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Distributed Finite-Time Coordination Control for Networked Euler-Lagrange Systems Under Directed Graphs ⋆

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Abstract: The distributed coordination problems for networked Euler-Lagrange systems are investigated in this paper, where both the distributed synchronization control and the distributed containment control are considered. Compared with the existing traditional asymptotically stable control laws, the desired cooperative control objectives of this paper can be realized in finite time, and the estimate of the settling times are explicitly provided. Another distinct feature of our work is that the communication interactions between neighboring agents are unidirectional, which is more practical in real applications. Finally, some simulation results are shown to validate the feasibility of the theoretical schemes.

Keywords: Distributed finite-time control, synchronization control, containment control, Euler-Lagrange system, directed graph.

1. INTRODUCTION

Recently, distributed finite-time coordination control has become an interesting research topic due to the growing demands on the fast convergence rate of the networked systems. Unlike traditional asymptotic stability, finite-time convergence promises advantages in disturbance rejection and robustness against uncertainties.

Driven by the fact that a large class of dynamical systems involving robotic manipulators, marine vehicles, spacecraft, and so forth, can be modeled by the Euler-Lagrange systems, much attention has been paid on the cooperative control for networked Euler-Lagrange systems. The asymptotically convergent distributed synchronization, tracking and containment control schemes were developed in Mei et al. (2013); Zhao et al. (2015); Xu et al. (2019), Lu et al. (2019); Lu and Liu (2019) and Mei et al. (2012); Cheng et al. (2017), respectively. When satisfactory coordination control performance with finite-time convergence is considered, distributed finite-time tracking problems with unknown bounds of the model uncertainties and external disturbances were addressed in Chen et al. (2013). In Zhao et al. (2015), distributed finite-time tracking control in the absence of relative velocity measurements was solved. By using the back-stepping approach, the distributed tracking control was also investigated in He et al. (2018). It is worth noting that the communication networks between adjacent followers in Chen et al. (2013); Zhao et al. (2015); He et al. (2018) are undirected.

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Motivated by the discussions mentioned above, this paper aims at dealing with the distributed synchronization and containment control problems for multiple Euler-Lagrange systems, where the communication graphs are presumed to be directed. The main contributions of our work are two-fold. First, different from the asymptotically stable control strategies proposed in Mei et al. (2013); Zhao et al. (2015); Xu et al. (2019); Lu et al. (2019); Lu and Liu (2019); Mei et al. (2012); Cheng et al. (2017), the control objectives of this paper can be achieved in finite time, and the settling times are explicitly provided. Second, the information flows in the communication topologies are unidirectional as compared with Chen et al. (2013); Zhao et al. (2015); He et al. (2018).

The rest of this paper is arranged as follows. The preliminaries are presented in Section 2. Distributed finite-time synchronization control and containment control are discussed in Section 3 and Section 4, respectively. Simulation examples are provided in Section 5 and conclusions are drawn in Section 6.

2. PRELIMINARIES

2.1 Notations

Given a constant $\alpha \in \mathbb{R}$ and a vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, the symbol $x^{[\alpha]}$ is defined as $x^{[\alpha]} = \mathrm{sig}(x)^\alpha = (\mathrm{sgn}(x_1)|x_1|^\alpha, \ldots, \mathrm{sgn}(x_n)|x_n|^\alpha)^T$ with $\mathrm{sgn}(x_i)$, $i = 1, \ldots, n$, being a sign function. The notation $\mathrm{diag}\{x_i\}$ is denoted as a diagonal matrix with the elements $x_i, i = 1, \ldots, n$, on its diagonal. The minimum eigenvalue of a real symmetric matrix A is represented by $\lambda_m(A)$. The convex hull of the set $X = \{x_1, \ldots, x_n\}$ is defined as $Co(X) = \{\sum_{i=1}^n \alpha_i x_i | \alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1, x_i \in X\}$.

2.2 Euler-Lagrange system

The Euler-Lagrange equality is introduced in the following:

$$H_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + g_i(q_i) = \tau_i, \tag{1}$$

with $q_i, \dot{q}_i, \ddot{q}_i \in \mathbb{R}^p$ being the generalized position, velocity and acceleration vectors, respectively. The notations $H_i(q_i) \in \mathbb{R}^{p\times p}$, $C_i(q_i, \dot{q}_i) \in \mathbb{R}^{p\times p}$, $g_i(q_i) \in \mathbb{R}^p$ and $\tau_i \in \mathbb{R}^p$ represent the symmetric positive definite inertia matrix, the Coriolis and centrifugal torque matrix, the gravitational torque vector and the control torque vector, respectively.

2.3 Graph theory

Consider a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with the node set $\mathcal{V} = (v_1, \ldots, v_n)$, the edge set $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ and the adjacency matrix $\mathcal{A} = [a_{ij}]_{n\times n}$. An edge (v_i, v_j) in $\mathcal{G}$ means that the information of v_i can be received by v_j, but not vice versa. The adjacency matrix $\mathcal{A} = [a_{ij}]_{n\times n}$ is defined such that $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix $L = [l_{ij}]_{n\times n}$ associated with $\mathcal{A}$ can be defined as $l_{ii} = \sum_{j=1,j\neq i}^{n} a_{ij}$ and $l_{ij} = -a_{ij}, j \neq i, i, j = 1, \ldots, n$. The directed graph is strongly connected if there exists a directed path between every pair of distinct nodes. A directed graph contains a directed spanning tree if there exist a root node as well as directed paths from the root node to all other nodes.

