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Zonotope-based Interval Estimation for Discrete-Time Linear Switched Systems ⋆

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Abstract: This paper is concerned with the interval estimation for discrete-time linear switched systems affected by unknown disturbances and noises. A novel interval estimation approach is proposed by integrating robust observer design with zonotopic techniques. By introducing L_∞ technique into observer design, the proposed approach is effective in attenuating the influence of unknown disturbances and noises, and improving the accuracy of interval estimation. Based on the designed observer, the interval estimation can be obtained by using zonotopic analysis. Numerical simulation results are conducted to demonstrate the feasibility and effectiveness of the proposed approach.

Keywords: Zonotopic techniques, interval estimation, switched systems, observer design.

1. INTRODUCTION

Switched system, which is an important class of hybrid dynamical system, consists of continuous or discrete-time subsystems and a switching signal which determines the switching from one mode to another at every switching point. Switched system is an effective tool for describing practical industrial systems, including flight control systems (Vu and Morgansen, 2010) and network control systems (Donkers et al., 2011). Due to their powerful modelling capability, the stability analysis and control synthesis for switched systems have been extensively studied in the literature, see, e.g. Liberzon (2003); Zhao et al. (2012); Niu and Zhao (2013).

Apart from the stability analysis and controller synthesis issues, state estimation is also very important for switched systems. State estimation has widely investigated in the control community such as fault diagnosis techniques (Wang et al., 2017), unknown input observer design (Guo et al., 2018) and robust controller design (Aslam et al., 2019). Consequently, many scholars work on the state estimation for switched systems in the past decades. Specifically, the robust state estimation techniques have drawn growing attention due to the uncertainties such as process disturbances and measurement noises always exist in practical systems. In Bejarano and Pisano (2011), the authors proposed a reduced-order unknown input switched observer for uncertain switched systems. For switched systems with unknown inputs, a high-order robust observer synthetic method was presented in Ríos et al. (2012). For a class of nonlinear switched systems, the state estimation was achieved in Ríos et al. (2015) with the aid of hybrid observer design and parameter identification. In Yang et al. (2017), a robust switched observer was designed to estimate state via the augmented approach. A robust estimator design method was proposed for switched systems in Delshada et al. (2018), which can attenuate the influence of unknown input on state estimation. However, these above-mentioned methods all use the H_∞ technique to reduce the effects of uncertainties and improve the estimation accuracy. Note that H_∞ norm is a measurement of energy-to-energy gain. As pointed out in Wang et al. (2017), the practical signals are not necessarily energy-bounded but have bounded peak values. Consequently, L_∞ norm, which aims to minimize the peak-to-peak gain, can be considered as an alternative solution to analyse the state estimation robustness performance.

On the other hand, the above-mentioned results are all point-estimation of state. Since the existence of model and/or signals uncertainties in practical systems, the point-estimation usually cannot converge to the real state. Thus, interval estimation approaches via interval observer and zonotopic techniques get more attention in recent years. The fundamental idea of interval observer is to design two sub-observers such that their error dynamics are both cooperative and stable. The two sub-observers

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can provide the upper and lower bounds of the real system states. During the past decade, several interval observer design works have been devoted to various linear or nonlinear regular systems (Raïssi et al., 2012; Efimov and Raïssi, 2016; Meslem et al., 2018). Specially, Ethabet et al. (2017) addressed the interval observer design issue for continuous-time linear switched systems. The interval observer for discrete-time linear switched systems was designed in D-inh et al. (2019). Nevertheless, it is not a trivial work to construct a cooperative and stable error system, and even impossible for some dynamical systems. Although the cooperative constraint can be relaxed by coordinate transformations, it still has several deficiencies. First, the coordinate transformations may lead to some conservatisms in the interval estimation. Second, the performance of the interval observer heavily depends on a predefined matrix in the design, but there is no systematic and effective approach proposed on how to choose this predefined matrix. Fortunately, the zonotope-based interval estimation methods can provide a good balance between computation complexity and estimation accuracy, and have gained much attention by many researchers (Tang et al., 2019). Especially, Alamo et al. (2005) proposed a guaranteed state estimation approach for nonlinear systems with the aid of zonotopic techniques. Le et al. (2013) presented a novel interval estimation method via zonotopes for uncertain multivariable systems. For time-varying dynamics systems with measurement uncertainties, Combastel (2015) designed a zonotopic Kalman observer to achieve the interval estimation. In addition, some criteria such as P-radius (Le et al., 2013) and F-radius (Combastel, 2015) to decrease the size of zonotopes have been used to improve the estimation accuracy. However, these methods by using zonotopic technique are all the state estimation for state-space systems. In fact, there are only a limited number of papers on the case of zonotope-based interval estimation for switched systems.

