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Integral control of stable nonlinear systems based on singular perturbations ⋆

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Abstract: One of the main issues related to integral control is windup, which occurs when, possibly due to a fault, the input signal u of the plant reaches a value outside the allowed input range U. This paper presents an integral controller with anti-windup, called saturating integrator, for a single-input single-output nonlinear plant having a curve of locally exponentially stable equilibrium points that correspond to constant inputs in U. A closed-loop system is formed by connecting the saturating integrator in feedback with the plant. The control objective is to make the output signal y of the plant track a constant reference r, while not allowing its input signal u to leave U. Using singular perturbation methods, we prove that, under reasonable assumptions, the equilibrium point of the closed-loop system is exponentially stable, with a "large" region of attraction. Moreover, when the state of the closed-loop system converges to this equilibrium point, then the tracking error tends to zero. A step-by-step procedure is presented to perform the closed-loop stability analysis, by finding separately a Lyapunov function for the reduced (slow) model and a Lyapunov function for the boundary-layer (fast) system. Afterwards, a Lyapunov function for the closed-loop system is built as a convex combination of the two previous ones, and an upper bound on the controller gain is found such that closed-loop stability is guaranteed. Finally, we show that if certain stronger conditions hold, then the domain of attraction of the stable equilibrium point of the closed-loop system can be made large by choosing a small controller gain.

Keywords: nonlinear systems, integral control, singular perturbation method, windup, Lyapunov methods.

1. INTRODUCTION

Integral control is an important topic in the control literature, being extensively used to achieve robust asymptotic regulation and disturbance rejection (it is the simplest instance of the internal model principle). When the plant is an uncertain linear system, closed-loop stability can be achieved if the control gain is sufficiently small and the plant fulfils certain stability conditions, as discussed in Morari (1985) and in Fliegner et al. (2003). The theory has been extended to nonlinear systems, as discussed in Desoer and Lin (1985) for PI controllers, or in Isidori and Byrnes (1990), where local stability results can be found for the more general internal model principle. Later, regional and semiglobal results have been presented in Isidori (1997) and in Khalil (2000) for specific classes of nonlinear systems using high-gain observers. In addition, in Singh and Khalil (2005) and in Seshagiri and Khalil (2005) *conditional integrators* are presented, which provide integral action inside a boundary layer while acting as stable systems outside of it. Besides, recent application of integral control to port-Hamiltonian systems can be found in Ferguson et al. (2018) and in the references therein.

In the presence of an integral controller, the windup problem can occur if the integrator's state reaches a value far from its normal operating range, for instance, due to a fault. This may cause long transients and oscillations that could lead the system to instability. In order to prevent the windup phenomenon, several anti-windup techniques have been investigated, resulting in a rich literature on the topic. Initially, anti-windup methods were developed only for specific problems and they lacked a rigorous stability analysis. In the field of linear systems, one of the first systematic methods was the *conditioning technique* introduced in Hanus et al. (1987). Later, this and other results have been collected in Kothare et al. (1994), where a coprime-factor framework has been introduced, and then in Edwards and Postlethwaite (1998), where a generic approach in the form of an H_∞ optimization framework has been proposed. A survey of this area is Peng et al. (1996), where the most relevant PID anti-windup techniques are summarized. A treatment of the $\mathscr{L}_2$ anti-windup problem can be found in Zaccarian and Teel (2002). In recent years,

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more attention has been devoted to LMIs to tackle more complex anti-windup scenarios. To this aim, it is worth mentioning the work of Wu and Lu (2004), where regional stability is guaranteed in the presence of input saturation for an exponentially unstable linear system, and the result in Turner et al. (2007), where uncertainty in the system model is taken into account. These and other LMI-based methods can be found in the survey paper of Tarbouriech and Turner (2009). In addition, an interesting anti-windup result for PI controllers is in Choi and Lee (2009), where the conditional integrator scheme is improved by setting a specific initial value for the integrator when switching from P to PI control mode.

Nonlinear systems have received less attention in the anti-windup literature. For Euler-Lagrange systems, an anti-windup scheme inducing global asymptotic stability and local exponential stability is proposed in Morabito et al. (2004), while for affine nonlinear systems an anti-windup controller based on nonlinear dynamic inversion is proposed in Herrmann et al. (2010). Other remarkable results are presented in Rehan et al. (2013) and in da Silva et al. (2016), where specific classes of nonlinear systems are addressed, and in the references therein. A recent result on anti-windup integral control for nonlinear systems is from Konstantopoulos et al. (2016), where a *bounded integral controller* (BIC) is presented. Under suitable assumptions, the BIC is able to generate a bounded control output independently from the plant parameters and states, guaranteeing closed-loop stability in the sense of boundedness. As claimed in that paper, most of the control algorithms depend on the system structure (relative degree etc.) and they usually lead to complex control schemes that require a saturation unit to avoid windup scenarios.

The objective of this paper is to provide a simple integral controller with anti-windup for stable nonlinear systems, called *saturating integrator*, which is straightforward to implement. The saturating integrator is effective for a wide range of applications, see for instance Natarajan and Weiss (2017) for an application of our main result in power electronics. It also allows us to derive rigorous stability results using the singular perturbation approach.

Let us now state the problem in precise terms. The nonlinear plant $\mathbf{P}$ to be controlled is described by:

$$\dot{x} = f(x,u), \qquad y = g(x), \tag{1.1}$$

where $f \in C^2(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n)$ and $g \in C^1(\mathbb{R}^n; \mathbb{R})$. The control objective is to make the output signal y track a constant reference signal $r \in Y := (y_{min}, y_{max}) \subset \mathbb{R}$, using an input signal that takes values in the range $U := [u_{min}, u_{max}] \subset \mathbb{R}$ (here $u_{min} < u_{max}$). In order to achieve this goal, a type of anti-windup integral controller is used, which we call the *saturating integrator*. We define the positive (negative) part of a real number w by $w^+ = \max\{w, 0\}$ ($w^- = \min\{w, 0\}$). The *saturating integrator* is a system with input w and state u, described by

$$\dot{u} = \mathscr{S}(u,w),$$

$$\text{where} \quad \mathscr{S}(u,w) = \begin{cases} w^+ & u \le u_{min}, \\ w & u \in (u_{min}, u_{max}), \\ w^- & u \ge u_{max}. \end{cases} \tag{1.2}$$

The plant (1.1) and the saturating integrator (1.2) are connected according to the feedback loop shown in Figure

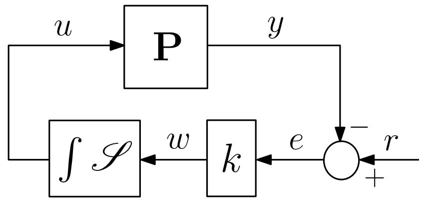

Fig. 1. Closed-loop system formed by the plant $\mathbf{P}$, the saturating integrator $\int \mathscr{S}$ and the constant gain $k > 0$, with the reference r.

