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Abstraction of Monotone Systems Based on Feedback Controllers ⋆

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Abstract: In this paper, we consider the problem of computation of efficient symbolic abstractions for a certain subclass of continuous-time monotone control systems. The new abstraction algorithm utilizes the properties of such systems to build symbolic models with the same number of states but fewer transitions in comparison to the one produced by the standard algorithm. At the same time, the new abstract system is at least as controllable as the standard one. The proposed algorithm is based on the solution of a region-to-region control synthesis problem. This solution is formally obtained using the theory of viscosity solutions of the dynamic programming equation and the theory of differential equations with discontinuous right-hand side. In the new abstraction algorithm, the symbolic controls are essentially the feedback controllers that solve this control synthesis problem. The improvement in the number of transitions is achieved by reducing the number of successors for each symbolic control. The approach is illustrated by an example that compares the two abstraction algorithms.

1. INTRODUCTION

Synthesis of feedback controllers for nonlinear dynamical systems is one of the key problems in control theory. Formal methods approach suggests splitting this problem into several subproblems with the first one being the contruction of a symbolic abstract system (or abstraction), which is usually a system with finite number of states and transitions (see Tabuada (2008); Belta et al. (2017)). These abstractions capture the behavior of the original system in such a way that a controller built to solve the control problem for an abstract system can be refined to a respective controller for the original system. The notions of an alternating simulation relation, an approximate alternating simulation relation and a feedback refinement relation are used to formalize such properties.

There are several known methods of abstraction. Some of those methods require the control system to satisfy certain sets of conditions to be applicable. One of the more general methods is based on partitioning of the state space and on discretizing the control space (Reissig et al. (2016)). This abstraction method utilizes the notion of alternating simulation relation and can be applied to a very wide class of systems but is especially efficient when the reachable sets originated from partition elements can be efficiently computed or approximated (see Scott and Barton (2013); Kurzhanski and Varaiya (2014); Kostousova (2014); Sinyakov (2015); Meyer et al. (2019)). One of such types of control systems is monotone systems or, more generally, mixed-monotone systems. Due to its generality and popularity we will refer to this method as "standard" throughout the paper. In this paper we specify a subclass of monotone systems for which there is a more efficient abstraction algorithm. This new algorithm and the formal proof of its correctness constitute the main contribution of the paper.

The method we present here also utilizes the partitioning of the state space. Unlike in the standard algorithm, each symbolic control in this method corresponds to a certain feedback controller for the original system as opposed to an open-loop control function (see e.g. Caines and Wei (1998)). Intuitively, we use a feedback controller such that the interval approximation of the reachable set (of the closed-loop system) from a partition element is the smallest in size or, more precisely, that it is minimal with respect to inclusion in a certain class $\mathscr{A}$ of interval sets for which we are able to construct the respective controllers. That way we expect to have fewer transitions correponding to a single symbolic control. The considered class $\mathscr{A}$ of interval sets has a description in terms of viscosity solutions of the related Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation (see Crandall and Lions (1983); Fleming and Soner (1995)). These intervals can be also described by certain differential equations with discontinuous right-hand side (see Filippov (1988)). We utilize both frameworks to establish the existence and uniqueness of the minimal element as well as the method of its practical construction.

The problem of polytope-to-polytope control for nonlinear control systems in relation with symbolic control has been considered extensively in the literature (see Belta and Habets (2006); Girard and Martin (2012); Ben Sassi and Girard (2013); Sloth and Wisniewski (2014); Meyer et al. (2016)). It has been shown (see e.g. Saoud et al. (2018)) that for controllability reasons it is sometimes important to consider "flat" partition elements. Moreover, depending on the system and the partition element, a minimal reachable set may be also flat. These considerations pose the

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main technical difficulty in the proof of correctness of our construction.

The paper is structured as follows. In Section 2 we define the problem of calculating the minimal (in a certain class $\mathscr{A}$) target set to which we can control the system from a given initial set (Problem 1). The main result of Section 3 suggests that every target set in the considered class $\mathscr{A}$ corresponds to a viscosity supersolution (upper solution) of the related backward HJBI equation. Once we have a supersolution, the feedback controller can be constructed (or verified) using the idea of extremal aiming (see, e.g. Subbotin (1995)). In Section 4 we first obtain the description of $\mathscr{A}$ in terms of differential equations with discontinuous right-hand side. Then we prove the existence and uniqueness of the minimal element of class $\mathscr{A}$. Finally, we define the controller and prove that it solves Problem 1.

In Section 5 we utilize the controllers obtained in Section 4 to define the new abstraction. Each symbolic control input v is associated with a particular controller $u(t,x)$ (instead of an open-loop or a constant control as in the standard algorithm). The transitions from a state q with a control v in the abstract system are enabled for every partition element that intersects the respective reachable set overapproximation. In Section 6 we compare the standard and the new abstraction algorithms on a 3-dimensional example of a temperature regulation problem. Due to the lack of space, the proofs are omitted in the conference version of this paper.