Motivated by the above-mentioned discussion, this paper proposes a zonotope-based interval estimation approach for linear switched systems with unknown inputs. The main contributions are summarized as follows:

1) An interval estimation method that combines the observer design with zonotopic technique is proposed for linear switched systems.
2) By using zonotope approach, the proposed method can overcome the cooperative constraints used in the interval observer design.
3) With the L_∞ technique used to reduce the influence of unknown inputs, the proposed method provides a systematic way to improve the estimation accuracy.

2. PRELIMINARIES

Notation: $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ are the n and $m \times n$ dimensional Euclidean space, respectively. For a matrix $X \in \mathbb{R}^{m\times n}$, X^T and $X(i,j)$ represent its transpose and the element of X in the i-th row and the j-th column. The matrix $Y \succ 0$ ($Y \prec 0$) indicates that $Y \in \mathbb{R}^{n\times n}$ is a positive (negative) definite matrix. For a vector $\zeta \in \mathbb{R}^n$, $\|\zeta\|$ and $\zeta(i)$ denote its Euclidean norm and the i-th component. The symbols $\oplus$ and $\odot$ represent the Minkowski sum and the linear image operators. The asterisk $\star$ represents the symmetric term in a symmetric matrix. For a signal $x_k \in \mathbb{R}^n$, its L_∞ norm is defined as:

$$\|x\|_\infty = \sup_{k\geq 0}\|x_k\|,\ \|x_k\| = \sqrt{x_k^T x_k}. \tag{1}$$

In this paper, the following definitions and properties will be used.

Definition 1. (Zhang et al., 2020) Given a set $\mathbf{\Phi} \subset \mathbb{R}^n$, its interval hull Box($\mathbf{\Phi}$) is defined as follows:

$$\phi \in \mathbf{\Phi} \subseteq \mathrm{Box}(\mathbf{\Phi}) = [\underline{\phi}, \overline{\phi}], \tag{2}$$

where $[\underline{\phi}, \overline{\phi}]$ is the smallest interval vector containing $\mathbf{\Phi}$. $\underline{\phi} \in \mathbb{R}^n$ and $\overline{\phi} \in \mathbb{R}^n$ are the upper and lower bounds of ϕ, which satisfy $\underline{\phi} \leq \phi \leq \overline{\phi},\ \phi \in \mathbf{\Phi}$.

Definition 2. (Combastel, 2005) An m-order zonotope $\mathcal{Z} \subset \mathbb{R}^n\,(n \leq m)$ is a linear image of a hypercube $\mathbf{B}^m = [-1,\ +1]^m$, which can be defined as follows:

$$\mathcal{Z} = \langle c, Z\rangle = c \oplus Z\mathbf{B}^m = \{z = c + Zb, b \in \mathbf{B}^m\} \tag{3}$$

where $c \in \mathbb{R}^n$ is the center of $\mathcal{Z}$, and $Z \in \mathbb{R}^{n\times m}$ is its generation matrix, which defines the shape of $\mathcal{Z}$.

Property 1. (Scott et al., 2014) For zonotpes, the following properties hold:

$$\Gamma \odot \langle c, Z\rangle = \langle \Gamma c, \Gamma Z\rangle \tag{4}$$

$$\langle c_1, Z_1\rangle \oplus \langle c_2, Z_2\rangle = \langle c_1 + c_2, [Z_1 \quad Z_2]\rangle \tag{5}$$

where $c,\ c_1,\ c_2 \in \mathbb{R}^n$ are known vectors, $Z \in \mathbb{R}^{n\times m}$, $Z_1 \in \mathbb{R}^{n\times m_1}$, $Z_2 \in \mathbb{R}^{n\times m_2}$ and $\Gamma \in \mathbb{R}^{l\times n}$ are determined matrices.