1, where $k > 0$ is a constant gain. The closed-loop system is described by

$$\dot{x} = f(x,u), \qquad \dot{u} = \mathscr{S}(u, k(r - g(x))), \tag{1.3}$$

with state space $X := \mathbb{R}^n \times U$. An informal statement of our main result is that, under reasonable assumptions on the plant $\mathbf{P}$, for any constant reference $r \in Y$ the following holds: For any small enough feedback gain $k > 0$, the closed-loop system (1.3) shown in Figure 1 is locally exponentially stable around an equilibrium point, with a "large" region of attraction. When the state converges to this equilibrium point, then the tracking error $e := r - y$ tends to zero at an exponential rate.

A similar result but without limitations on u and without anti-windup, assuming global exponential stability of the plant's equilibrium point for any constant input u in the input space $\mathbb{R}^m$, has been given in Desoer and Lin (1985).

A preliminary version of this work, with many proofs missing, and following a certain line of reasoning that we now consider to be out of date, has been presented as a conference paper, see Weiss and Natarajan (2016). We believe that the new approach here is much more elegant.

The paper is organized as follows. In Section 2 the state trajectories of the closed-loop system are studied. In Section 3 the closed-loop system equations are rewritten as a singular perturbation model. In Section 4 the main result is presented, where the stability analysis of the closed-loop system is proved using singular perturbation methods. Finally, in Section 5 the domain of attraction of the closed-loop system equilibrium point is investigated.

2. CLOSED-LOOP TRAJECTORIES

Some care has to be taken to define the closed-loop trajectories of the system (1.3). First, it has to be ensured that the state trajectories u of the saturating integrator (1.2) are well-defined for any input $w \in L^1[0,t]$. For this we resort to a density argument: Note that for a polynomial input w, the state trajectory u is easy to define.

Let u_1 and u_2 be the state trajectories of (1.2) corresponding to the polynomials w_1 and w_2. Then by an elementary argument

$$|u_2(t) - u_1(t)| \le |u_2(0) - u_1(0)| + \int_0^t |w_2(\sigma) - w_1(\sigma)| d\sigma.$$

This shows that $u(t)$ depends Lipschitz continuously both on $u(0)$ and also on w considered with the L^1 norm. For $u_2(0) = u_1(0)$ we can write the last estimate as

$$|u_2(t) - u_1(t)| \le \|w_2 - w_1\|_{L^1[0,t]}. \tag{2.1}$$

Hence, by continuous extension, we can define $u(t)$ for any initial state $u(0)$ and for any input $w \in L^1[0,t]$ (because the polynomials are dense in $L^1[0,t]$). Next, we show the local existence and uniqueness of state trajectories for the closed-loop system (1.3), which is not trivial due to the discontinuity of $\mathscr{S}$.

Notation. For any $R>0$ let B_R be the closed ball of radius R in $\mathbb{R}^n$. For any $\tau>0$ we denote by C_τ the set of all the continuous functions on the interval $[0,\tau]$, with values in U. This is a complete metric space with the distance induced by the supremum norm, denoted by $\|\cdot\|_\infty$.

Lemma 2.1. Take $x_0 \in \mathbb{R}^n$ and $R>0$. Denote $M = \max\{\|f(x,v)\| \mid x \in x_0+B_R,\ v\in U\}$ and let $\tau \in (0, R/M]$. Then, for every $u \in C_\tau$, (1.1) with the initial state $x(0)=x_0$ has a unique solution $x \in C^1([0,\tau];\mathbb{R}^n)$ and $x(t) \in x_0+B_R$ for all $t\in[0,\tau]$.

Denote by T_τ the (nonlinear) operator determined by $\mathbf{P}$, that maps any input function $u \in C_\tau$ into an output function $y \in C[0,\tau]$ (corresponding to $x(0)=x_0$). Then T_τ is Lipschitz continuous.

Proof. It follows from the mean value theorem that for any $u\in C_\tau$, the state trajectory $x(t)$ of $\mathbf{P}$ exists and remains in x_0+B_R for all $t\le\tau$. Let $L_1, L_2>0$ be such that for any $z_1,z_2 \in x_0+B_R$, and for any $v_1,v_2\in U$

$$\|f(z_2,v_2)-f(z_1,v_1)\| \le L_1\|z_2-z_1\| + L_2\|v_2-v_1\|. \quad (2.2)$$

Such L_1, L_2 exist since $f\in C^2$ and $\{x_0+B_R\}\times U$ is compact. Take two state trajectories of $\mathbf{P}$, x_1 and x_2, starting from the same initial state x_0 and corresponding to inputs u_1 and u_2. Then

$$x_2(t)-x_1(t) = \int_0^t \left[f(x_2(\sigma),u_2(\sigma)) - f(x_1(\sigma),u_1(\sigma))\right] d\sigma,$$

for any $t\in[0,\tau]$, whence, using (2.2),

$$\|x_2(t)-x_1(t)\| \le L_1\int_0^t \|x_2(\sigma)-x_1(\sigma)\|d\sigma + \tau L_2\|u_2-u_1\|_\infty.$$

It follows from Gronwall's inequality that

$$\|x_2(t)-x_1(t)\| \le \tau L_2\|u_1-u_2\|_\infty\, e^{L_1 t} \qquad \forall\, t\in[0,\tau],$$

which implies that

$$\|x_2-x_1\|_\infty \le \tau L_2\|u_1-u_2\|_\infty\, e^{L_1\tau}.$$

Finally, since $g\in C^1$, we get

$$\|y_2-y_1\|_\infty \le L_T\|u_1-u_2\|_\infty,$$

where L_T is the product of $\tau L_2 e^{L_1\tau}$ and the Lipschitz constant of g. □

Proposition 2.2. Let $\mathbf{P}$ be described by (1.1) and let $\int\mathscr{S}$ be the saturating integrator from (1.2). For every $x_0\in\mathbb{R}^n$, every $u_0\in U$, every $k\ge 0$ and every $r\in\mathbb{R}$ there exists a $\tau\in(0,\infty]$ such that the closed-loop system from Figure 1 has a unique state trajectory (x,u) defined on $[0,\tau)$, such that $x(0)=x_0$ and $u(0)=u_0$. If τ is finite and maximal (i.e., the state trajectory cannot be continued beyond τ) then $\limsup_{t\to\tau}\|x(t)\| = \infty$.

Proof. We use the notation from Lemma 2.1, in particular, $R>0$ is arbitrary and x_0, u_0 are fixed. The Lipschitz bound of T_τ, which we denote by L_T, can be chosen to be independent of $\tau\in(0,R/M]$ (simply choosing the maximum, which occurs at $\tau=R/M$). Let us denote by S_τ the input to output map of the saturating integrator on the time interval $[0,\tau]$. The estimate (2.1) shows that $S_\tau : C([0,\tau];\mathbb{R}) \mapsto C_\tau$ is Lipschitz continuous, with the Lipschitz bound τ. If (x,u) is a state trajectory of the closed-loop system which is defined on $[0,\tau]$, then we must have (see Figure 1)

$$u = S_\tau k(r - T_\tau u).$$

This can be regarded as a fixed point equation on C_τ. For τ sufficiently small so that $L_T\cdot k\cdot\tau<1$, the above equation has a unique solution according to the Banach fixed point theorem, see for instance Section 3 in Brooks and Schmitt (2009). It is easy to see that if u is a solution of the fixed point equation and x is the corresponding state trajectory of $\mathbf{P}$ starting from x_0, then (x,u) is the desired state trajectory of the closed-loop system on $[0,\tau]$. The τ that we have just found is not maximal, because if the solution exists on the closed interval $[0,\tau]$, then we can repeat the same argument starting at time τ, and we get a larger interval of existence of the state trajectory.

