-uniform weight for the ℓ^1 norm in the cost function (28) instead of single λ. Namely, we can minimize

$$\rho + \sum_{k=1}^{N} \lambda_k |c_k|,$$

with

$$\lambda_1 > 0, \lambda_2 > 0, \dots, \lambda_N > 0,$$
$$\lambda_1 + \lambda_2 + \dots + \lambda_N = 1.$$

With this cost function, the optimization is still convex.

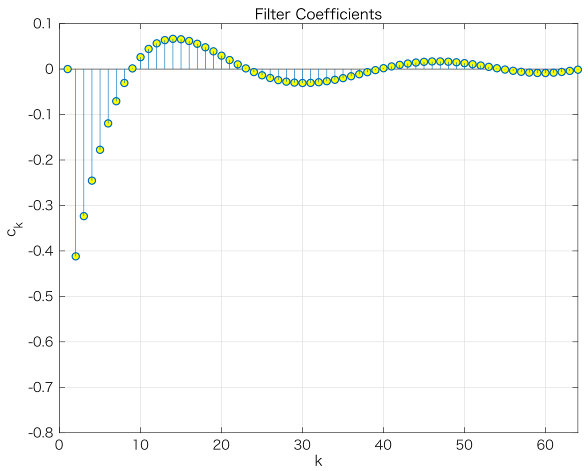

Fig. 5. Filter coefficients of $F(z)$ with $\lambda = 0$ (full-coefficient filter)

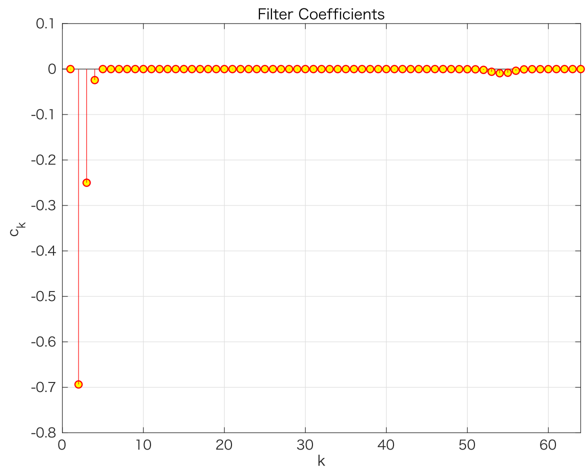

Fig. 6. Filter coefficients of $F(z)$ with $\lambda = 0.5$ (sparse-coefficient filter)

5. DESIGN EXAMPLE

In this section, we show a design example of the delta-sigma modulator. We assume that the pre-filter $P(z) = 1$, and the feedback filter $F(z)$ to be an FIR filter of order $N = 64$. The cut-off frequency Ω is chosen as $\pi/32 \approx 0.0982$ (rad/sec). We set the upper bound γ of the H^∞ norm of the NTF to be 1.5, that is, we use the Lee criterion $\|H\|_\infty < 1.5$ for the stability.

Under these parameters, we compute three filters:

(i) The optimal filter that solves (29) with $\lambda = 0$, combined with the zero constraint (31) with $m = 1$. This is a conventional filter proposed in Nagahara and Yamamoto (2012), which uses all 64 coefficients.
(ii) The optimal filter that solves (29) with $\lambda = 10$, cutting very small coefficients (i.e. $|c_k| < 10^{-4}$) to zero, combined with the zero constraint (31) with $m = 1$. This is the proposed sparse-coefficient filter.
(iii) The truncated 15-th order FIR filter from the full-coefficient filter computed in (i).

Figure 5 shows the filter coefficients of the full-coefficient filter. We see that all the coefficients of this filter are non-zero, and hence we need to implement the full 64-tap filter. On the other hand, Figure 6 shows the filter coefficients of the proposed filter. We can see that this filter has very sparse coefficients. In fact, the absolute values of 49 coefficient out of 64 are zero. In other words, we have $\|c\|_0 = 15 \ll 64$, and we only need to implement these 15 nonzero coefficients.

Figure 7 shows the gain of the frequency response of the NTF. We can see that the truncated filter, which has the same number of non-zero coefficients as the sparse filter, shows worse response than the sparse filter at low frequencies around $\omega = 0$.

Remark 3. One can download MATLAB codes to run the numerical example. See Appendix.

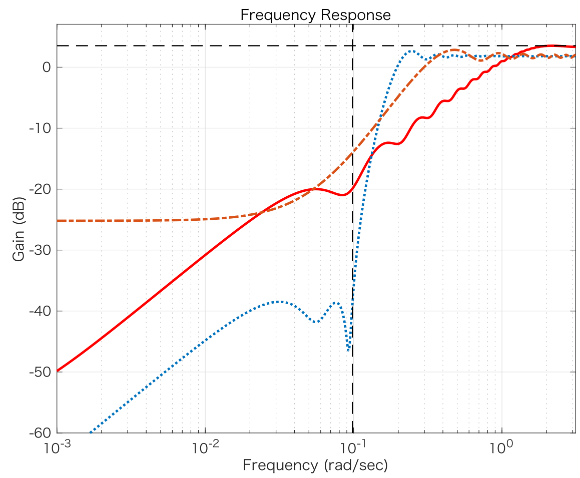

Fig. 7. Frequency response of NTF: sparse filter (solid), dense filter (dash), truncated filter (dash-dotted)

6. CONCLUSION

In this paper, we have proposed a sparse filter design of FIR feedback filters in delta-sigma modulators. By the KYP and generalized KYP lemmas, and ℓ^1 norm relaxation of ℓ^0 norm, the design problem of the FIR coefficients can be written as a convex optimization problem with LMIs, which can be efficiently solved by numerical softwares. Future work includes the design of FIR feedback filters for band-pass and multi-band delta-sigma modulators with sparse representation.

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Appendix A. MATLAB CODES

MATLAB function `NTF_sparse.m` to obtain a sparse feedback filter by the proposed method can be downloaded from

`https://nagahara-masaaki.github.io/ds.html`

Here is an example to use the function:

```
>lam = 10; %Regularization parameter
>ord = 64; %FIR filter order
>Om = pi/32; %Cut-off frequency
>Hf = 1.5; %H-infinity norm of NTF
>m = 1; %Zero placement of NTF
>th = 1e-4; %Thresholding parameter
>[ntf, F] = NTF_sparse(lam,ord,Om,Hf,m,th);
```

By running the above codes, you get `ntf`, the transfer function of NTF, and `F`, the filter coefficients of $F(z)$.