2.4 Some basic lemmas

Lemma 1. (Mei et al. (2012)) The matrix $A \in \mathbb{R}^{n\times n}$ is a nonsingular M-matrix if and only if A^{-1} exists and each entry of A^{-1} is non-negative. Additionally, there exists a diagonal matrix $D = \text{diag}(d_i)$ with $d_i > 0$, $i = 1, \ldots, n$, such that $DA + A^T D$ is a symmetric positive definite matrix.

Lemma 2. (Meng et al. (2010)) For $x \in \mathbb{R}$, $\alpha \in \mathbb{R}_{>0}$, it holds that $\frac{\text{d}|x|^{\alpha+1}}{\text{d}t} = (\alpha + 1)\text{sig}(x)^\alpha \dot{x}$ and $\frac{\text{d}x^{[\alpha+1]}}{\text{d}t} = (\alpha + 1)|x|^\alpha \dot{x}$.

Lemma 3. (Meng et al. (2010)) Let $0 < \alpha \leq 1$ and $x_1, \ldots, x_p \geq 0$. Then, $\sum_{i=1}^{p}(x_i^\alpha) \geq (\sum_{i=1}^{p} x_i)^\alpha$.

Lemma 4. (Meng et al. (2010)) Consider a continuous system $\dot{x} = f(x, t)$ with $f(0, t) = 0$. Suppose that there exist $0 < \alpha < 1$, $\beta > 0$, $c > 0$ and a radially unbounded positive definite continuous function $V(x, t)$ such that $\dot{V}(x, t) + cV(x, t) + \beta V(x, t)^\alpha \leq 0$ (or $\dot{V}(x, t) + cV(x, t)^\alpha \leq 0$). Then, $V(x, t)$ converge to zero in finite time and the settling time is bounded by $T \leq \frac{1}{c(1-\alpha)} \ln \frac{cV(x(t_0),t_0)^{1-\alpha}+\beta}{\beta}$ (or $T \leq \frac{V(x(t_0),t_0)^{1-\alpha}}{c(1-\alpha)}$).

3. DISTRIBUTED FINITE-TIME SYNCHRONIZATION CONTROL

3.1 Problem formulation

The finite-time distributed synchronization control is investigated in this section. Suppose that there exist n agents with the following communication graphs:

Assumption 1. The communication topology among all agents is directed that contains a directed spanning tree.

Lemma 5. (Mei et al. (2013)) Suppose that Assumption 1 holds. By adjusting the order of the agents, the Laplacian matrix L can be expressed by the Frobenius normal form:

$$L = \begin{bmatrix} L_{11} & 0 & \cdots & 0 \\ L_{21} & L_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{k1} & L_{k2} & \cdots & L_{kk} \end{bmatrix},$$

where $1 \leq k \leq n$, $L_{ii} \in \mathbb{R}^{r_i\times r_i}$ is either a scalar or a square irreducible matrix and $\sum_{i=1}^{k} r_i = n$. The communication network is strong connected when $k = 1$. If $k > 1$, the subgraph associated with L_{11} is strongly connected and L_{ii}, $i = 2, \ldots, k$ are nonsingular M-matrices.

Lemma 6. (Wang et al. (2018)) If the communication graph $\mathcal{G}$ is directed strongly connected, there exists an eigenvector $\xi = (\xi_1, \ldots, \xi_n)^T$ corresponding to the zero eigenvalue of L, where $\xi_i > 0, i = 1, \ldots, n$. For any fixed $e > 0$, $\varepsilon > 0$ and $\delta > 0$ satisfying $\varepsilon \neq \delta$, denote $\Delta(e, \varepsilon, \delta) = \{z : \exists\rho \perp \xi, \gamma \geq 0,$ such that $z = e\rho^{[\varepsilon]} + \gamma\rho^{[\delta]}$, and $\|z\| = 1\}$. Then, it holds that $\inf_{z\in\Delta(e,\varepsilon,\delta)} z^T \hat{L} z = \chi > 0$, where $\hat{L} = \frac{1}{2}(\Xi L + L^T \Xi)$ with $\Xi = \text{diag}\{\xi_i\}$, $i = 1, \ldots, n$.

In this section, we aims at designing control protocols such that $q_i(t) \to q_j(t)$, $i, j = 1, \ldots, n$, in finite time.

3.2 Controller design and analysis

Some auxiliary variables are designed in the following:

$$\psi_i(t) = \sum_{j=1}^{n} a_{ij}(q_i(t) - q_j(t)), \tag{2}$$

$$\bar{s}_i(t) = \psi_i(t) + k_1\dot{q}_i^{[\alpha_1]}(t), \tag{3}$$

where the constants $k_1 > 0$ and $1 < \alpha_1 < 2$.