Using standard arguments, see for instance Exercise 3.26 in Khalil (2002), we get that if $\tau>0$ is finite and maximal, then $\limsup_{t\to\tau}\|x(t)\|=\infty$. □

Remark 2.3. An alternative way to prove existence (but not uniqueness) of the solution to the closed-loop system (1.3) is to use the tools from *differential inclusions* theory. In particular, (1.2) can be regarded as a *constrained differential inclusion* (see for example (5.4) in Goebel et al. (2012)), where the state u is constrained in the closed set U and the map $\mathscr{S}$ is replaced with the Krasovskii set-valued map

$$\mathscr{S}_K(u,w) = \begin{cases} \{w^+\} & u<u_{min},\\ [w,w^+] & u=u_{min},\\ \{w\} & u\in(u_{min},u_{max}),\\ [w^-,w] & u=u_{max},\\ \{w^-\} & u>u_{max}.\end{cases}$$

The set-valued map describing the closed-loop system is *outer semicontinuous* (see Definition 5.9 of Goebel et al. (2012)). Therefore, the closed-loop system (1.3) satisfies Assumption 6.5 of Goebel et al. (2012) and, according to Theorem 6.30 of Goebel et al. (2012), it is *well-posed.*

3. THE CLOSED-LOOP SYSTEM AS A STANDARD SINGULAR PERTURBATION MODEL

The idea is to regard the constant gain $k>0$ from the closed-loop system of Figure 1 as a "sufficiently small" parameter such that the closed-loop system (1.3) can be rewritten as a standard singular perturbation model (see Chapter 1 of Kokotović et al. (1999) or Chapter 11 of Khalil (2002)). Then, we can separately analyze the stability of the reduced (slow) model and of the boundary-layer (fast) system, and, using singular perturbation theory, obtain stability results for (1.3). The stability analysis is performed following the approach of Khalil (2002) and Kokotović et al. (1999). Note that in the less general formulation in Section 11.5 of Khalil (2002), the functions determining the closed-loop system are required to be locally Lipschitz, while in Chapter 7 of Kokotović et al. (1999) it is only required that a unique solution exists for the closed-loop system, which we have proved in Proposition 2.2.

The following assumption is common in the singular perturbation theory (see Chapter 11 of Khalil (2002), Kokotović et al. (1999)) and in the theory for nonlinear systems with slowly varying inputs (see Section 9.6 of Khalil (2002), Kelemen (1986), Lawrence and Rugh (1990)). Recall that $U = [u_{min}, u_{max}]$.

Assumption 1. There exists a function $\Xi \in C^1(U; \mathbb{R}^n)$ such that

$$f(\Xi(u), u) = 0 \qquad \forall\, u \in U, \tag{3.1}$$

i.e., for each $u_0 \in U$, $\Xi(u_0)$ is an equilibrium point that corresponds to the constant input u_0. Moreover, $\mathbf{P}$ is *uniformly exponentially stable* around these equilibrium points. This means that there exist $\varepsilon_0 > 0$, $\lambda > 0$ and $m \geq 1$ such that for each constant input function $u_0 \in U$, the following holds:

$$\text{If } \|x(0) - \Xi(u_0)\| \leq \varepsilon_0, \text{ then for every } t \geq 0,$$
$$\|x(t) - \Xi(u_0)\| \leq m e^{-\lambda t} \|x(0) - \Xi(u_0)\|. \tag{3.2}$$

Remark 3.1. The uniform exponential stability condition above can be checked by linearization: If the Jacobian matrices

$$A(u_0) = \left. \frac{\partial f(x,u)}{\partial x} \right|_{\substack{x=\Xi(u_0)\\ u=u_0}} \in \mathbb{R}^{n\times n}$$

have eigenvalues bounded away from the right half-plane,

$$\max \operatorname{Re} \sigma(A(u_0)) \leq \lambda_0 < 0 \qquad \forall\, u_0 \in U,$$

then $\mathbf{P}$ is uniformly exponentially stable, see (11.16) in Khalil (2002). From the first part of Assumption 1, $\max \operatorname{Re} \sigma(A(u_0))$ is a continuous function of u_0. Hence, if this function is always negative, then by the compactness of U, its maximum is also negative. Thus, for the uniform exponential stability of $\mathbf{P}$ we only have to check that each of the matrices $A(u_0)$ is stable (where $u_0 \in U$).

Remark 3.2. If there exists a function Ξ with the properties (3.1) and (3.2), then automatically $\Xi \in C^1$ according to the implicit function theorem since $f \in C^2$. The stability property (3.2) guarantees that all the eigenvalues of the Jacobians are in the open left half-plane, as discussed in Remark 3.1.

Assumption 2. The system $\mathbf{P}$ satisfies Assumption 1 and, moreover, the function

$$G(u) := g(\Xi(u)) \qquad \forall\, u \in U,$$

is monotone increasing, i.e., there exists $\mu > 0$ such that

$$G'(u) \geq \mu \qquad \forall\, u \in U.$$

Note that the function $G \in C^1(U, \mathbb{R})$, since it is defined as the composition of two class C^1 functions ($G := g \circ \Xi$). We denote $y_{min} := G(u_{min})$ and $y_{max} := G(u_{max})$, where clearly $y_{min} < y_{max}$. Moreover, using the notation $Y = (y_{min}, y_{max})$, for any $r \in Y$, we define $u_r := G^{-1}(r)$, which is well-defined in U since G strictly monotone.

In the following, we manipulate the closed-loop equation (1.3) to rewrite it as a standard singular perturbation model (see (11.33), (11.34) in Khalil (2002)). We define

$$h(x, u) := \mathscr{S}(u, r - g(x)), \tag{3.3}$$

and then the closed-loop system (1.3) can be rewritten as

$$\dot{u} = k \cdot h(x, u), \qquad \dot{x} = f(x, u). \tag{3.4}$$

Using that $k > 0$, we change the *time-scale* of (3.4) introducing $s := k \cdot t$ (it is a "slower" time-scale because k is small). Rewriting the system (3.4) in the new time-scale s, we get

$$\frac{\mathrm{d}u}{\mathrm{d}s} = h(x, u), \qquad k\frac{\mathrm{d}x}{\mathrm{d}s} = f(x, u). \tag{3.5}$$

Table 1. Correspondence of our notation with the one used in Khalil (2002)

this paper	$\tilde{u}$	$\tilde{x}$	$\tilde{h}$	$\tilde{f}$	k	z	$\tilde{\Xi}$	s	t
Khalil (2002)	x	z	f	g	ε	y	h	t	τ