Notations: For $x \in \mathbb{R}^n$, $\|x\|_\infty = \max_i |x_i|$ is the infinity norm. Let $d(x,X)$ denote the distance $\inf_{z\in X}\|x-z\|_\infty$ between $x \in \mathbb{R}^n$ and $X \subseteq \mathbb{R}^n$. Given vectors x, $x' \in \mathbb{R}^n$, $x \preceq x'$ stands for $x_i \le x'_i$ for all $i = 1,\dots,n$. Using this partial order, we define multi-dimensional interval sets as follows: for $\underline{x},\overline{x} \in \mathbb{R}^n$, $[\underline{x},\overline{x}] = \{x \mid x \succeq \underline{x},\ x \preceq \overline{x}\}$. For a set-valued map $W\colon [0,T] \rightrightarrows \mathbb{R}^{n_w}$, the space of all Lebesgue measurable functions $w(\cdot)$ on $[0,T]$ such that $w(t) \in W(t)$ a.e. is denoted by $L^\infty([0,T],W(\cdot))$.

2. CONTROLLER SYNTHESIS PROBLEM FOR MONOTONE SYSTEMS

Consider a nonlinear system of the following type ($i = 1,\dots,n_x$):

$$\dot{x}_i = f_i(t,x,u_i,w), \quad t \in [0,T]. \tag{1}$$

Here $x \in \mathbb{R}^{n_x}$ is the state, $u \in U = [\underline{u},\overline{u}] \subset \mathbb{R}^{n_x}$ is the control and $w \in W = [\underline{w},\overline{w}] \subset \mathbb{R}^{n_w}$ is the disturbance. Here we allow the case when $\underline{u}_j = \overline{u}_j$ for some indices j. In this case, there are essentially less control parameters than n_x.

The set of admissible open-loop controls is $\mathcal{U}(t,\tau) = L^\infty([t,\tau],U)$. The set of admissible realizations of the disturbance is $\mathcal{W}(t,\tau) = L^\infty([t,\tau],W)$. Let $\mathbf{x}(t;\tau,x,u(\cdot),w(\cdot))$ denote a trajectory of the system satisfying the initial condition $x(\tau) = x$ and corresponding to the control $u(\cdot)$ and disturbance $w(\cdot)$. Finally, let $X^{u(\cdot)}(t;t_0,X^0)$ denote the reachable set

$$\{x \in \mathbb{R}^{n_x} \mid \exists x^0 \in X^0,\ \exists w(\cdot) \in \mathcal{W}(t_0,t) : \\ \mathbf{x}(t;t_0,x^0,u(\cdot),w(\cdot)) = x\}.$$

The conditions on the considered class of systems are summarized in the following.

Assumption 1. The right-hand side of system (1) is continuous in (t,x,u,w), globally Lipschitz in (x,u) uniformly in (t,w) and satisfies the following monotonicity condition: f_i is nondecreasing in x_j, u_i and w_k for all $i,j = 1,\dots,n_x$, $i \ne j$ and all $k = 1,\dots,n_w$. Such systems are called *monotone* with respect to state x and input (u,w).

Let us consider a class $\mathscr{A}$ of target interval sets X^1 that we will define below. For a controller $u\colon [0,T]\times\mathbb{R}^{n_x} \to U$ and a disturbance realization $w(\cdot)$, we will consider the closed-loop system:

$$\dot{x}_i = f_i(t,x,u_i(t,x),w(t)), \quad t \in [0,T]. \tag{2}$$

Problem 1. Given a system (1) satisfying Assumption 1, an initial interval set $X^0 = [\underline{x}^0,\overline{x}^0] \subset \mathbb{R}^{n_x}$ and a time horizon $T > 0$, find a minimal by inclusion set X^1 in a class $\mathscr{A}$ and a controller $u(t,x)$ such that

- the closed-loop system has a solution for all initial data and all admissible disturbances and every solution exists on the whole interval $[0,T]$;
- all trajectories of the closed-loop system originated from X^0 at $t = 0$ reach X^1 at $t = T$.

Since the inclusion relation $\subseteq$ induces only a partial order on subsets of $\mathbb{R}^{n_x}$, a minimal by inclusion set X^1 may be not unique in general. However, it will be unique in the case discussed in this paper.