The distributed finite-time synchronization control law is proposed as follows:

$$\tau_i(t) = \bar{\tau}_{i_1}(t) + \bar{\tau}_{i_2}(t), \tag{4}$$

with

$$\bar{\tau}_{i_1}(t) = -\frac{H_i(q_i)}{k_1\alpha_1}\Omega^{-1}\dot{\psi}_i(t) + C_i(q_i, \dot{q}_i)\dot{q}_i(t) + g_i(t), \tag{5}$$

$$\bar{\tau}_{i_2}(t) = -\frac{H_i(q_i)}{k_1\alpha_1}(\bar{s}_i(t) + \bar{s}_i^{[\alpha_2]}(t)), \tag{6}$$

where $0 < \alpha_2 < 1$ and $\Omega = \text{diag}\{|\dot{q}_{i(\nu)}(t)|^{\alpha_1-1}\}$ with $\dot{q}_{i_{(\nu)}}(t)$ being the ν-th entry of $\dot{q}_i(t)$, $\nu = 1, \ldots, p$.

Theorem 1. Consider the networked Euler-Lagrange systems (1) with Assumptions 1, the synchronization can be achieved in finite time by using the distributed control strategies (4)-(6).

Proof 1. The time variable t will be omitted to facilitate the analysis. Substituting (4)-(6) into the Euler-Lagrange equality (1) yields that

$$\begin{aligned}\ddot{q}_i &= H_i^{-1}(q_i)[\bar{\tau}_{i_1} + \bar{\tau}_{i_2} - C_i(q_i, \dot{q}_i)\dot{q}_i - g_i(q_i)] \\ &= -\frac{1}{k_1\alpha_1}(\Omega^{-1}\dot{\psi}_i + \bar{s}_i + \bar{s}_i^{[\alpha_2]}).\end{aligned} \tag{7}$$

The Lyapunov function is chosen as

$$V_a = \frac{1}{2}\bar{s}_i^T\bar{s}_i. \tag{8}$$

Taking the derivative of V_a along the trajectories (3) and (7), one has

$$\begin{aligned}\dot{V}_a &= \bar{s}_i^T\dot{\bar{s}}_i = \bar{s}_i^T(\dot{\psi}_i(t) + k_1\alpha_1\Omega\ddot{q}_i)\\ &= -\bar{s}_i^T\Omega\bar{s}_i - \bar{s}_i^T\Omega\bar{s}_i^{[\alpha_2]},\end{aligned} \tag{9}$$

where Lemma 2 is applied. When $\dot{q}_i \neq 0$, it follows from Lemma 3 that $\bar{s}_i^T\Omega\bar{s}_i^{[\alpha_2]} \geq \lambda_m(\Omega)\sum_{\nu=1}^p|\bar{s}_{i_{(\nu)}}|^{\alpha_2+1} \geq \lambda_m(\Omega)(\sum_{\nu=1}^p|\bar{s}_{i_{(\nu)}}|^2)^{\frac{\alpha_2+1}{2}} \geq \lambda_m(\Omega)\|\bar{s}_i\|^{\alpha_2+1}$. Then, it can be further deduced from (9) that

$$\begin{aligned}\dot{V}_a &\leq -\lambda_m(\Omega)\|\bar{s}_i\|^2 - \lambda_m(\Omega)\|\bar{s}_i\|^{\alpha_2+1}\\ &= -2\lambda_m(\Omega)V_a - 2^{\frac{\alpha_2+1}{2}}\lambda_m(\Omega)V_a^{\frac{\alpha_2+1}{2}},\end{aligned} \tag{10}$$

where $\bar{\Omega} = \text{diag}\{\omega_\nu^{\alpha_1-1}\}$, $0 < \frac{\alpha_2+1}{2} < 1$ since $0 < \alpha_2 < 1$. It follows from Lemma 4 that $V_a \to 0$ in finite time, which indicates that $\bar{s}_i \to 0$ in finite time. Furthermore, the settling time T_1 is bounded by

$$T_1 \leq \frac{1}{\lambda_m(\Omega)(1-\alpha_2)}\ln(2^{\frac{1-\alpha_2}{2}}\lambda_m(\Omega)V_a(0)^{\frac{1-\alpha_2}{2}} + 1) \tag{11}$$

When $\dot{q}_i = 0$, it follows from (7) that $\ddot{q}_i = -\frac{1}{k_1\alpha_1}(\bar{s}_i + \bar{s}_i^{[\alpha_2]}) \neq 0$ if $\bar{s}_i \neq 0$. Also note that $\ddot{q}_i > 0$ and $\ddot{q}_i < 0$ for $\bar{s}_i < 0$ and $\bar{s}_i > 0$, respectively. Following the similar analysis as shown in Feng et al. (2002) and Yu et al. (2005), it is derived that the sliding mode $\bar{s}_i = 0$ can be reached in finite time that can be defined as T_2.

When $t > \max\{T_1, T_2\}$, it is easy to deduce from (3) that $\psi_i = -k_1\dot{q}_i^{[\alpha_1]}$, which is equivalent to

$$\dot{q}_i = -\bar{k}_1\psi_i^{[\bar{\alpha}_1]}, \tag{12}$$

where $\bar{k}_1 = \frac{1}{k_1^{\bar{\alpha}_1}} > 0$ and $\frac{1}{2} < \bar{\alpha}_1 = \frac{1}{\alpha_1} < 1$. In the following, this proof will be divided into three steps.