To simplify the stability analysis, we move the equilibrium point of the closed-loop system (3.5) to the origin. It follows from Assumption 1 and the notation introduced after Assumption 2 that the equilibrium point of (3.5) is (x_r, u_r), where $x_r := \Xi(u_r)$. Introducing the variables

$$\tilde{x} := x - x_r, \qquad \tilde{u} := u - u_r, \tag{3.6}$$

and the functions

$$\tilde{h}(\tilde{u}, \tilde{x}) := h(\tilde{x} + x_r, \tilde{u} + u_r), \tag{3.7a}$$
$$\tilde{f}(\tilde{u}, \tilde{x}) := f(\tilde{x} + x_r, \tilde{u} + u_r), \tag{3.7b}$$

the system (3.5) can be rewritten as

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}s} = \tilde{h}(\tilde{u}, \tilde{x}), \qquad k\frac{\mathrm{d}\tilde{x}}{\mathrm{d}s} = \tilde{f}(\tilde{u}, \tilde{x}). \tag{3.8}$$

For $k > 0$ is small, this is a standard singular perturbation model according to Section 11.5 of Khalil (2002) (see equations (11.33) and (11.34) there). Recalling Ξ from Assumption 1, we introduce the function

$$\tilde{\Xi}(\tilde{u}) := \Xi(\tilde{u} + u_r) - x_r \tag{3.9}$$

and following the guidelines of Khalil (2002), we define

$$z := \tilde{x} - \tilde{\Xi}(\tilde{u}). \tag{3.10}$$

Using the notation introduced above, we reformulate our (3.5) like (11.35), (11.36) of Khalil (2002), i.e.,

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}s} = \tilde{h}(\tilde{u}, z + \tilde{\Xi}(\tilde{u})), \tag{3.11a}$$

$$k\frac{\mathrm{d}z}{\mathrm{d}s} = \tilde{f}(\tilde{u}, z + \tilde{\Xi}(\tilde{u})) - k\frac{\partial \tilde{\Xi}}{\partial \tilde{u}}\tilde{h}(\tilde{u}, z + \tilde{\Xi}(\tilde{u})), \tag{3.11b}$$

which has an equilibrium point at $(\tilde{u}, z) = (0, 0)$. Finally, in accordance with the change of variables (3.6), we define

$$\tilde{u}_{min} := u_{min} - u_r\,, \qquad \tilde{u}_{max} := u_{max} - u_r$$

and the set $\tilde{U} := [\tilde{u}_{min}, \tilde{u}_{max}] \subset \mathbb{R}$, which contains the origin. As a consequence, the state space of the closed-loop system (3.11a), (3.11b) is $\tilde{X} := \tilde{U} \times \mathbb{R}^n$. To facilitate the comparison of our equations with those in Khalil (2002), the relation between the notation in these two works is shown in Table 1.

Remark 3.3. Since $\tilde{\Xi} \in C^1(\tilde{U}, \mathbb{R}^n)$, the change of variables (3.10) is stability preserving, i.e., the origin of (3.11a), (3.11b) is asymptotically (exponentially) stable, if and only if the origin of (3.8) is asymptotically (exponentially) stable.

3.1 Formulation of the reduced (slow) model

The *reduced (slow) model* is obtained setting $k = 0$ in (3.11b), and solving the resulting algebraic equation

$$0 = \tilde{f}(\tilde{u}, z + \tilde{\Xi}(\tilde{u})),$$

which yields $z = 0$. Substituting $z = 0$ in (3.11a), the following reduced (slow) model is obtained:

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}s} = \tilde{h}(\tilde{u}, \tilde{\Xi}(\tilde{u})) \tag{3.12}$$

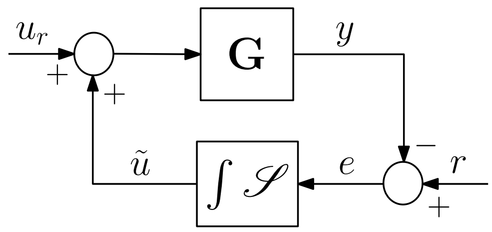

Fig. 2. Closed-loop representation of the reduced (slow) model (3.13).

Note that using the definition of h from (3.3) and the definition of G from Assumption 2, the reduced (slow) model (3.12) can be described equivalently by

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}s} = \mathscr{S}(\tilde{u}+u_r, r - G(\tilde{u}+u_r)). \qquad (3.13)$$

The reduced closed-loop system is shown in Figure 2.

3.2 Formulation of the boundary-layer (fast) system

The *boundary-layer (fast) system* is obtained by rewriting the second equation in (3.8) in the original fast time-scale $t = \frac{s}{k}$ as

$$\dot{z} = \tilde{f}(\tilde{u}, z + \tilde{\Xi}(\tilde{u})), \qquad (3.14)$$

where $\tilde{u} \in \tilde{U}$ is treated as a fixed parameter.

4. STABILITY ANALYSIS - SINGULAR PERTURBATION ANALYSIS

In order to perform the stability analysis of the closed-loop system (3.11a), (3.11b), we want to use Theorem 2.1 from Kokotović et al. (1999), or, equivalently, Theorem 11.3 from Khalil (2002). As already mentioned at the beginning of Section 3, the smoothness assumptions in Khalil (2002) are more stringent than those in Kokotović et al. (1999) and our system fits only into the latter's framework. However, the proofs of the two aforementioned theorems are the same.

The analysis will be performed following the guidelines in Section 11.5 of Khalil (2002), by first finding two distinct Lyapunov functions V and W, respectively for the reduced (slow) model (3.12) and the boundary-layer (fast) system (3.14). Then, the *interconnection conditions* (see Section 11.5 of Khalil (2002)) are verified, proving the stability of the closed-loop system. (A similar approach was taken in the recent paper Weiss et al. (2019).)

4.1 Stability of the reduced (slow) model

We have to find a Lyapunov function $V(\tilde{u})$ for (3.12) such that (11.39) in Khalil (2002) holds, i.e.,

$$\frac{\mathrm{d}V}{\mathrm{d}s} = \frac{\partial V}{\partial \tilde{u}}\tilde{h}(\tilde{u}, \tilde{\Xi}(\tilde{u})) \leq -\alpha_1\psi_1^2(\tilde{u}), \qquad (4.1)$$

for all $\tilde{u} \in \tilde{U}$, where $\alpha_1 > 0$ and ψ_1 is a positive definite function. Consider the candidate

$$V(\tilde{u}) = \tfrac{1}{2}\tilde{u}^2 \qquad \forall\, \tilde{u} \in \tilde{U}. \qquad (4.2)$$

It follows from (3.13) that

$$\frac{\mathrm{d}V}{\mathrm{d}s} = \tilde{u}\frac{\mathrm{d}\tilde{u}}{\mathrm{d}s} = \tilde{u}\mathscr{S}(\tilde{u}+u_r, r - G(\tilde{u}+u_r)).$$

Recalling that $\tilde{U} = [u_{min} - u_r, u_{max} - u_r]$ and $0 \in \tilde{U}$, if $\tilde{u}(s) = 0$, then the system (3.13) has reached its equilibrium point (recall $r = G(u_r)$). If $\tilde{u}(s) > 0$, then recalling that G is monotone increasing (see Assumption 2), it follows that

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}s} = G(u_r) - G(\tilde{u}(s)+u_r) < 0,$$

therefore

$$\frac{\mathrm{d}V}{\mathrm{d}s} = \tilde{u}\frac{\mathrm{d}\tilde{u}}{\mathrm{d}s} < 0\,.$$

If $\tilde{u}(s) < 0$, then by a similar argument, the same conclusion is obtained. This proves that (4.2) is indeed a Lyapunov function for (3.12), defined for all $\tilde{u} \in \tilde{U}$.