Property 2. (Combastel, 2005) For an m-order zonotope $\mathcal{Z} = \langle c, Z\rangle \subset \mathbb{R}^n$, its interval hull Box($\mathcal{Z}$) $= [\underline{z}, \overline{z}]$ can be obtained by

$$\begin{cases} \overline{z}(i) = c(i) + \sum\limits_{j=1}^{m}|Z(i,j)|,\ i = 1,\cdots,n, \\ \underline{z}(i) = c(i) - \sum\limits_{j=1}^{m}|Z(i,j)|,\ i = 1,\cdots,n. \end{cases} \tag{6}$$

According to the Definitions 1 and 2, the interval hull of zonotope $\mathcal{Z} = \langle c, Z\rangle$ can also be computed by

$$\mathcal{Z} \subseteq \mathrm{Box}(\mathcal{Z}) = c \oplus \Lambda(Z)\mathbf{B}^n, \tag{7}$$

where $\Lambda(Z) \in \mathbb{R}^{n\times n}$ is a diagonal matrix satisfying the following form

$$\Lambda(Z) = \begin{bmatrix} \sum\limits_{j=1}^{m}|Z(1,j)| & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum\limits_{j=1}^{m}|Z(n,j)| \end{bmatrix}.$$

Remark 1. *In the Minkowski sum operation of zonotopes, the column number of the generator matrix will increase linearly, which may cause the curse of dimensionality. Fortunately, the reduction operator proposed in Combastel (2005) can use a lower-dimensional zonotope to contain a higher-dimensional one, which are summarized as*

$$\mathcal{Z} = \langle c, Z\rangle \subseteq \langle c, Re_{\downarrow}(Z)\rangle$$

where $Re_{\downarrow}(Z) \in \mathbb{R}^{n\times s}$ represents the generator matrix of the lower-dimensional zonotope and $s\,(n \leq s \leq m)$ is the maximum number of columns of $Re_{\downarrow}(Z)$. The $Re_{\downarrow}(Z)$ can be computed as follows:

• *Arrange the column vectors of matrix Z in descending order according to their Euclidean norm and obtain a new matrix $\bar{Z}$.*

$$\bar{Z} = [z_1 \ \cdots \ z_m], \| z_j \| \geq \| z_{j+1} \|, \ j = 1, \cdots m-1.$$

• *If $m \leq s$, then $Re_{\downarrow}(Z) = Z$.*

• *If $m > s$, then $Re_{\downarrow}(Z) = [Z_> \quad \Lambda(Z_<)] \in \mathbb{R}^{n \times s}$, where $Z_> = [z_1 \ \cdots z_{m-n}]$, $Z_< = [z_{m-n+1} \ \cdots z_m]$.*

3. PROBLEM FORMULATION

Consider the following discrete-time linear switched system with unknown disturbances and noises

$$\begin{cases} x_{k+1} = A_{\sigma(k)}x_k + B_{\sigma(k)}u_k + D_{\sigma(k)}w_k \\ y_k = C_{\sigma(k)}x_k + E_{\sigma(k)}v_k \end{cases}, \tag{8}$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$, $y_k \in \mathbb{R}^{n_y}$, $w_k \in \mathbb{R}^{n_w}$ and $v_k \in \mathbb{R}^{n_v}$ represent the vectors of state, control input, measurement output, unknown disturbances and measurement noises, respectively. $\sigma(k)$ is a known piecewise constant function which denotes the switching signal. $\{(A_{\sigma(k)}, B_{\sigma(k)}, C_{\sigma(k)}, D_{\sigma(k)}, E_{\sigma(k)}) : \sigma(k) \in \mathscr{N}\}$ are a family of matrices parameterized by an index set $\mathscr{N} = \{1, \cdots, N\}$ and N is the number of subsystems. Let $q = \sigma(k)$ be the index of the active subsystem, A_q, B_q, C_q, D_q and E_q are constant matrices with the corresponding dimensions.

The following assumptions will be used in this paper.

Assumptions 1. The switching signal $\sigma(k)$ in (8) can be available in real-time.