Step 1: Let $\Psi_1 = (\psi_1^T, \ldots, \psi_{r_1}^T)^T$, $Q_1 = (q_1^T, \ldots, q_{r_1}^T)^T$. It follows from (2) and (12) that

$$\Psi_1 = (L_{11} \otimes I_p)Q_1, \tag{13}$$
$$\dot{Q}_1 = -\bar{k}_1\Psi_1^{[\bar{\alpha}_1]}. \tag{14}$$

Note from Lemmas 5 and 6 that the subgraph associated with the matrix L_{11} is strongly connected and there exists an eigenvector $\xi_a = (\xi_1, \ldots, \xi_{r_1})^T$ corresponding to the eigenvalue zero with $\xi_i > 0, i = 1, \ldots, r_1$. The Lyapunov function is designed as

$$V_b = \sum_{i=1}^{r_1}\sum_{\nu=1}^{p}\frac{\xi_i}{1+\bar{\alpha}_1}|\psi_{i_{(\nu)}}|^{1+\bar{\alpha}_1}, \tag{15}$$

where $\psi_{i_{(\nu)}}$ represents the ν-th entry of ψ_i, $\nu = 1, \ldots, p$. Based on Lemma 6, the derivative of V_b along the trajectories (13) and (14) can be calculated by

$$\begin{aligned}\dot{V}_b &= \sum_{i=1}^{r_1}\sum_{\nu=1}^{p}\xi_i\psi_{i_{(\nu)}}^{[\bar{\alpha}_1]}\dot{\psi}_{i_{(\nu)}} = -\bar{k}_1(\Psi_1^{[\bar{\alpha}_1]})^T[\hat{L}_a \otimes I_p]\Psi_1^{[\bar{\alpha}_1]}\\ &= -\bar{k}_1\|\Psi_1^{[\bar{\alpha}_1]}\|^2(\frac{\Psi_1^{[\bar{\alpha}_1]}}{\|\Psi_1^{[\bar{\alpha}_1]}\|})^T[\hat{L}_a \otimes I_p](\frac{\Psi_1^{[\bar{\alpha}_1]}}{\|\Psi_1^{[\bar{\alpha}_1]}\|})\\ &\leq -\bar{k}_1\chi_a\sum_{i=1}^{r_1}\sum_{\nu=1}^{p}|\psi_{i_{(\nu)}}|^{2\bar{\alpha}_1},\end{aligned} \tag{16}$$

where $\hat{L}_a = \frac{1}{2}(\Xi_a L_{11} + L_{11}^T\Xi_a)$ with $\Xi_a = \text{diag}\{\xi_i\}$, $i = 1, \ldots, r_1$. Besides, the notation $\chi_a = \inf_{z_a \in \Delta(1,\bar{\alpha}_1,\delta)} z_a^T\hat{L}_a z_a$ with $z_a = \frac{\Psi_1^{[\bar{\alpha}_1]}}{\|\Psi_1^{[\bar{\alpha}_1]}\|}$, $\delta \neq \bar{\alpha}_1$. Note that $0 < \frac{2\bar{\alpha}_1}{1+\bar{\alpha}_1} < 1$ since $\frac{1}{2} < \bar{\alpha}_1 < 1$. According to Lemma 4, one can obtain that

$$\begin{aligned}\sum_{\nu=1}^{p}|\psi_{i_{(\nu)}}|^{2\bar{\alpha}_1} &= \sum_{\nu=1}^{p}(|\psi_{i_{(\nu)}}|^{1+\bar{\alpha}_1})^{\frac{2\bar{\alpha}_1}{1+\bar{\alpha}_1}}\\ &\geq (\sum_{\nu=1}^{p}|\psi_{i_{(\nu)}}|^{1+\bar{\alpha}_1})^{\frac{2\bar{\alpha}_1}{1+\bar{\alpha}_1}}.\end{aligned} \tag{17}$$

It also can be derived from Lemma 4 that

$$\sum_{i=1}^{m}(\sum_{\nu=1}^{p}|\psi_{i_{(\nu)}}|^{1+\bar{\alpha}_1})^{\frac{2\bar{\alpha}_1}{1+\bar{\alpha}_1}} \geq (\sum_{i=1}^{m}\sum_{\nu=1}^{p}|\psi_{i_{(\nu)}}|^{1+\bar{\alpha}_1})^{\frac{2\bar{\alpha}_1}{1+\bar{\alpha}_1}} \tag{18}$$

Combining with (17) and (18) yields

$$\sum_{i=1}^{m}\sum_{\nu=1}^{p}|\psi_{i_{(\nu)}}|^{2\bar{\alpha}_1} \geq (\sum_{i=1}^{m}\sum_{\nu=1}^{p}|\psi_{i_{(\nu)}}|^{1+\bar{\alpha}_1})^{\frac{2\bar{\alpha}_1}{1+\bar{\alpha}_1}}. \tag{19}$$

Let $\xi_a = \max_{i=1,\ldots,r_1}\{\xi_i\}$. It follows from (15), (16) and (19) that

$$\dot{V}_b \leq -\bar{k}_1\chi_a(\frac{1+\bar{\alpha}_1}{\xi_a})^{\frac{2\bar{\alpha}_1}{1+\bar{\alpha}_1}}V_b^{\frac{2\bar{\alpha}_1}{1+\bar{\alpha}_1}}. \tag{20}$$