Remark 4.1. Note that the above argument tells us also that if $\tilde{u}(0) \in \tilde{U}$, then the saturating integrator $\int \mathscr{S}$ of the reduced model in Figure 2 is behaving like a standard integrator, since $\tilde{u}(s) \in (\tilde{u}_{min}, \tilde{u}_{max})$ for all $s \geq 0$.

According to the mean value theorem, there exists a $v \in (u_r, \tilde{u}+u_r)$ if $\tilde{u} > 0$ (or $v \in (\tilde{u}+u_r, u_r)$ if $\tilde{u} < 0$), such that $|G(\tilde{u}+u_r) - G(u_r)| = |G'(v)\tilde{u}| \geq \mu|\tilde{u}|$ (recall μ from Assumption 2), for all $\tilde{u} \in \tilde{U}$. Then, we can write

$$\frac{\mathrm{d}V}{\mathrm{d}s} = -\left|\tilde{u}[G(u_r) - G(\tilde{u}+u_r)]\right| \leq -|\tilde{u}|\mu|\tilde{u}| = -\mu\tilde{u}^2,$$

for all $\tilde{u} \in \tilde{U}$. Therefore, condition (4.1) holds with

$$\psi_1(\tilde{u}) = |\tilde{u}| \quad \text{and} \quad \alpha_1 = \mu. \qquad (4.3)$$

4.2 Stability of the boundary-layer (fast) system

We have to find a Lyapunov function $W(\tilde{u}, z)$ for (3.14) such that (11.40) and (11.41) in Khalil (2002) hold, i.e.,

$$\begin{aligned} \frac{\mathrm{d}W}{\mathrm{d}t} &= \frac{\partial W}{\partial z}\tilde{f}(\tilde{u}, z + \tilde{\Xi}(\tilde{u})) \leq -\alpha_2\psi_2^2(z), \\ W_1(z) &\leq W(\tilde{u}, z) \leq W_2(z), \end{aligned} \qquad (4.4)$$

for all $(\tilde{u}, z) \in \tilde{U} \times B_{\varepsilon_0}$ (recall ε_0 from Assumption 1), where $\alpha_2 > 0$ and ψ_2, W_1, W_2 are positive definite continuous functions. We use Lemma 9.8 of Khalil (2002), which, under Assumption 1 and some smoothness requirements on $\tilde{f}$, guarantees the existence of W such that the conditions in (4.4) hold, together with additional properties. The aforementioned requirements on $\tilde{f}$ are: Denoting $p(z, \tilde{u}) := \tilde{f}(\tilde{u}, z + \tilde{\Xi}(\tilde{u}))$ (this function is denoted by g in Lemma 9.8 of Khalil (2002)), it should hold that

$$\left\|\frac{\partial p}{\partial z}(z, \tilde{u})\right\| \leq L_1 \quad \text{and} \quad \left\|\frac{\partial p}{\partial \tilde{u}}(z, \tilde{u})\right\| \leq L_2\|z\|, \qquad (4.5)$$

for all $(z, \tilde{u}) \in B_r \times \tilde{U}$ (where $\varepsilon_0 \leq r < \infty$). The first condition follows from $p \in C^2$ and $B_r \times \tilde{U}$ being compact. For the second condition, for any $j \in \{1, 2, \ldots n\}$, we define $F_j(z, \tilde{u}) := \frac{\partial p_j}{\partial \tilde{u}}(z, \tilde{u}) \in \mathbb{R}$. Since $\tilde{f}(\tilde{u}, \tilde{\Xi}(\tilde{u})) = 0$ for all $\tilde{u} \in \tilde{U}$, then $F_j(0, \tilde{u}) = 0$ for all $\tilde{u} \in \tilde{U}$. For every fixed $(z, \tilde{u}) \in B_r \times \tilde{U}$, we introduce the function $\tilde{F}_j : [0, 1] \to \mathbb{R}$ such that $\tilde{F}_j(\sigma) = F_j(\sigma z, \tilde{u})$, hence $\tilde{F}_j(0) = 0$ and $\tilde{F}_j(1) = F_j(z, \tilde{u})$. According to the mean value theorem, there exists $\xi \in (0, 1)$ such that $F_j(z, \tilde{u}) = \tilde{F}_j(1) - \tilde{F}_j(0) = \tilde{F}_j'(\xi)$. Therefore, we get that for all $(z, \tilde{u}) \in B_r \times \tilde{U}$

$$|F_j(z, \tilde{u})| \leq \left\|\frac{\partial F_j}{\partial z}(\xi z, \tilde{u})\right\| \cdot \|z\| = \left\|\frac{\partial p_j}{\partial z \partial \tilde{u}}(\xi z, \tilde{u})\right\| \cdot \|z\|,$$

which holds because $p \in C^2$ and $B_r \times \tilde{U}$ is compact. Recall that $F_j(z, \tilde{u})$ is the j-th component of $\frac{\partial p}{\partial \tilde{u}}(z, \tilde{u})$.

Thus, the above inequality implies the second estimate in (4.5). Therefore, we can apply Lemma 9.8 of Khalil (2002), which yields the existence of W such that

$$c_1\|z\|^2 \le W(\tilde{u},z) \le c_2\|z\|^2, \tag{4.6a}$$

$$\frac{\partial W}{\partial z}\tilde{f}(\tilde{u},z+\tilde{\Xi}(\tilde{u})) \le -c_3\|z\|^2, \tag{4.6b}$$

$$\left\|\frac{\partial W}{\partial z}\right\| \le c_4\|z\|, \quad \left\|\frac{\partial W}{\partial \tilde{u}}\right\| \le c_5\|z\|^2, \tag{4.6c}$$

for all $(\tilde{u},z) \in \tilde{U} \times B_{\varepsilon_0}$, where c_i, $i = 1, \dots 5$ are positive constants. It follows that the conditions (4.4) (and more) are satisfied with

$$\psi_2(z) = \|z\|, \quad \alpha_2 = c_3, \quad W_1(z) = c_1\|z\|^2 \quad \text{and} \quad W_2(z) = c_2\|z\|^2. \tag{4.7}$$

4.3 Interconnection conditions

The interconnection conditions to be verified are (11.43) and (11.44) in Khalil (2002), i.e.,

$$\frac{\partial V}{\partial \tilde{u}}[\tilde{h}(\tilde{u},z+\tilde{\Xi}(\tilde{u}))-\tilde{h}(\tilde{u},\tilde{\Xi}(\tilde{u}))] \le \beta_1\psi_1(\tilde{u})\psi_2(z), \tag{4.8}$$

$$\left[\frac{\partial W}{\partial \tilde{u}} - \frac{\partial W}{\partial z}\frac{\partial \tilde{\Xi}}{\partial \tilde{u}}\right]\tilde{h}(\tilde{u},z+\tilde{\Xi}(\tilde{u})) \le \beta_2\psi_1(\tilde{u})\psi_2(z) + \gamma\psi_2^2(z), \tag{4.9}$$