Assumptions 2. The initial state x_0, disturbances w_k and noises v_k are assumed to be unknown but bounded as

$$|x_0 - c_0| \leq \tilde{x}_0, \ |w_k| \leq \tilde{w}, \ |v_k| \leq \tilde{v}, \tag{9}$$

where $|\cdot|$ denotes the absolute value operator, $c_0 \in \mathbb{R}^{n_x}$, $\tilde{x}_0 > 0 \in \mathbb{R}^{n_x}$, $\tilde{w} > 0 \in \mathbb{R}^{n_w}$ and $\tilde{v} > 0 \in \mathbb{R}^{n_v}$ are known vectors.

According to Definition 2, (9) can be reformulated as

$$\begin{cases} x_0 \in \mathcal{X}_0 = \langle c_0, Z_0 \rangle, \\ w_k \in \mathcal{W} = \langle 0, W \rangle, \\ v_k \in \mathcal{V} = \langle 0, V \rangle, \end{cases} \tag{10}$$

where $c_0 \in \mathbb{R}^{n_x}$ is a known vector, $Z_0 = \text{diag}(\tilde{x}_0) \in \mathbb{R}^{n_x \times n_x}$, $W = \text{diag}(\tilde{w}) \in \mathbb{R}^{n_w \times n_w}$ and $V = \text{diag}(\tilde{v}) \in \mathbb{R}^{n_v \times n_v}$ are determined diagonal matrices, respectively.

The interval estimation techniques aim to obtain an interval vector $[\underline{x}_k, \overline{x}_k]$, which can contain the real state x_k, i.e.

$$\underline{x}_k \leq x_k \leq \overline{x}_k.$$

In this paper, an interval estimation approach is proposed for linear switched systems by combining the robust observer design with the zonotopic analysis. First, a class of Luenberger observers for system (8) are designed via the L_∞ techniques. Based on the L_∞ observers, the state interval estimation will be obtained with the aid of zonotopic analysis.

4. ROBUST STATE OBSERVER DESIGN

Consider the following structure of observer

$$\hat{x}_{k+1} = A_q\hat{x}_k + B_qu_k + L_q(y_k - C_q\hat{x}_k), \tag{11}$$

where $\hat{x}_k$ denotes the vector of state estimation and $L_q \in \mathbb{R}^{n_x \times n_y}$, $q \in \mathscr{N}$ represents the observer gain matrix to be determined.

Define the estimation error as

$$e_k = x_k - \hat{x}_k, \tag{12}$$

then the following error dynamics systems are obtained

$$e_{k+1} = (A_q - L_qC_q)e_k + D_qw_k - L_qE_qv_k, \tag{13}$$

which can be rewritten as

$$e_{k+1} = A_{eq}e_k + D_qw_k + L_{eq}v_k, \tag{14}$$

where $A_{eq} = A_q - L_qC_q$ and $L_{eq} = -L_qE_q$, $q \in \mathscr{N}$.

To improve the estimation accuracy, L_∞ technique is used to attenuate the influence of unknown inputs. Thus, the following theorem is proposed to design L_q for the observer in (11) based on error dynamic systems (13).

Theorem 1. Given constants $0 < \lambda < 1$, $\gamma_w > 0$ and $\gamma_v > 0$, if there exist a scalar $\mu > 0$, matrices $P = P^T \succ 0 \in \mathbb{R}^{n_x \times n_x}$ and $W_q \in \mathbb{R}^{n_x \times n_y}$ for $\forall q \in \mathscr{N}$ such that

$$\begin{bmatrix} (\lambda-1)P & \star & \star & \star \\ 0 & -\mu I_{n_w} & \star & \star \\ 0 & 0 & -\mu I_{n_v} & \star \\ PA_q - W_qC_q & PD_q & -W_qE_q & -P \end{bmatrix} \prec 0, \tag{15}$$

$$\begin{bmatrix} \lambda P & \star & \star & \star \\ 0 & (\gamma_w - \mu)I_{n_w} & \star & \star \\ 0 & 0 & (\gamma_v - \mu)I_{n_v} & \star \\ I_{n_x} & 0 & 0 & (\gamma_w + \gamma_v)I_{n_x} \end{bmatrix} \succ 0, \tag{16}$$