It can be further deduced from Lemma 4 that $V_b \to 0$, i.e., $\psi_i \to 0, i = 1, \ldots, r_1$, in finite time. It thus follows from (2) that $q_i \to q_j$, $i, j = 1, \ldots, r_1$, in finite time with the settling time bounded by

$$T_3 \leq \max\{T_1, T_2\} + \frac{\xi_a^{\frac{2\bar{\alpha}_1}{1+\bar{\alpha}_1}}(1+\bar{\alpha}_1)^{\frac{1-\bar{\alpha}_1}{1+\bar{\alpha}_1}}V_b(0)^{\frac{1-\bar{\alpha}_1}{1+\bar{\alpha}_1}}}{\bar{k}_1\chi_a(1-\bar{\alpha}_1)}. \tag{21}$$

Step 2: Denote $\Psi_2 = (\psi_{r_1+1}^T, \ldots, \psi_{r_1+r_2}^T)^T$, $Q_2 = (q_{r_1+1}^T, \ldots, q_{r_1+r_2}^T)^T$. In light of (2) and (3), it is not difficult to get that

$$\Psi_2 = (L_{21} \otimes I_p)Q_1 + (L_{22} \otimes I_p)Q_2, \tag{22}$$
$$\dot{Q}_2 = -k_2\Psi_2^{[\bar{\alpha}_1]}. \tag{23}$$

It follows from Lemma 5 that L_{22} is a non-singular M-matrix. From Lemma 1, there exists a diagonal matrix $D_a = \text{diag}\{d_i\}$ with $d_i > 0, i = r_1+1, \ldots, r_1+r_2$, such that $D_a L_{22} + L_{22}^T D_a$ is symmetric positive definite. Consider the following Lyapunov function:

$$V_c = \sum_{i=r_1+1}^{r_1+r_2}\sum_{\nu=1}^{p}\frac{d_i}{1+\bar{\alpha}_1}|\psi_{i_{(\nu)}}|^{1+\bar{\alpha}_1}. \tag{24}$$

Note that $\Psi_1 \to 0$ in finite time T_3. When $t > T_3$, by adopting the similar analysis as presented in Step 1, the derivative of V_c along the trajectories (22) and (23) is computed as follows:

$$\begin{aligned}\dot{V}_c &= \sum_{i=r_1+1}^{r_1+r_2}\sum_{\nu=1}^{p} d_i \psi_{i_{(\nu)}}^{[\bar{\alpha}_1]}\dot{\psi}_{i_{(\nu)}} = -\bar{k}_1(\Psi_2^{[\bar{\alpha}_1]})^T[P_a\otimes I_p]\Psi_2^{[\bar{\alpha}_1]} \\ &\le -\bar{k}_1\lambda_m(P_a)\sum_{i=r_1+1}^{r_1+r_2}\sum_{\nu=1}^{p}|\psi_{i_{(\nu)}}|^{2\bar{\alpha}_1} \\ &\le -\bar{k}_1\lambda_m(P_a)(\sum_{i=r_1+1}^{r_1+r_2}\sum_{\nu=1}^{p}|\psi_{i_{(\nu)}}|^{1+\bar{\alpha}_1})^{\frac{2\bar{\alpha}_1}{1+\bar{\alpha}_1}} \\ &\le -\bar{k}_1\lambda_m(P_a)(\frac{1+\bar{\alpha}_1}{d_a})^{\frac{2\bar{\alpha}_1}{1+\bar{\alpha}_1}}V_c^{\frac{2\bar{\alpha}_1}{1+\bar{\alpha}_1}}, \qquad (25)\end{aligned}$$

where $P_a = \frac{1}{2}(D_aL_{22}+L_{22}^TD_a)$, $d_a = \max_{i=r_1+1,\ldots,r_1+r_2}\{d_i\}$. Note that $0 < \frac{2\bar{\alpha}_1}{1+\bar{\alpha}_1} < 1$ since $\frac{1}{2} < \bar{\alpha}_1 < 1$. It follows from Lemma 4 that $V_c \to 0$, i.e., $\psi_i \to 0$, $i = r_1+1,\ldots,r_1+r_2$, in finite time. It further can be derived from (2) that $q_i \to q_j$, $i,j = r_1+1,\ldots,r_1+r_2$, in finite time with the settling time estimated by

$$T_4 \le T_3 + \frac{d_a^{\frac{2\bar{\alpha}_1}{1+\bar{\alpha}_1}}(1+\bar{\alpha}_1)^{\frac{1-\bar{\alpha}_1}{1+\bar{\alpha}_1}}V_c(0)^{\frac{1-\bar{\alpha}_1}{1+\bar{\alpha}_1}}}{\bar{k}_1\lambda_m(P_a)(1-\bar{\alpha}_1)}. \qquad (26)$$

Step 3: Note that the matrices L_{ii}, $i = 3,\ldots,k$ are nonsingular M-matrices. Therefore, following the similar analysis as shown in Step 2, the finite-time synchronization can also be realized when the subgraphs are associated with L_{ii}, $i = 3,\ldots,k$.

Combining with the aforementioned discussions, one can conclude that $q_i \to q_j$, $i,j = 1,\ldots,n$, in finite time, which indicates that all agents can reach an agreement in finite time and thereby achieving the goal of finite-time synchronization. □

4. DISTRIBUTED FINITE-TIME CONTAINMENT CONTROL

4.1 Problem formulation

The distributed finite-time containment control with multiple dynamic leaders are considered in this section, where agent 1 to agent m are followers and agent $m+1$ to agent n are leaders. Let q_F and q_L be the column stack vectors of $q_i, i = 1,\ldots,m$ and $q_i, i = m+1,\ldots,n$, respectively.