for some constants β_1,β_2,γ and for all $(\tilde{u},z) \in \tilde{U} \times B_{\varepsilon_0}$. Substituting V from (4.2), $\tilde{\Xi}$ from (3.9), $\tilde{h}$ from (3.7a) and h from (3.3), the left-hand side of (4.8) reduces to
$\tilde{u}[\mathscr{S}(\tilde{u}+u_r, r-g(z+\Xi(\tilde{u}+u_r)))-\mathscr{S}(\tilde{u}+u_r, r-g(\Xi(\tilde{u}+u_r)))]$.
The function $\mathscr{S}$ is Lipschitz in the second argument with constant 1, i.e., $|\mathscr{S}(u,e_1) - \mathscr{S}(u,e_2)| \le |e_1 - e_2|$, for all $u, e_1, e_2 \in \mathbb{R}$. Therefore, we can write

$$\begin{aligned}&\tilde{u}[\mathscr{S}(\tilde{u}+u_r, r-g(z+\Xi(\tilde{u}+u_r))) \\ &\qquad - \mathscr{S}(\tilde{u}+u_r, r-g(\Xi(\tilde{u}+u_r)))] \\ &\le |\tilde{u}|\left|g(z+\Xi(\tilde{u}+u_r)) - g(\Xi(\tilde{u}+u_r))\right| \le L_g|\tilde{u}|\|z\|,\end{aligned}$$

where L_g, the Lipschitz constant of g is well-defined since $g \in C^1$ and $B_{\varepsilon_0} \subset \mathbb{R}^n$ is compact. Hence, choosing

$$\beta_1 = L_g, \tag{4.10}$$

the interconnection condition (4.8) holds. For what concerns (4.9), since $\mathscr{S}(u,0) = 0$ for all $u \in \mathbb{R}$ and $\mathscr{S}$ is Lipschitz in the second argument, the following holds:

$$\begin{aligned}&|\tilde{h}(\tilde{u},z+\tilde{\Xi}(\tilde{u}))| \\ &= \left|\mathscr{S}(\tilde{u}+u_r, r-g(z+\Xi(\tilde{u}+u_r))) - \mathscr{S}(\tilde{u}+u_r,0)\right| \\ &\le |r-g(z+\Xi(\tilde{u}+u_r))|.\end{aligned} \tag{4.11}$$

Denoting $q(\tilde{u},z) := r - g(z+\Xi(\tilde{u}+u_r))$, it is clear that $q \in C^1$ and $q(0,0) = 0$. It follows that there exists $C_q > 0$ such that

$$|q(\tilde{u},z)| \le C_q\left\|\begin{matrix}\tilde{u}\\ z\end{matrix}\right\| \le C_q(|\tilde{u}| + \|z\|), \tag{4.12}$$

for all $(\tilde{u},z) \in \tilde{U} \times B_{\varepsilon_0}$. On the other hand, from (4.6c), it follows that:

$$\begin{aligned}\left|\frac{\partial W}{\partial \tilde{u}} - \frac{\partial W}{\partial z}\frac{\partial \tilde{\Xi}}{\partial \tilde{u}}\right| &\le \left(c_5\|z\|^2 + c_4\|z\|\left\|\frac{\partial \tilde{\Xi}}{\partial \tilde{u}}\right\|\right) \\ &\le C_W\|z\|,\end{aligned} \tag{4.13}$$

for all $(\tilde{u},z) \in \tilde{U} \times B_{\varepsilon_0}$, where $C_W > 0$. Therefore, using (4.11), (4.12) and (4.13) on the left hand side of (4.9), we get

$$\left[\frac{\partial W}{\partial \tilde{u}} - \frac{\partial W}{\partial z}\frac{\partial \tilde{\Xi}}{\partial \tilde{u}}\right]\tilde{h}(\tilde{u},z+\tilde{\Xi}(\tilde{u})) \le C_qC_W\|z\|(|\tilde{u}| + \|z\|),$$

for all $(\tilde{u},z) \in \tilde{U} \times B_{\varepsilon_0}$. Hence, choosing

$$\beta_2 = \gamma = C_qC_W, \tag{4.14}$$

the interconnection condition (4.9) holds.

4.4 Closed-loop stability analysis

We use the result from Theorem 11.3 of Khalil (2002) to prove that the equilibrium point $(\Xi(u_r), u_r)$ of the closed-loop system (1.3) is locally exponentially stable and, consequently, the output y of $\mathbf{P}$ converges to $r \in Y$ (recall that $Y = (y_{min}, y_{max})$ and $u_r = G^{-1}(r)$).

Theorem 4.2. Consider the closed-loop system (1.3), where $\mathbf{P}$ satisfies Assumption 2 and $r \in Y$. Then there exists a $\kappa^* > 0$ such that if the gain $k \in (0,\kappa^*)$, then $(\Xi(u_r), u_r)$ is a (locally) exponentially stable equilibrium point of the closed-loop system (1.3), with state space $X = \mathbb{R}^n \times U$. If the initial state $(x(0), u(0)) \in X$ of the closed-loop system satisfies $\|x(0) - \Xi(u(0))\| \le \varepsilon_0$, then

$$x(t) \to \Xi(u_r), \qquad u(t) \to u_r, \qquad y(t) \to r,$$

and this convergence is at an exponential rate.

Proof. Consider the singularly perturbed system (3.11a), (3.11b) with state space $\tilde{X} = \tilde{U} \times \mathbb{R}^n$, and recall the Lyapunov functions $V(\tilde{u})$ and $W(\tilde{u},z)$ that satisfy (4.1), (4.4), (4.8) and (4.9). According to Theorem 11.3 of Khalil (2002), it follows that the function defined as

$$\nu(\tilde{u},z) := (1-d)V(\tilde{u}) + dW(\tilde{u},z), \quad 0 < d < 1, \tag{4.15}$$

is a Lyapunov function for the closed-loop system (3.11a), (3.11b), for every $d \in (0,1)$, as long as

$$k < \kappa_d := \frac{\alpha_1\alpha_2}{\alpha_1\gamma + \frac{1}{4d(1-d)}[(1-d)\beta_1 + d\beta_2]^2},$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma$ are from (4.3), (4.7), (4.10), (4.14). In particular, the maximum value of κ_d occurs at $d^* := \frac{\beta_1}{\beta_1+\beta_2}$ and it is given by $\kappa^* := \frac{\alpha_1\alpha_2}{\alpha_1\gamma+\beta_1\beta_2}$. As a consequence, choosing $d = d^*$ in (4.15), the origin of the closed-loop system (3.11a), (3.11b) is asymptotically stable for all $k \in (0,\kappa^*)$. Moreover, since $V(\tilde{u})$ and $W(\tilde{u},z)$ are quadratic, the origin of the closed-loop system (3.11a), (3.11b) is exponentially stable (see Corollary 2.2 of Kokotović et al. (1999)). To investigate the region of attraction, we proceed as follows. From (4.2), it is clear that

$$V(\tilde{u}) \le \frac{1}{2}\max\{\tilde{u}_{min}^2, \tilde{u}_{max}^2\} \qquad \forall\, \tilde{u} \in \tilde{U},$$

and from (4.6a), it follows that

$$W(\tilde{u},z) \le c_2\varepsilon_0^2 \qquad \forall\, (\tilde{u},z) \in \tilde{U} \times B_{\varepsilon_0}. \tag{4.16}$$