Let us now introduce the type of classes $\mathscr{A}$ of target sets under consideration. Fix a trajectory $\hat{x}(\cdot) = \mathbf{x}(\cdot;0,x^0,\hat{u}(\cdot),\hat{w}(\cdot))$ of system (1) such that $x^0 \in X^0$. Consider a class $\mathscr{A}^{\hat{x}(\cdot)}$ consisting of all interval sets X^1 for which there exists a Lipschitz continuous interval-valued map $X(t)$ satisfying the following properties:

(a) $X(0) = X^0$, $X(T) = X^1$;
(b) for all $t \in [0,T]$, $x \in X(t)$, and all $w(\cdot) \in \mathcal{W}(t,T)$ there exists $u(\cdot) \in \mathcal{U}(t,T)$ such that $\mathbf{x}(\tau;t,x,u(\cdot),w(\cdot)) \in X(\tau)$ for all $\tau \in [t,T]$;
(c) $\hat{x}(t) \in X(t)$ for all $t \in [0,T]$.

Remark 1. Minimal interval over-approximation $X^+(t) = [\mathbf{x}(t;0,\underline{x}_0,\hat{u}(.),\underline{w}),\mathbf{x}(t;0,\overline{x}_0,\hat{u}(.),\overline{w})]$ of the reachable set $X^{\hat{u}(\cdot)}(t;0,X^0)$ gives an example of such interval-valued map.

Remark 2. Property (b) is sometimes called weak invariance of $X(t)$ with respect to differential inclusion $\dot{x}_i \in f_i(t,x,U,w(t))$. Every *recursively computed* over-approximation (backward in time) of the dynamics (1) possesses this property. Property (c) implies the following: consider $X(t) = [\underline{x}(t),\overline{x}(t)]$ and let $\overline{x}_j(\tau) = \hat{x}_j(\tau)$ for some j and $\tau \in [0,T)$. If $\overline{x}_j(\cdot)$ and $\hat{x}_j(\cdot)$ are differentiable at τ then $\dot{\hat{x}}_j(\tau) \le \dot{\overline{x}}_j(\tau)$. Similarly, one may prove that if $\underline{x}_j(\tau) = \hat{x}_j(\tau)$ then $\dot{\underline{x}}_j(\tau) \le \dot{\hat{x}}_j(\tau)$ if both derivatives exist at τ.

It is known that the problem of controller synthesis for a reachability specification can be solved by considering the corresponding problem of dynamic optimization (see Subbotin (1995); Kurzhanski and Varaiya (2001)). Namely, given a supersolution of the backward Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation, a reachability controller can be obtained, for example, by utilizing the idea of extremal aiming. With this in mind, let us formally translate our description of the problem into the Hamilton-Jacobi setting.

3. PRELIMINARIES ON THE HAMILTONIAN FORMALISM

Discussion in this section is applicable to a general nonlinear system with uncertainty

$$\dot{x} = f(t, x, u, w), \quad t \in [0, T]. \tag{3}$$

Here $x \in \mathbb{R}^{n_x}$ is the state, $u \in U \subset \mathbb{R}^{n_u}$ is the control and $w \in W \subset \mathbb{R}^{n_w}$ is the disturbance. We assume that sets U and W are convex and compact. The respective sets of admissible open-loop controls and disturbance realizations are $L^\infty([t, \tau], U)$ and $L^\infty([t, \tau], W)$ as before.

Consider an initial set X^0 and let us represent it as a sublevel set of some function $\sigma(\cdot)$:

$$X^0 = \{x \in \mathbb{R}^{n_x} \mid \sigma(x) \le 0\}.$$

Similarly, given a target set X^1, let us represent it as a sublevel set of some other function $\psi(\cdot)$:

$$X^1 = \{x \in \mathbb{R}^{n_x} \mid \psi(x) \le 0\}.$$

Consider now the HJBI equation

$$V_t + \min_{u \in U} \max_{w \in W} \big\langle V_x, f(t, x, u, w)\big\rangle = 0. \tag{4}$$

Assumption 2. We impose the following assumptions.

(1) The right-hand side of (3) is continuous in (t, x, u, w), globally Lipschitz in (x, u) uniformly in (t, w);
(2) For system (3), Isaacs minimax condition is satisfied: for all $p \in \mathbb{R}^{n_x}$

$$\begin{aligned} &\min_{u \in U} \max_{w \in W} \big\langle p, f(t, x, u, w)\big\rangle = \\ &\max_{w \in W} \min_{u \in U} \big\langle p, f(t, x, u, w)\big\rangle. \end{aligned} \tag{5}$$

In the following let $H(t, x, p)$ denote the expression

$$\min_{u \in U} \max_{w \in W} \big\langle p, f(t, x, u, w)\big\rangle.$$

As mentioned above, we may obtain a controller, which steers system (3) to X^1 at $t = T$, by computing a supersolution (or the actual solution) of equation (4) with the terminal condition

$$V(T, x) = \psi(x) \tag{6}$$

backwards in time. To guarantee that every point of X^0 is controllable, condition $V(0, x) \le \sigma(x)$ for all $x \in \mathbb{R}^{n_x}$ must be satisfied.