Assumption 2. The leaders have no neighbors and at least one leader have directed pathes to all followers. Besides, only a portion of followers have access to the leaders.

Lemma 7. (Mei et al. (2012)) If Assumption 2 holds, one can get the following Laplacian matrix L by proper decomposition:

$$L = \begin{bmatrix} L_1 & L_2 \\ \mathbf{0}_{(n-m)\times m} & \mathbf{0}_{(n-m)\times(n-m)}\end{bmatrix},$$

where $L_1 \in \mathbb{R}^{m\times m}$ and $L_2 \in \mathbb{R}^{m\times(n-m)}$. Moreover, the matrix L_1 is a nonsingular M-matrix, each entry of the matrix $-L_1^{-1}L_2$ is non-negative and all row sums of $-L_1^{-1}L_2$ equal one.

Denote $q_d(t) = (q_{d_1}^T(t),\ldots,q_{d_m}^T(t))^T = -(L_1^{-1}L_2\otimes I_p)q_L(t)$ with $q_{d_i}(t)\in\mathbb{R}^p$, $i = 1,\ldots,m$. It is not difficult to deduce from Lemma 6 that $q_d(t)$ is within the convex hull spanned by the leaders. Consequently, the control objective of this section is to design distributed control strategies such that $q_F(t) - q_d(t) \to 0$ in finite time.

A commonly-used estimator $\hat{v}_i(t)$, which is used to estimate $\dot{q}_{d_i}(t)$, is introduced as follows:

$$\dot{\hat{v}}_i(t) = -\beta sgn[\sum_{j=1}^{m} a_{ij}(\hat{v}_i(t)-\hat{v}_j(t)) + \sum_{j=m+1}^{n} a_{ij}(\hat{v}_i(t)-\dot{q}_j(t))], \qquad (27)$$

where β is a positive constant.

Lemma 8. (Mei et al. (2012)) Suppose that Assumption 2 holds and the velocity and acceleration vectors of the leaders are all bounded. By choosing $\beta > \|\ddot{q}_d(t)\|$, it can be obtained that $\hat{v}_i(t) - \dot{q}_{d_i}(t) \to 0$ in finite time.

4.2 Controller design and analysis

Some auxiliary variables are given by

$$\phi_i(t) = \sum_{j=1}^{n} a_{ij}(q_i(t)-q_j(t)), \qquad (28)$$

$$\tilde{s}_i(t) = \phi_i(t) + k_2\dot{\tilde{q}}_i^{[\alpha_3]}(t), \qquad (29)$$

where $\dot{\tilde{q}}_i(t) = \dot{q}_i(t) - \hat{v}_i(t)$ with $\hat{v}_i(t)$ introduced in (27), the constants $k_2 > 0$ and $1 < \alpha_3 < 2$.

The distributed finite-time containment control algorithm is developed as follows:

$$\tau_i(t) = \tilde{\tau}_{i_1}(t) + \tilde{\tau}_{i_2}(t), \qquad (30)$$

with

$$\tilde{\tau}_{i_1}(t) = -\frac{H_i(q_i)}{k_2\alpha_3}\Theta^{-1}\dot{\phi}_i(t) + C_i(q_i,\dot{q}_i)\dot{q}_i(t) + g_i(t), \qquad (31)$$

$$\tilde{\tau}_{i_2}(t) = -\frac{H_i(q_i)}{k_2\alpha_3}(\tilde{s}_i(t) + \tilde{s}_i^{[\alpha_4]}(t)), \qquad (32)$$

where $0 < \alpha_2 < 1$ and $\Theta = \mathrm{diag}\{|\dot{\tilde{q}}_{i(\nu)}(t)|^{\alpha_3-1}\}$ with $\dot{\tilde{q}}_{i_{(\nu)}}(t)$ being the ν-th entry of $\dot{\tilde{q}}_i(t)$, $\nu = 1,\ldots,p$.

Theorem 2. Suppose that Assumptions 2 hold. The finite-time containment control problem with dynamic leaders for multiple Euler-Lagrange systems (1) can be solved under the proposed distributed control laws (30)-(32).

Proof 2. The time variable t will be omitted for simplicity if no confusion occurs. Based the the Euler-Lagrange equation (1) and the distributed finite-time control schemes (30)-(32), it can be deduced that

$$\begin{aligned}\ddot{q}_i &= H_i^{-1}(q_i)[\tilde{\tau}_{i_1} + \tilde{\tau}_{i_2} - C_i(q_i,\dot{q}_i)\dot{q}_i - g_i(q_i)] \\ &= \frac{1}{k_2\alpha_3}(\Theta^{-1}\dot{\tilde{q}}_i - \tilde{s}_i - \tilde{s}_i^{[\alpha_4]}). \qquad (33)\end{aligned}$$

The Lyapunov function is constructed by

$$V_d = \frac{1}{2}\tilde{s}_i^T\tilde{s}_i. \qquad (34)$$

When $\dot{q}_i \ne 0$, following the similar analysis as provided in the proof of Theorem 1, one has

$$\dot{V}_d \leq -2\lambda_m(\Theta)V_d - 2^{\frac{\alpha_4+1}{2}}\lambda_m(\Theta)V_d^{\frac{\alpha_4+1}{2}}. \quad (35)$$