We introduce the compact positively-invariant set

$$L := \left\{(\tilde{u},z) \in \tilde{U} \times \mathbb{R}^n \mid \nu(\tilde{u},z) \le (1-d^*)\frac{1}{2}\max\{\tilde{u}_{min}^2, \tilde{u}_{max}^2\} + d^*c_2\varepsilon_0^2\right\}. \tag{4.17}$$

It is clear from the above reasoning that $\tilde{U} \times B_{\varepsilon_0} \subset L$, and, therefore, that $\tilde{U} \times B_{\varepsilon_0}$ is contained in the domain of attraction of the origin of (3.11a), (3.11b) for all $k < \kappa^*$. Finally, the change of variable (3.10) is stability preserving (see Remark 3.3), hence the equilibrium point $(\Xi(u_r), u_r)$ of the system (1.3) is locally exponential stable with region of attraction containing all $(x(0), u(0)) \in \mathbb{R}^n \times U$ such that $\|x(0) - \Xi(u(0))\| \leq \varepsilon_0$. Since $y(t) = g(x(t))$ and g is a C^1 function, we have that $y(t)$ converges to $g(\Xi(u_r)) = G(u_r) = r$ at an exponential rate. □

Remark 4.3. A proportional block τ_p can be added in parallel to the saturating integrator, after the block k in the block diagram of Figure 1. In this way a feedback loop with a PI controller is obtained. The resulting closed-loop system is given by

$$\dot{x} = f(x, u + k\tau_p(r - g(x))), \quad \dot{u} = \mathscr{S}(u, k(r - g(x))),$$

where $k \in (0, \kappa^*)$ (κ^* from Theorem 4.2). Defining

$$\beta(x, u) := f(x, u + k\tau_p(r - g(x))) - f(x, u),$$

the above closed-loop system can be rewritten as

$$\dot{x} = f(x, u) + \beta(x, u), \quad \dot{u} = \mathscr{S}(u, k(r - g(x))). \quad (4.18)$$

The term $\beta(x, u)$ is vanishing at the equilibrium point, $(\beta(\Xi(u_r), u_r) = 0)$, therefore Lemma 9.1 from Khalil (2002) ("vanishing perturbations") can be applied with the Lyapunov function (4.15). It follows that if τ_p is sufficiently small, then the perturbed closed-loop system (4.18) is locally exponentially stable around the equilibrium point $(\Xi(u_r), u_r)$. The proportional term may improve the transient response, but the domain of attraction of the equilibrium point may shrink. We cannot prove the existence of a relatively large domain of attraction, as we did in Theorem 4.2 without the proportional block.

5. STABILITY ANALYSIS - DOMAIN OF ATTRACTION

Assumption 3. There exists $\kappa_0 > 0$ such that for any $k \in [0, \kappa_0]$, the closed-loop system formed by **P** and the saturating integrator, as shown in Figure 1, with any $r \in Y$, has a unique state trajectory on the interval $[0, \infty)$, for any initial state in X. Moreover, at any time $t \geq 0$, the state $(x(t), u(t))$ depends continuously on the initial state $(x(0), u(0))$.

The above assumption is not trivial, because the differential equations describing the closed-loop system are not continuous (the discontinuity is in $\mathscr{S}$). We note that the saturating integrator is irreversible (in time) and hence the closed-loop system usually has no uniquely defined backwards (in time) state trajectories.

Theorem 5.1. Assume that **P** satisfies Assumption 2 and the closed-loop system from Figure 1 satisfies Assumption 3. Let $r \in Y$, let $T > 0$ and let $X_T \subset X$ be a compact set such that if $(x_0, u_0) \in X_T$, then the state trajectory x of **P** starting from $x(0) = x_0$, with constant input u_0, satisfies

$$\|x(T) - \Xi(u_0)\| \leq \varepsilon_0 .$$

Then there exists a $\kappa_T^* \in (0, \kappa_0]$ such that for any $k \in (0, \kappa_T^*)$, if the initial state (x_0, u_0) of the closed-loop system is in X_T, then the state trajectory (x, u) of (1.3) satisfies

$$x(t) \to \Xi(u_r), \qquad u(t) \to u_r, \qquad y(t) \to r,$$

and this convergence is at an exponential rate.

Proof. Recall the changes of variables (3.6), (3.10) and the state space $\tilde{X} = \tilde{U} \times \mathbb{R}^n$ of the closed-loop system (3.11a), (3.11b). Define the set $\tilde{X}_T \subset \tilde{X}$ as the image of the set X_T through the change of variables just mentioned. Then for every $(\tilde{u}_0, z_0) \in \tilde{X}_T$, state trajectory z of the boundary-layer system (3.14) starting from $z(0) = z_0$, with fixed input $\tilde{u}_0$, satisfies $\|z(T)\| \leq \varepsilon_0$, i.e., z reaches B_{ε_0} in time T. Clearly $\tilde{X}_T$ is compact.

The convergence of z to B_{ε_0} can be regarded as an exponential one (since T is finite, there exist $\tilde{m}_T \geq 1$ and $\tilde{\lambda}_T > 0$ such that $\|z(t)\| \leq \tilde{m}_T e^{-\tilde{\lambda}_T t}\|z(0)\|$). Therefore, applying Lemma 9.8 of Khalil (2002), we get a Lyapunov function $W(\tilde{u}, z)$ for the boundary-layer system (3.14) such that (4.6a) to (4.6c) hold for all $(\tilde{u}, z) \in \tilde{X}_T$. Note that the constants c_i, $i = 1, \ldots 5$ in this case are different, since $\tilde{m}_T$ and $\tilde{\lambda}_T$ are, in general, different from m and λ of Assumption 1. From this point on, everything proceeds as in the proof of Theorem 4.2, substituting the inequality (4.16) with

$$W(\tilde{u}, z) \leq c_2 \max\{\|z\|^2 \mid z \in \Pi\tilde{X}_T\} \qquad \forall\, (\tilde{u}, z) \in \tilde{X}_T,$$

where Π denotes the projection onto the second component in the product $\tilde{U} \times \mathbb{R}^n$, and modifying the definition of L in (4.17) accordingly. □

Remark 5.2. The reason why we may call X_T a possibly "large" domain of attraction is the following: If **P** happens to be globally asymptotically stable (GAS) for every constant input $u_0 \in U$, then *every initial state* of the closed-loop system is contained in one of the sets X_T of the form

$$X_T = \{(x_0, u_0) \in X \mid \|x(T) - \Xi(u_0)\| \leq \varepsilon_0, \ \|x_0\| \leq T\},$$

if we choose T large enough. (In the above formula, $x(\cdot)$ denotes the state trajectory of **P**, starting from $x(0) = x_0$ and with constant input u_0.) If we choose a "region of interest" $\mathcal{K} \subset X$ that is compact, then there exists a $k > 0$ such that all the closed-loop state trajectories starting from $\mathcal{K}$ will converge to the unique equilibrium point. Indeed, the interiors of the sets X_T as defined above are an open covering of $\mathcal{K}$, so that $\mathcal{K} \subset X_T$ if T is large enough. Then we have to choose a gain $k \leq \kappa_T^*$. Of course, the price for choosing a very large T is that we may have to choose a very small gain k, and this may deteriorate the dynamic response of the closed-loop system.