However, since X^1 is an unknown part of the solution of Problem 1, we have to employ another approach. Intuitively, one may try to consider equation (4) forward in time with the initial condition

$$V(0, x) = \sigma(x) \tag{7}$$

and put

$$X^1 = \{x \in \mathbb{R}^{n_x} \mid V(T, x) \equiv \psi(x) \le 0\}.$$

In general, the forward solution V of (4), (7) is not even a supersolution of (4), (6). However, for system (1) the forward subsolutions, which we construct below, turn out to be backward supersolutions indeed.

Let us now remind precisely the definitions of viscosity solutions in the considered cases (see Crandall and Lions (1983); Fleming and Soner (1995)). For equation (4) considered in forward time we have

- A function V is a forward viscosity subsolution of (4) if and only if for all $(t, x) \in (0, T] \times X$
$$q + H(t, x, p) \le 0 \quad \forall (q, p) \in D^+ V(t, x); \tag{8}$$
- A function V is a forward viscosity supersolution of (4) if and only if for all $(t, x) \in (0, T] \times X$
$$q + H(t, x, p) \ge 0 \quad \forall (q, p) \in D^- V(t, x); \tag{9}$$
- V is a forward viscosity solution if it is both a sub- and a supersolution.

Here $D^+V(t, x)$ denotes the superdifferential of V at (t, x) and $D^-V(t, x)$ denotes the subdifferential of V at (t, x).

For equation (4) considered in backward time we have

- A function V is a backward viscosity subsolution of (4) if and only if for all $(t, x) \in [0, T) \times X$
$$q + H(t, x, p) \ge 0 \quad \forall (q, p) \in D^+ V(t, x); \tag{10}$$
- A function V is a backward viscosity supersolution of (4) if and only if for all $(t, x) \in [0, T) \times X$
$$q + H(t, x, p) \le 0 \quad \forall (q, p) \in D^- V(t, x); \tag{11}$$
- V is a backward viscosity solution if it is both a sub- and a supersolution.

The next lemma and the following corollary show the connection between Problem 1 and the HJBI equation (4).

Lemma 1. Consider a continuous set-valued map $X(t)$, $t \in [0, T]$ with closed values and let $L > 0$ be the Lipschitz constant of the right-hand side of (3). Under Assumption 2, $X(t)$ satisfies property (b) of the definition of class $\mathscr{A}^{\hat{x}(\cdot)}$ if and only if the function

$$V(t, x) = e^{-Lt} d(x, X(t))$$

is a backward supersolution of equation (4).

Corollary 1. Let the assumptions of Lemma 1 hold. If $X(t)$ satisfies property (b) of the definition of class $\mathscr{A}^{\hat{x}(\cdot)}$ and $X(\cdot)$ is Lipschitz continuous and convex-valued then the function $V(t, x) = e^{-Lt} d(x, X(t))$ is a forward subsolution of equation (4).

In the next section we utilize this corollary to obtain a description of $\mathscr{A}^{\hat{x}(\cdot)}$ in terms of equations with discontinuous right-hand side (Corollary 2).

4. SOLUTION OF THE SYNTHESIS PROBLEM

In this section we provide the solution to Problem 1. From this point onward we consider Assumptions 1 and 2 being satisfied.

4.1 Minimal reachable sets

In this subsection we find equations that define the minimal target set X^1 in Problem 1.

Given an arbitrary interval $X^0 = [\underline{x}^0, \overline{x}^0]$, let us consider a Lipschitz continuous interval-valued map $X(t) = [\underline{x}(t), \overline{x}(t)]$ such that $X(0) = X^0$. We introduce the function $\sigma(\cdot)$:

$$\sigma(x) \equiv d(x, X^0) = \max_i \max\{x_i - \overline{x}_i^0,\ \underline{x}_i^0 - x_i, 0\}. \tag{12}$$

Now let us define the function

$$V(t,x) = e^{-Lt} \max_i \max\{x_i - \overline{x}_i(t),\ \underline{x}_i(t) - x_i, 0\} \tag{13}$$

where $L > 0$ is the Lipschitz constant of the right-hand side of (1):

$$|f_i(t,x,u_i,w) - f_i(t,y,u_i,w)| \le L\|x-y\|_\infty.$$

As mentioned in the previous section, to obtain a controller that solves the reachability problem for a target set $X^1 = [\underline{x}(T), \overline{x}(T)]$, we need a backward supersolution of (4), (6). Under the assumptions of Corollary 1, a backward supersolution of the form (13) is also a forward subsolution of (4), (7). Therefore, let us now give a criterion for (13) to be a forward subsolution.

Lemma 2. Function V is a viscosity subsolution of (4), (7) in forward time if and only if

$$\begin{aligned} \dot{\overline{x}}_i(t) &\ge f_i(t, \overline{x}(t), \underline{u}_i, \overline{w}), \\ \dot{\underline{x}}_i(t) &\le f_i(t, \underline{x}(t), \overline{u}_i, \underline{w}) \end{aligned} \tag{14}$$

a.e. on $[0,T]$.