Note that $0 < \frac{\alpha_4+1}{2} < 1$ since $0 < \alpha_4 < 1$. By invoking Lemma 4, it can be concluded that $V_d \to 0$ in finite time, which leads to $\tilde{s}_i \to 0$ in finite time. In addition, the settling time can be estimated by

$$T_5 \leq \frac{1}{\lambda_m(\Theta)(1-\alpha_4)} \ln(2^{\frac{1-\alpha_4}{2}}\lambda_m(\Theta)V_d(0)^{\frac{1-\alpha_4}{2}}+1) \quad (36)$$

When $\dot{\tilde{q}}_i = 0$, it can be deduced from (33) that $\ddot{q}_i = -\frac{1}{k_2\alpha_3}(\tilde{s}_i + \tilde{s}_i^{[\alpha_4]}) \neq 0$ if $\tilde{s}_i \neq 0$. It also can be observed that $\ddot{q}_i > 0$ and $\ddot{q}_i < 0$ if $\tilde{s}_i < 0$ and $\tilde{s}_i > 0$, respectively. It thus follows that the finite-time reachability of $\tilde{s}_i = 0$ can be guaranteed with the settling time bounded by T_6.

When $t > \max\{T_5, T_6\}$, it follows from (29) that $\phi_i = -k_2\dot{\tilde{q}}_i^{[\alpha_3]}$, i.e., $\dot{\tilde{q}}_i = \dot{q}_i - \hat{v}_i = -\tilde{k}_2\phi_i^{[\tilde{\alpha}_3]}$ with $\tilde{k}_2 = \frac{1}{k_2^{\tilde{\alpha}_3}} > 0$ and $\frac{1}{2} < \tilde{\alpha}_3 = \frac{1}{\alpha_3} < 1$. From Lemma 8, one obtains that $\hat{v}_i \to \dot{q}_{di}$ in finite time and the settling time can be further defined by T_7. When $t > \max\{T_5, T_6, T_7\}$. it can be derived from (28) that

$$\begin{aligned}\phi &= (L_1 \otimes I_p)q_F + (L_2 \otimes I_p)q_L \\ &= (L_1 \otimes I_p)(q_F - q_d), \end{aligned} \quad (37)$$

and thus

$$\begin{aligned}\dot{\phi} &= (L_1 \otimes I_p)(\dot{q}_F - \dot{q}_d) = (L_1 \otimes I_p)(\dot{q}_F - \hat{v}) \\ &= -\tilde{k}_2(L_1 \otimes I_p)\phi^{[\tilde{\alpha}_3]}. \end{aligned} \quad (38)$$

Due to the fact that the matrix L_1 is a M-matrix, it thus can be followed from Lemma 1 that there exists a diagonal matrix $D_b = \mathrm{diag}\{d_i\}$ with $d_i > 0$, $i = 1, \ldots, m$, such that $D_bL_1 + L_1^TD_b$ is symmetric positive definite. The Lyapunov function is given by

$$V_e = \sum_{i=1}^{m}\sum_{\nu=1}^{p} \frac{d_i}{1+\tilde{\alpha}_3}|\phi_{i_{(\nu)}}|^{1+\tilde{\alpha}_3}, \quad (39)$$

where $\phi_{i_{(\nu)}}$ represents the ν-th entry of ϕ_i, $\nu = 1, \ldots, p$. Note that $0 < \frac{2\tilde{\alpha}_3}{1+\tilde{\alpha}_3} < 1$ since $0 < \tilde{\alpha}_3 < 1$. By applying Lemma 2 and following the similar discussions as in (15)-(20), the time derivative of V_e along with (38) can be calculated as

$$\begin{aligned}\dot{V}_e &= \sum_{i=1}^{m}\sum_{\nu=1}^{p} d_i\phi_{i_{(\nu)}}^{[\tilde{\alpha}_3]}\dot{\phi}_{i_{(\nu)}} = -k_2(\phi^{[\tilde{\alpha}_3]})^T[P_b \otimes I_p]\phi^{[\tilde{\alpha}_3]} \\ &\leq -k_2\lambda_m(P_b)\sum_{i=1}^{m}\sum_{\nu=1}^{p}|\phi_{i_{(\nu)}}|^{2\tilde{\alpha}_3} \\ &\leq -k_2\lambda_m(P_b)(\sum_{i=1}^{m}\sum_{\nu=1}^{p}|\phi_{i_{(\nu)}}|^{1+\tilde{\alpha}_3})^{\frac{2\tilde{\alpha}_3}{1+\tilde{\alpha}_3}} \\ &\leq -k_2\lambda_m(P_b)(\frac{1+\tilde{\alpha}_3}{d_b})^{\frac{2\tilde{\alpha}_3}{1+\tilde{\alpha}_3}}V_e^{\frac{2\tilde{\alpha}_3}{1+\tilde{\alpha}_3}}, \end{aligned} \quad (40)$$

where $P_b = \frac{1}{2}(D_bL_1 + L_1^TD_b)$, $d_b = \max_{i=1,\ldots,m}\{d_i\}$. Therefore, it can be concluded from Lemma 4 that $V_e \to 0$ in finite time, i.e., $\phi \to 0$ in finite time. Besides, the setting time is bounded by

$$T_8 \leq \max\{T_5, T_6, T_7\} + \frac{d_b^{\frac{2\tilde{\alpha}_3}{1+\tilde{\alpha}_3}}(1+\tilde{\alpha}_3)^{\frac{1-\tilde{\alpha}_3}{1+\tilde{\alpha}_3}}V_e(0)^{\frac{1-\tilde{\alpha}_3}{1+\tilde{\alpha}_3}}}{k_2\lambda_m(P_b)(1-\tilde{\alpha}_3)} \quad (41)$$