then error system (13) is asymptotically stable and satisfies the following L_∞ performance

$$\|e_k\|^2 \leq (\gamma_w + \gamma_v)(\lambda(1-\lambda)^kV_0 + \gamma_w\|w\|^2 + \gamma_v\|v\|^2), \tag{17}$$

where $V_0 = e_0^TPe_0$ and $P \succ 0 \in \mathbb{R}^{n_x \times n_x}$ being a designed matrix. Moreover, if the LMIs in (15) and (16) are solvable, the matrix L_q can be determined by $L_q = P^{-1}W_q$, $q \in \mathscr{N}$.

Proof : Choose the following common quadratic Lyapunov function

$$V_k = e_k^TPe_k, P = P^T \succ 0, \tag{18}$$

Then, the difference of V_k is

$$\Delta V_k = V_{k+1} - V_k = \begin{bmatrix} e_k \\ w_k \\ v_k \end{bmatrix}^T \Omega \begin{bmatrix} e_k \\ w_k \\ v_k \end{bmatrix}, \tag{19}$$

where

$$\Omega = \begin{bmatrix} A_{eq}^TPA_{eq} - P & \star & \star \\ D_q^TPA_{eq} & D_q^TPD_q & \star \\ L_{eq}^TPA_{eq} & L_{eq}^TPD_q & L_{eq}^TPL_{eq} \end{bmatrix}.$$

By setting $W_q = PL_q$, $q \in \mathscr{N}$ then according to $A_{eq} = A_q - L_qC_q$ and $L_{eq} = -L_qE_q$, the inequality in (15) is refactored as

$$\begin{bmatrix} (\lambda-1)P & \star & \star & \star \\ 0 & -\mu I_{n_w} & \star & \star \\ 0 & 0 & -\mu I_{n_v} & \star \\ PA_{eq} & PD_q & PL_{eq} & -P \end{bmatrix} \prec 0, \tag{20}$$

Pre- and post- multiplying (20) with

$$\begin{bmatrix} I_{n_x} & 0 & 0 & A_{eq}^T \\ 0 & I_{n_w} & 0 & D_q^T \\ 0 & 0 & I_{n_v} & L_{eq}^T \end{bmatrix}$$

and its transpose, respectively, we have

$$\Omega + \begin{bmatrix} \lambda P & \star & \star \\ 0 & -\mu I_{n_w} & \star \\ 0 & 0 & -\mu I_{n_v} \end{bmatrix} \prec 0. \tag{21}$$

Pre-multiplying and post-multiplying inequality (20) with $[e_k^T \quad w_k^T \quad v_k^T]$ and its transpose, we can obtain

$$\Delta V_k < -\lambda V_k + \mu w_k^T w_k + \mu v_k^T v_k. \tag{22}$$

When $w_k = 0$ and $v_k = 0$, (22) implies that

$$\Delta V_k = V_{k+1} - V_k < -\lambda V_k < 0 \tag{23}$$

Consequently, error system (13) is asymptotically stable.

On the other hand, inequality (22) implies that

$$V_{k+1} < (1-\lambda)V_k + \mu\|\tilde{w}\|^2 + \mu\|\tilde{v}\|^2,$$

which means

$$\begin{aligned} V_k &< (1-\lambda)^k V_0 + \sum_{\varsigma=0}^{k-1}(1-\lambda)^{\varsigma}(\mu\|\tilde{w}\|^2 + \mu\|\tilde{v}\|^2) \\ &\le (1-\lambda)^k V_0 + \frac{1-\lambda^k}{\lambda}(\mu\|\tilde{w}\|^2 + \mu\|\tilde{v}\|^2) \\ &\le (1-\lambda)^k V_0 + \frac{\mu\|\tilde{w}\|^2}{\lambda} + \frac{\mu\|\tilde{v}\|^2}{\lambda}. \end{aligned} \tag{24}$$

Applying the Schur complement lemma (Boyd et al., 1994), inequality (16) is equivalent to

$$\begin{bmatrix} \lambda P & \star & \star \\ 0 & (\gamma_w - \mu) I_{n_w} & \star \\ 0 & 0 & (\gamma_v - \mu) I_{n_v} \end{bmatrix} - \frac{1}{\gamma_w + \gamma_v} \begin{bmatrix} I_{n_x} \\ 0 \\ 0 \end{bmatrix} [I_{n_x} \ 0 \ 0] \succ 0, \tag{25}$$