Thus, for every interval-valued map $X(t)$ in the definition of class $\mathscr{A}^{\hat{x}(\cdot)}$ inequalities (14) must hold. This observation leads to the following.

Corollary 2. If $X^1 \in \mathscr{A}^{\hat{x}(\cdot)}$ then there exist $X(t) = [\underline{x}(t), \overline{x}(t)]$ and $\xi(\cdot) = (\overline{\xi}(\cdot), \underline{\xi}(\cdot)) \in L^\infty([0,T], \mathbb{R}^{2n_x})$ with $\overline{\xi}(t) \succeq 0$, $\underline{\xi}(t) \preceq 0$ satisfying equations

$$\begin{aligned} \dot{\overline{x}}_i &= \begin{cases} f_i(t, \overline{x}, \underline{u}_i, \overline{w}) + \overline{\xi}_i(t),\ \hat{x}_i(t) < \overline{x}_i, \\ \max\{f_i(t, \overline{x}, \underline{u}_i, \overline{w}) + \overline{\xi}_i(t), \dot{\hat{x}}_i(t)\},\ \hat{x}_i(t) \ge \overline{x}_i, \end{cases} \\ \dot{\underline{x}}_i &= \begin{cases} f_i(t, \underline{x}, \overline{u}_i, \underline{w}) + \underline{\xi}_i(t),\ \underline{x}_i < \hat{x}_i(t), \\ \min\{f_i(t, \underline{x}, \overline{u}_i, \underline{w}) + \underline{\xi}_i(t), \dot{\hat{x}}_i(t)\},\ \underline{x}_i \ge \hat{x}_i(t) \end{cases} \end{aligned} \tag{15}$$

a.e. on $[0,T]$, initial conditions

$$\underline{x}(0) = \underline{x}^0, \quad \overline{x}(0) = \overline{x}^0 \tag{16}$$

and such that $X(T) = X^1$.

This result gives a useful description of the considered class $\mathscr{A}^{\hat{x}(\cdot)}$. Intuitively, the interval-valued map $X(t)$ that satisfies differential equations (15) with $\xi(t) \equiv 0$ should produce the minimal element of the respective class $\mathscr{A}^{\hat{x}(\cdot)}$. To formally establish it, we need to prove that (15) is monotone in state $(\overline{x}, \underline{x})$ and input $(\overline{\xi}, \underline{\xi})$ and has a solution for $\xi(t) \equiv 0$. First, we provide the following two lemmas.

Lemma 3. (1) System of equations (15) has a unique solution on $[0,T]$ in the sense of Filippov (see Filippov (1988), §4, definition a)). Moreover, the solution is Lipschitz continuous.

(2) For any solution of (15), (16), the following relation holds:

$$\underline{x}(t) \preceq \hat{x}(t) \preceq \overline{x}(t).$$

Lemma 4. The function V defined by (13), (15), (16) is a viscosity supersolution of (4), (6) in backward time.

Thus, for every solution of (15), (16) the corresponding set $X(T) \in \mathscr{A}^{\hat{x}(\cdot)}$. Now we present the main result of this subsection.

Theorem 1. Consider the solution $(\underline{x}(\cdot), \overline{x}(\cdot))$ of (15), (16) with $\xi(t) \equiv 0$. The set $X^1 = [\underline{x}(T), \overline{x}(T)]$ is the unique minimal element of class $\mathscr{A}^{\hat{x}(\cdot)}$.

Corollary 3. For $X(t) = [\underline{x}(t), \overline{x}(t)]$ defined by (15), (16) with $\xi(t) \equiv 0$ and for any interval-valued map $X^+(t)$ such that $X^{\hat{u}(\cdot)}(t; 0, X^0) \subseteq X^+(t)$, the inclusion holds

$$X(t) \subseteq X^+(t), \quad t \in [0,T].$$

4.2 Controller construction

Let us now consider the interval-valued map $X(t) = [\underline{x}(t), \overline{x}(t)]$ defined by (15), (16) with $\xi(t) \equiv 0$. We define the following controller:

$$\begin{aligned} x_i^c(t) = (\underline{x}_i(t) + \overline{x}_i(t))/2, &\quad x_i^r(t) = (\overline{x}_i(t) - \underline{x}_i(t))/2, \\ u_i^c = (\underline{u}_i + \overline{u}_i)/2, &\quad u_i^r = (\overline{u}_i - \underline{u}_i)/2, \end{aligned}$$

$$u_i(t,x) = \begin{cases} \underline{u}_i,\ x_i > \overline{x}_i(t), \\ u_i^c + u_i^r \dfrac{x_i - x_i^c(t)}{x_i^r(t)},\ \underline{x}_i(t) \le x \le \overline{x}_i(t), \\ \overline{u}_i,\ x_i < \underline{x}_i(t). \end{cases} \tag{17}$$

If $\underline{x}_i(t) = \overline{x}_i(t)$ we formally put $u_i(t, \hat{x}(t)) = \hat{u}_i(t)$.