Since L_1 is a nonsingular M-matrix, one can deduce that L_1 is invertible from Lemma 1. Consequently, it can be further derived from (37) that $q_F - q_d \to 0$ within T_4. In other words, the multiple followers can converge to a convex hull spanned by the multiple dynamic leaders in finite time, which implies that the objective of the finite-time containment control with dynamic leaders for multiple Euler-Lagrange systems can be achieved. □

5. SIMULATIONS

The efficiency of the proposed control strategies is demonstrated in this section. Each agent is modeled by two-link robot manipulators whose dynamics can be referred to Lu and Liu (2019).

Example 1: The correctness of the distributed finite-time synchronization control laws (4)-(6) is confirmed in this example. The communication graph and initial position and velocity vectors of five agents (agent 1 to agent 5) is shown in Fig. 1 and Table 1, respectively. The parameters are chosen as $k_1 = 0.85$, $\alpha_1 = 1.1$ and $\alpha_2 = 0.8$. In order to show the synchronization performance, the evolution of each agent is provided in Fig. 2.

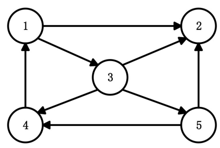

Fig. 1. A communication network including five agents.

Table 1. The initial position and velocity vectors of the five agents.

follower	initial position (rad)	initial velocity (rad/s)
agent 1	$(-3.0, -4.0)^T$	$(0.15, -0.10)^T$
agent 2	$(-6.5, -7.5)^T$	$(0.45, 0.05)^T$
agent 3	$(8.5, -1.5)^T$	$(-0.25, -0.15)^T$
agent 4	$(-4.0, 5.0)^T$	$(0.25, 0.20)^T$
agent 5	$(5.5, -2.0)^T$	$(-0.15, 0.25)^T$

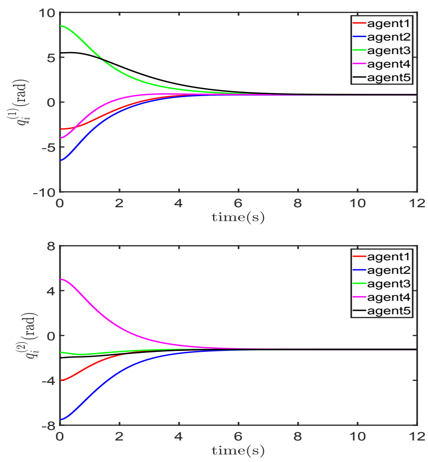

Fig. 2. The synchronization performance of the five agents.

It can be observed from Fig. 2 that the five agents achieve synchronization within finite time 8.0 s, and thus pursuing the finite-time synchronization control objectives.

Example 2: In this example, the effectiveness of the distributed finite-time control laws (30)-(32) is validated. A communication topology among five followers (agents 1-5) and four leaders (agents 6-9) is exhibited in Fig. 3. The position vectors of the leaders are give by $q_6 = [-0.04sin(t), -0.02sin(t)]^T$, $q_7 = [0.05cos(t), -0.02sin(t)]^T$, $q_8 = [0.02sin(t), 0.05cos(t)]$, $q_9 = [0.02cos(t), -0.03sin(t)]^T$ and the initial position and velocity vectors of five followers can be referred to Table 1. We choose $k_2 = 0.75$, $\alpha_3 = 1.1$, $\alpha_4 = 0.9$ and $\beta = 0.8$. The containment error of each follower is displayed in Fig. 4 to show the containment performance.

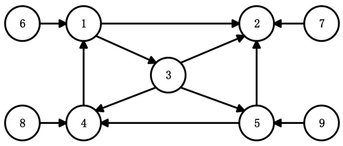

Fig. 3. A communication network including five followers and four leaders.

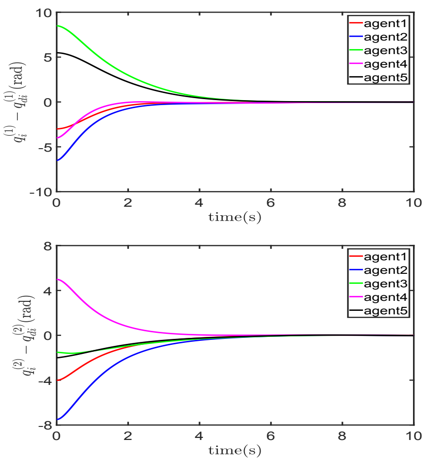

Fig. 4. The containment errors of the five followers.

It follows from Fig. 4 that the containment errors of the followers are converged to zeros within finite time 8.0 s. Therefore, the containment control problem is addressed in finite time.

6. CONCLUSION

The distributed synchronization and containment control problems for multiple Euler-Lagrange systems are addressed in this paper, where the communication networks are presumed to be directed. The satisfactory coordination control objectives of our work are achieved in finite time, and the estimate of the settling times are explicitly presented.

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