A similar, but more complicated statement can be made for plants that are almost globally asymptotically stable for every constant input $u_0 \in U$, as encountered for instance in Barabanov et al. (2017) and Natarajan and Weiss (2018).

6. CONCLUSION

This paper presented an integral controller with anti-windup (saturating integrator), for a single-input single-output nonlinear plant **P**, based on singular perturbations theory. Under reasonable assumptions, it was proved that the closed-loop system from Figure 1 is able to track a constant reference signal r, while not allowing the input u to exit the range $U = [u_{min}, u_{max}]$. Moreover, it has been shown that the equilibrium point of the closed-loop system has a "large" region of attraction.

In the future, we plan to formulate a similar theory for multi-input multi-output (MIMO) nonlinear plants. In fact, this extension to MIMO plants would open a new range of possibilities, for instance in the stability analysis of electric power grids.

REFERENCES

Barabanov, N., Schiffer, J., Ortega, R., and Efimov, D. (2017). Conditions for almost global attractivity of a synchronous generator connected to an infinite bus. *IEEE Trans. on Automatic Control*, 62(10), 4905–4916.

Brooks, R. and Schmitt, K. (2009). The contraction mapping principle and some applications. *Electronic Journal of Differential Equations*, 9.

Choi, J. and Lee, S. (2009). Antiwindup strategy for PI-type speed controller. *IEEE Trans. on Industrial Electronics*, 56(6), 2039–2046.

da Silva , J.M.G., Longhi, M.B., and Oliveira, M.Z. (2016). Dynamic anti-windup design for a class of nonlinear systems. *Int. Journal of Control*, 89(12), 2406–2419.

Desoer, C. and Lin C.A. (1985). Tracking and disturbance rejection of MIMO nonlinear systems with PI controller. *IEEE Trans. on Automatic Control*, 30(9), 861–867.

Edwards, C. and Postlethwaite, I. (1998). Anti-windup and bumpless-transfer schemes. *Automatica*, 34(2), 199 – 210.

Ferguson, J., Donaire, A., Ortega, R., and Middleton, R.H. (2018). Robust integral action of port-Hamiltonian systems. *IFAC-PapersOnLine*, 51(3), 181 – 186. 6th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control LHMNC 2018.

Fliegner, T., Logemann, H., and Ryan, E.P. (2003). Low-gain integral control of continuous-time linear systems subject to input and output nonlinearities. *Automatica*, 39(3), 455–462.

Goebel, R., Sanfelice, R.G., and Teel, A.R. (2012). *Hybrid Dynamical Systems: Modeling, Stability, and Robustness.* Princeton University Press.

Hanus, R., Kinnaert, M., and Henrotte, J.L. (1987). Conditioning technique, a general anti-windup and bumpless transfer method. *Automatica*, 23(6), 729–739.

Herrmann, G., Menon, P.P., Turner, M.C., Bates, D.G., and Postlethwaite, I. (2010). Anti-windup synthesis for nonlinear dynamic inversion control schemes. *Int. Journal of Robust and Nonlinear Control*, 20(13), 1465–1482.

Isidori, A. (1997). A remark on the problem of semiglobal nonlinear output regulation. *IEEE Trans. on Automatic Control*, 42(12), 1734–1738.

Isidori, A. and Byrnes, C.I. (1990). Output regulation of nonlinear systems. *IEEE Trans. on Automatic Control*, 35(2), 131–140.

Kelemen, M. (1986). A stability property. *IEEE Trans. on Automatic Control*, 31(8), 766–768.

Khalil, H.K. (2000). Universal integral controllers for minimum-phase nonlinear systems. *IEEE Trans. on Automatic Control*, 45(3), 490–494.

Khalil, H.K. (2002). *Nonlinear systems; 3rd ed.* Prentice-Hall, Upper Saddle River, NJ.

Kokotović, P., Khalil, H., and O'Reilly, J. (1999). *Singular Perturbation Methods in Control: Analysis and Design.* Society for Industrial and Applied Mathematics.

Konstantopoulos, G.C., Zhong, Q.C., Ren, B., and Krstic, M. (2016). Bounded integral control of input-to-state practically stable nonlinear systems to guarantee closed-loop stability. *IEEE Trans. on Automatic Control*, 61(12), 4196–4202.

Kothare, M.V., Campo, P.J., Morari, M., and Nett, C.N. (1994). A unified framework for the study of anti-windup designs. *Automatica*, 30(12), 1869–1883.

Lawrence, D.A. and Rugh, W.J. (1990). On a stability theorem for nonlinear systems with slowly varying inputs. *IEEE Trans. on Automatic Control*, 35(7), 860–864.

Morabito, F., Teel, A.R., and Zaccarian, L. (2004). Nonlinear antiwindup applied to Euler-Lagrange systems. *IEEE Trans. on Robotics and Automation*, 20(3), 526–537.

Morari, M. (1985). Robust stability of systems with integral control. *IEEE Trans. on Automatic Control*, 30(6), 574–577.

Natarajan, V. and Weiss, G. (2017). Synchronverters with better stability due to virtual inductors, virtual capacitors, and anti-windup. *IEEE Trans. on Industrial Electronics*, 64(7), 5994–6004.

Natarajan, V. and Weiss, G. (2018). Almost global asymptotic stability of a grid-connected synchronous generator. *Mathematics of Control, Signals, and Systems*, 30(2), 10.

Peng, Y. Vrancic, D., and Hanus, R. (1996). Anti-windup, bumpless, and conditioned transfer techniques for PID controllers. *IEEE Control Systems Magazine*, 16(4), 48–57.

Rehan, M., Khan, A.Q., Abid, M., Iqbal, N., and Hussain, B. (2013). Anti-windup-based dynamic controller synthesis for nonlinear systems under input saturation. *Applied Mathematics and Computation*, 220, 382 – 393.

Seshagiri, S. and Khalil, H.K. (2005). Robust output feedback regulation of minimum-phase nonlinear systems using conditional integrators. *Automatica*, 41(1), 43 – 54.

Singh, A. and Khalil, H.K. (2005). Regulation of nonlinear systems using conditional integrators. *Int. Journal of Robust and Nonlinear Control*, 15(8), 339–362.

Tarbouriech, S. and Turner, M. (2009). Anti-windup design: an overview of some recent advances and open problems. *IET control theory & applications*, 3(1), 1–19.

Turner, M.C., Herrmann, G., and Postlethwaite, I. (2007). Incorporating robustness requirements into antiwindup design. *IEEE Trans. on Automatic Control*, 52(10), 1842–1855.

Weiss, G. and Natarajan, V. (2016). Stability of the integral control of stable nonlinear systems. In *2016 IEEE Int. Conference on the Science of Electrical Engineering (ICSEE)*, 1–5.

Weiss, G., Dörfler, F., and Levron, Y. (2019). A stability theorem for networks containing synchronous generators. *Systems & Control Letters*, 134, 104561.

Wu, F. and Lu, B. (2004). Anti-windup control design for exponentially unstable LTI systems with actuator saturation. *Systems & Control Letters*, 52(3), 305 – 322.

Zaccarian, L. and Teel, A.R. (2002). A common framework for anti-windup, bumpless transfer and reliable designs. *Automatica*, 38(10), 1735 – 1744.