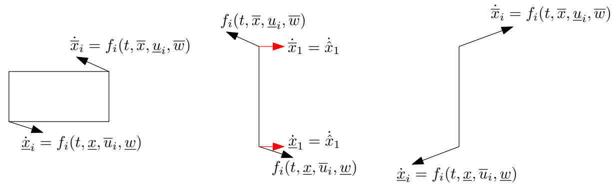

Fig. 1. Different scenarios of evolution of $X(t)$.

Theorem 2. The following propositions hold.

(1) Closed-loop system (2) has a unique solution on $[0,T]$ (in the sense of Filippov) for all admissible disturbances $w(\cdot)$. Every solution $x(\cdot)$ emanating from $X^0 = [\underline{x}^0, \overline{x}^0]$ satisfies the inclusions $x(t) \in [\underline{x}(t), \overline{x}(t)]$ for all $t \in [0,T]$;

(2) If the interior of $[\underline{x}(t), \overline{x}(t)]$ is not empty for all $t \in [0,T]$ then the closed-loop system (2) has a solution (in the sense of Carathéodory) for all admissible disturbances $w(\cdot)$. Every solution $x(\cdot)$ emanating from $X^0 = [\underline{x}^0, \overline{x}^0]$ satisfies the inclusions $x(t) \in [\underline{x}(t), \overline{x}(t)]$ for all $t \in [0,T]$.

5. ABSTRACTION ALGORITHM

In this section we consider the time-invariant version of system (1):

$$\dot{x}_i = f_i(x, u_i, w), \quad i = 1, \dots, n_x. \tag{18}$$

Here $u \in U = [\underline{u}, \overline{u}]$, $w \in W = [\underline{w}, \overline{w}]$ as before.

Given a controller u: $[0,T] \times \mathbb{R}^{n_x} \to U$, let $\mathbf{x}(t; x, u, w(\cdot))$ denote the set of all solution endpoints (in the sense of Filippov) of the closed-loop system satisfying the initial condition $x(0) = x$ and corresponding to the disturbance $w(\cdot) \in \mathcal{W}(0,T)$.

Let us denote $\mathscr{U}_T^0(x)$ the set of all controllers such that for $x(0) = x$ and for every $w(\cdot) \in \mathcal{W}(0,T)$ there is at least one Filippov solution of the closed-loop system and every such solution exists on $[0,T]$.

Let us consider a set $X \subseteq \mathbb{R}^{n_x}$, which we call the state space, and restrict the dynamics of system (18) to this set. Let the state space X be covered by a finite set of intervals $(X_q)_{q \in Q}$: $X = \cup_{q \in Q} X_q$, $X_q = [\underline{x}^q, \overline{x}^q]$.

Definition 1. A *transition system* is a tuple (X, U, Y, Δ, H), where

- X is a set of states;
- U is a set of inputs;
- Y is a set of outputs;
- $\Delta : X \times U \rightrightarrows X$ is a set-valued transition map;
- H: $X \longrightarrow Y$ is an output map.

An input $u \in U$ is called *enabled* at $x \in X$ if $\Delta(x, u) \neq \emptyset$. Let $\mathrm{enab}_\Delta(x) \subseteq U$ denote the set of all inputs enabled at x. If $\mathrm{enab}_\Delta(x) = \emptyset$ the state x is called *blocking*.

Given the cover $(X_q)_{q\in Q}$, system (18) may be written as a transition system as follows:

$$S = (X, \mathscr{U}, Q, \delta, H)$$

where

$$\mathscr{U} = \{(T, u),\ T \in [0, +\infty),\ u : \ [0, T] \times \mathbb{R}^{n_x} \to U\},$$
$$q = H(x) \quad \Leftrightarrow \quad x \in X_q$$

and transition relation δ is defined as follows:

$$x' \in \delta(x, T, u), \quad (T, u) \in \mathrm{enab}_\delta(x)$$

if and only if there exists $w \in \mathcal{W}(0, T)$ such that $x' \in \mathbf{x}(T; 0, x, u, w(\cdot))$. Here the set of enabled inputs is defined as follows

$$\mathrm{enab}_\delta(x) = \{(T, u) \in \mathscr{U} \mid u \in \mathscr{U}^0_T(x) \quad \text{and}$$
$$\forall w \in \mathcal{W}(0, T), \forall t \in [0, T],\ \mathbf{x}(t; 0, x, u, w(\cdot)) \subseteq X\}.$$