Pre-multiplying and post-multiplying inequality (25) with $[e_k^T \quad w_k^T \quad v_k^T]$ and its transpose, we have

$$e_k^T e_k \le (\gamma_w + \gamma_v)(\lambda V_k + (\gamma_w - \mu)\|\tilde{w}\|^2 + (\gamma_v - \mu)\|\tilde{v}\|^2). \tag{26}$$

Substituting (24) into (26) yields

$$\begin{aligned} e_k^T e_k &\le (\gamma_w + \gamma_v)(\lambda((1-\lambda)^k V_0 + \frac{\mu\|\tilde{w}\|^2}{\lambda} + \frac{\mu\|\tilde{v}\|^2}{\lambda}) \\ &\quad + (\gamma_w - \mu)\|\tilde{w}\|^2 + (\gamma_v - \mu)\|\tilde{v}\|^2) \\ &= (\gamma_w + \gamma_v)(\lambda(1-\lambda)^k V_0 + \gamma_w\|\tilde{w}\|^2 + \gamma_v\|\tilde{v}\|^2), \end{aligned}$$

which implies error system (13) satisfies the L_∞ performance in (17). □

Remark 2. *To attenuate the influence of w_k and v_k as much as possible, the minimal scalars γ_w and γ_v can be determined by the following optimization problem:*

$$\min \quad \gamma_w + \gamma_v, \tag{27a}$$

$$\text{s.t.} \quad (15) - (16) \tag{27b}$$

and the feasible solution provides L_q by $L_q = P^{-1} W_q$.

Remark 3. *For brevity, the robust observer in (11) is determined by a common Lyapunov function, which may result in some conservatism. In fact, the observer can be designed based on multiple Lyapunov functions, which can reduce such conservatism and further improve the estimation accuracy (Shi et al., 2015; Fei et al., 2017).*

5. INTERVAL ESTIMATION OF STATE

After getting observer gain matrices L_q, $q \in \mathscr{N}$, the design of the L_∞ observer is completed. The interval estimation of x_k will be obtained with the aid of zonotopic techniques in this section.

From (12), we can obtain

$$x_k = \hat{x}_k + e_k. \tag{28}$$

Thus, if an interval vector $[\underline{e}_k, \overline{e}_k]$ satisfying $\underline{e}_k \le e_k \le \overline{e}_k$ can be obtained, from (28), the interval vector $[\underline{x}_k, \overline{x}_k]$ are calculated as

$$\begin{cases} \overline{x}_k = \hat{x}_k + \overline{e}_k, \\ \underline{x}_k = \hat{x}_k + \underline{e}_k. \end{cases} \tag{29}$$

Therefore, in the sequel, we first get the interval estimation of e_k, then give that of x_k.

With the aid of zonotopic techniques, the following theorem is presented to realise the interval estimation of x_k.

Theorem 2. For observer (11) and error dynamics systems (13), given $c_0 = \hat{x}_0$, then state x_k is bounded in a zonotope $\mathcal{X}_k = \langle \hat{x}_k, Z_k \rangle$ and the interval estimation $[\underline{x}_k, \overline{x}_k]$ of x_k are determined as follows:

$$\begin{cases} \overline{x}_k(i) = \hat{x}_k(i) + \sum_{j=1}^{n_z} |Z_k(i,j)|, \ i = 1, \cdots, n_x, \\ \underline{x}_k(i) = \hat{x}_k(i) - \sum_{j=1}^{n_z} |Z_k(i,j)|, \ i = 1, \cdots, n_x, \end{cases} \tag{30}$$

where n_z is the column number of Z_k and Z_k satisfies the following iteration equation

$$Z_{k+1} = \left[(A_q - L_q C_q) Re_{\downarrow}(Z_k) \quad D_q W \quad -L_q E_q V \right]. \tag{31}$$