We now define an abstract transition system S_a using the cover $(X_q)_{q\in Q}$, a sampling parameter $\tau > 0$ and a finite set of control inputs $\mathscr{V}$:

$$S_a = (Q, \mathscr{V}, Q, \Delta, \mathrm{Id}).$$

Here Id is the identity map on Q. In a state $q \in Q$ a symbolic control $v \in \mathscr{V}$ corresponds to a pair $(\tau, u^{(q,v)}) \in \mathscr{U}$ such that $u^{(q,v)}$ is defined by (17) and the corresponding interval $X^{(q,v)}(t) = [\underline{x}^{(q,v)}(t), \overline{x}^{(q,v)}(t)]$ is defined by (15) with $\xi(t) \equiv 0$ and the initial conditions

$$\underline{x}^{(q,v)}(0) = \underline{x}^q \preceq \overline{x}^q = \overline{x}^{(q,v)}(0).$$

The corresponding reference trajectories $\hat{x}(\cdot)$ and reference controls $\hat{u}(\cdot)$ in (15) and (17) depend on the pair (q, v). Below we provide a particular choice of those that guarantees the comparison result in Theorem 4.

Observe that $u^{(q,v)} \in \mathscr{U}^0_\tau(x)$. Transition relation Δ is defined as follows: $q' \in \Delta(q, v)$ for $v \in \mathscr{V}$ if and only if

$$X_{q'} \cap [\underline{x}^{(q,v)}(\tau), \overline{x}^{(q,v)}(\tau)] \neq \emptyset$$

and

$$[\underline{x}^{(q,v)}(t), \overline{x}^{(q,v)}(t)] \subseteq X$$

for all $t \in [0, \tau]$.

Definition 2. Let $S_a = (X_a, U_a, Y_a, \Delta_a, H_a)$ and $S_b = (X_b, U_b, Y_b, \Delta_b, H_b)$ be two transition systems with $Y_a = Y_b$. A relation $R \subseteq X_a \times X_b$ is an alternating simulation relation from S_a to S_b if the following conditions are satisfied:

(1) for every $(x_a, x_b) \in R$ we have $H_a(x_a) = H_b(x_b)$;
(2) for every $(x_a, x_b) \in R$ and for every $u_a \in \mathrm{enab}_{\Delta_a}(x_a)$ there exists $u_b \in \mathrm{enab}_{\Delta_b}(x_b)$ such that for every $x'_b \in \Delta_b(x_b, u_b)$ there exists $x'_a \in \Delta_a(x_a, u_a)$ satisfying $(x'_a, x'_b) \in R$.

It is said that S_b alternatingly simulates S_a, denoted by $S_a \preceq_{AS} S_b$, if there exists an alternating simulation relation $R \neq \emptyset$ from S_a to S_b.

Theorem 3. Transition system S alternatingly simulates abstract system S_a: $S_a \preceq_{AS} S$.

Let us now introduce the standard abstract system S_{std}. Consider a finite approximation $\hat{U}$ of the control space: $\hat{U} \subset U$. We define the abstraction

$$S_{std} = (Q, \hat{U}, Q, \hat{\Delta}, \mathrm{Id})$$

where transition relation $\hat{\Delta}$ is defined as follows: $q' \in \hat{\Delta}(q, \hat{u})$ for $\hat{u} \in \hat{U}$ if and only if

$$X_{q'} \cap [\mathbf{x}(\tau; \underline{x}^q, \hat{u}, \underline{w}), \mathbf{x}(\tau; \overline{x}^q, \hat{u}, \overline{w})] \neq \emptyset$$

and

$$[\mathbf{x}(t; \underline{x}^q, \hat{u}, \underline{w}), \mathbf{x}(t; \overline{x}^q, \hat{u}, \overline{w})] \subseteq X$$

for all $t \in [0, \tau]$.

To provide a comparison result between S_a and S_{std}, let us specify the set $\mathscr{V}$ and the corresponding controls $u^{(q,v)}$. Let $\mathscr{V} = \hat{U}$ and $u^{(q,v)}$ corresponds to the reference trajectory $\hat{x}(\cdot)$ defined by the following:

$$\dot{\hat{x}}_i = f(\hat{x}, \hat{u}_i, (\overline{w} - \underline{w})/2), \quad \hat{x}(0) = (\overline{x}^q - \underline{x}^q)/2.$$

Theorem 4. Transition system S_a alternatingly simulates S_{std}: $S_{std} \preceq_{AS} S_a$.

Theorems 3 and 4 give us the relation

$$S_{std} \preceq_{AS} S_a \preceq_{AS} S.$$

Given an arbitrary control specification, every symbolic state $q \in Q$, which is controllable for S_{std}, is also controllable for S_a. We emphasize that by construction the number of transitions in the new abstraction does not exceed the number of transitions in the standard abstraction.