Proof : First, we prove that the interval vector $[\underline{x}_k, \overline{x}_k]$ of x_k can be obtained from (30). When $\mathcal{X}_0 = \langle \hat{x}_0, Z_0 \rangle$, then from (4) and (12), we have

$$e_0 \in \mathcal{E}_0 = \langle \hat{x}_0, Z_0 \rangle \oplus (-\hat{x}_0) = \langle 0, Z_0 \rangle. \tag{32}$$

Note that $w_k \in \langle 0, W \rangle$, $v_k \in \langle 0, V \rangle$ and $e_0 \in \langle 0, Z_0 \rangle$, hence we can conclude that $e_k \in \mathcal{E}_k = \langle 0, Z_k \rangle$. From (28), we have $x_k \in \mathcal{X}_k = \hat{x}_k \oplus \langle 0, Z_k \rangle = \langle \hat{x}_k, Z_k \rangle$. Using Property 2, the interval estimation of x_k are calculated as

$$\begin{cases} \overline{x}_k(i) = \hat{x}_k(i) + \sum_{j=1}^{n_z} |Z_k(i,j)|, \ i = 1, \cdots, n_x, \\ \underline{x}_k(i) = \hat{x}_k(i) - \sum_{j=1}^{n_z} |Z_k(i,j)|, \ i = 1, \cdots, n_x. \end{cases}$$

where n_z is the column number of Z_k.

Now, we are ready to prove the iteration equation in (31). Since $e_k \in \mathcal{E}_k = \langle 0, Z_k \rangle$, then according to (10) and (13), $e_{k+1} \in \hat{\mathcal{E}}_{k+1}$ is updated as follows:

$$\begin{aligned} \hat{\mathcal{E}}_{k+1} &= \left\langle 0, \hat{Z}_{k+1} \right\rangle \\ &= (A_q - L_q C_q) \odot \mathcal{E}_k \oplus D_q \odot \mathcal{W} \oplus (-L_q E_q) \odot \mathcal{V}. \end{aligned}$$

According to (4) and (5), $\hat{Z}_{k+1}$ can be written as

$$\hat{Z}_{k+1} = \left[(A_q - L_q C_q) Z_k \quad D_q W \quad -L_q E_q V \right].$$

Using the reduction operator in Remark 1, we can obtain $\langle 0, Z_k \rangle \subseteq \langle 0, Re_{\downarrow}(Z_k) \rangle$, and it follows that $\langle 0, \hat{Z}_{k+1} \rangle \subseteq \langle 0, Z_{k+1} \rangle$. Finally, we have $e_{k+1} \in \mathcal{E}_{k+1} = \langle 0, Z_{k+1} \rangle$. □

Remark 4. *Note that the proposed approach does not requires cooperative constraints and can avoid the additional conservatism caused by coordinate transformation. Therefore, the proposed approach provides a systematic way to improve the interval estimation accuracy by combining robust observer design and zonotopic techniques.*

6. SIMULATION

In this section, a numerical example adapted from Dinh et al. (2019) is utilized to demonstrate the viability and validity of the proposed interval estimation approach. Consider the following discrete-time linear switched system

$$\begin{cases} x_{k+1} = A_q x_k + B_q u_k + D_q w_k \\ y_k = C_q x_k + E_q v_k \end{cases}, \ q = 1, \cdots, 3. \quad (33)$$

where

$$A_1 = \begin{bmatrix} 0.2 & -0.5 \\ 0 & 0.2 \end{bmatrix}, A_2 = \begin{bmatrix} 0.3 & -2 \\ 0 & 0.6 \end{bmatrix}, A_3 = \begin{bmatrix} 0.5 & -1.1 \\ 0 & 0.16 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, B_2 = \begin{bmatrix} 6 \\ 1 \end{bmatrix}, B_3 = \begin{bmatrix} -2 \\ 2 \end{bmatrix},$$

$$C_1 = [0.2 \ 0.8], C_2 = [1 \ 0], C_3 = [0.1 \ 1],$$

$$D_1 = D_2 = D_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E_1 = E_2 = E_3 = 1.$$