6. EXAMPLE

Let us consider a temperature regulation model of a circular n_x room building, which was adapted from Girard et al. (2015). The system is given by equations:

$$\dot{\mathbf{T}}_i(t) = \alpha(\mathbf{T}_{i+1}(t) + \mathbf{T}_{i-1}(t) - 2\mathbf{T}_i(t))$$
$$+\beta(\mathbf{T}_e(t) - \mathbf{T}_i(t)) + \gamma(\mathbf{T}_h - \mathbf{T}_i(t))u_i(t).$$

Here $\mathbf{T}_i$ is the temperature in room i, $\mathbf{T}_e(t) \in [\mathbf{T}_e^{\min}, \mathbf{T}_e^{\max}]$ is the outside temperature, which is considered as disturbance, α, β and γ are the corresponding conduction factors. The heater powers $u_i(t) \in [0, 1]$ are the control parameters whereas the maximal heater temperature is $\mathbf{T}_h$. We utilize the following values for conduction factors: $\alpha = 0.05$, $\beta = 0.005$, $\gamma = 0.01$. The system is monotone in state and inputs.

We consider this system on the following state space:

$$X = [\mathbf{T}_1^{\min}, \mathbf{T}_1^{\max}] \times \cdots \times [\mathbf{T}_{n_x}^{\min}, \mathbf{T}_{n_x}^{\max}].$$

Let us introduce a partition for each coordinate ($i = 1, \ldots, n_x$):

$$[\mathbf{T}_i^{\min}, \mathbf{T}_i^{\min} + \frac{1}{N_i}(\mathbf{T}^{\max} - \mathbf{T}_i^{\min})], \ldots,$$
$$[\mathbf{T}_i^{\min} + \frac{N_i - 1}{N_i}(\mathbf{T}_i^{\max} - \mathbf{T}_i^{\min}), \mathbf{T}^{\max}].$$

Based on this partition we construct regions X_q as Cartesian products of the elements from those partitions. The

τ	Algorithm	# of transitions	# of cont. states
1	Standard	235221	1000
1	New	(< 34%) 79245	1000
5	Standard	404167	1000
5	New	(< 7.4%) 29592	1000
40	Standard	548940	0
40	New	(< 5%) 27081	1000

Table 1. Comparison between the two abstraction algorithms for 3 different values of τ.

set of all such X_q covers the whole state space X. For both abstraction algorithm we use sampling paramter τ. We compare the two algorithms for $\tau = 1, 5, 40$.

We utilize $\hat{U} = \{0, \frac{1}{2}, 1\}^{n_x}$ as a finite approximation of U in the standard abstraction algorithm. In the new algorithm we use $|\hat{U}| = 3^{n_x}$ reference trajectories each component of which is chosen according to one of the following three conditions ($i = 1, \ldots, n_x$):

$$\dot{\hat{\mathbf{T}}}_i = \alpha(\hat{\mathbf{T}}_{i+1}(t) + \hat{\mathbf{T}}_{i-1}(t) - 2\hat{\mathbf{T}}_i(t)) \\ +\beta(\mathbf{T}_e^{\min} - \hat{\mathbf{T}}_i(t)) + \gamma(\mathbf{T}_h - \hat{\mathbf{T}}_i(t)),$$

or

$$\dot{\hat{\mathbf{T}}}_i = \alpha(\hat{\mathbf{T}}_{i+1}(t) + \hat{\mathbf{T}}_{i-1}(t) - 2\hat{\mathbf{T}}_i(t)) + \beta(\mathbf{T}_e^{\max} - \hat{\mathbf{T}}_i(t)),$$

or

$$\dot{\hat{\mathbf{T}}}_i = \alpha(\hat{\mathbf{T}}_{i+1}(t) + \hat{\mathbf{T}}_{i-1}(t) - 2\hat{\mathbf{T}}_i(t)).$$

For the simulations below we choose the following parameters: $n_x = 3$, $\mathbf{T}_i^{\min} = 19°\text{C}$, $\mathbf{T}_i^{\max} = 23°\text{C}$, $\mathbf{T}_e^{\min} = -1°\text{C}$, $\mathbf{T}_e^{\max} = 10°\text{C}$, $\mathbf{T}_h = 50°\text{C}$, $N_i = 10$. Here we consider a simple safety problem of keeping trajectories of the system in X at all times.

Table 1 gives the total count of transitions and controllable states for the standard and the new abstraction algorithms. Both abstract systems utilize the same number of symbolic controls but the overall number of transitions is greatly reduced for the new abstraction. The higher reduction is achieved for bigger values of sampling parameter τ. Coincidentally, for big enough values of τ the standard abstract system in this example becomes completely uncontrollable while the new abstract system is still controllable.

7. CONCLUSION

In this paper we introduced a new abstraction algorithm for a certain subclass of continuous-time monotone control systems. This algorithm produces more efficient symbolic systems with fewer number of transitions than a standard algorithm used in the literature for such systems. The improvement is achieved by considering interval-to-interval feedback controllers instead of open-loop (or constant) controls. The extension of the method to more general classes of systems, including mixed-monotone systems, is planned for future research.

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