The switching signal $\sigma(k)$ between the three subsystems is plotted in Fig 1. By choosing $\lambda = 0.5$ and solving the optimization problem (27), we obtain $\mu = 5.7532$, $\gamma_w = 5.7548$, $\gamma_v = 5.7543$, and the gain matrices L_1, L_2 and L_3 as follows:

$$L_1 = \begin{bmatrix} -0.0953 \\ 0.1362 \end{bmatrix}, L_2 = \begin{bmatrix} 0.5519 \\ -0.0792 \end{bmatrix}, L_3 = \begin{bmatrix} -1.0969 \\ 0.1189 \end{bmatrix}.$$

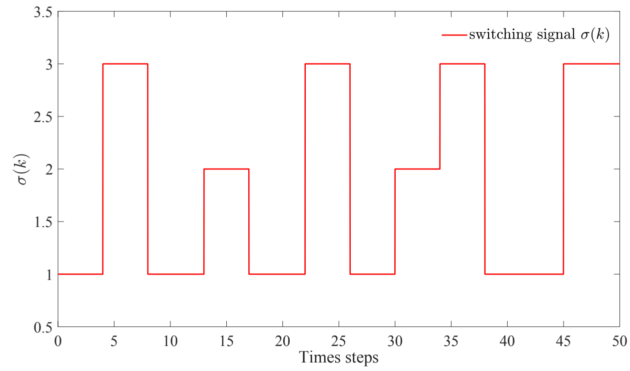

Fig. 1. Switching signal $\sigma(k)$

In the simulation, we choose the input $u_k = 0.5\sin(0.1k)$ and the initial state $x_0 = [1 \quad 2]^T$. Meanwhile, The unknown inputs are set as $w_k = [0.1\sin(0.5k) \quad 0.1\cos(0.5k)]^T$ and $v_k = 0.1\sin(0.5k)$. The initial zonotope of x_0 are set as $c_0 = [1 \quad 1]^T$ and $Z_0 = I_2$. The generation matrices of $\mathcal{W}$ and $\mathcal{V}$ are set as $W = 0.1I_2$ and $V = 0.1I_1$. The reduction order of $Re_\downarrow(Z_k)$ is set as $m = 20$ to avoid the curse of dimensionality.

The simulation results are shown in Fig 2. As we can see, although there is initial estimation error, the states estimate can quickly track the real states and provide accurate interval estimation. To further demonstrate the superiority of the proposed approach, the zonotope-based method is compared with the optimal interval observers proposed in Dinh et al. (2019). Note that the optimal interval observers proposed in Dinh et al. (2019) have not considered the influence of unknown measurement noises. Therefore, we set $E_1 = E_2 = E_3 = 0$ and $v_k = 0$ of system (33). By choosing $\lambda = 0.5$ and solving (27), we have $\mu = 7.5644$, $\gamma_w = 7.5654$. The observer gain matrices L_1, L_2 and L_3 are determined as

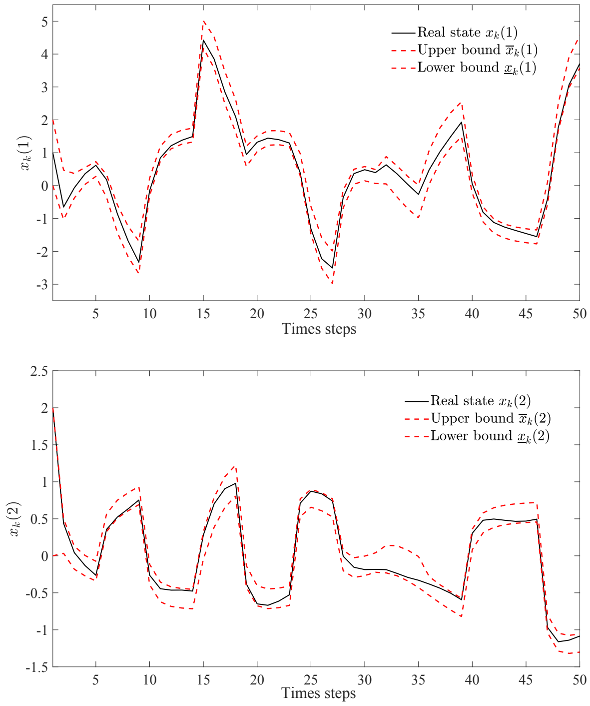

Fig. 2. States and their interval estimations by the proposed approach

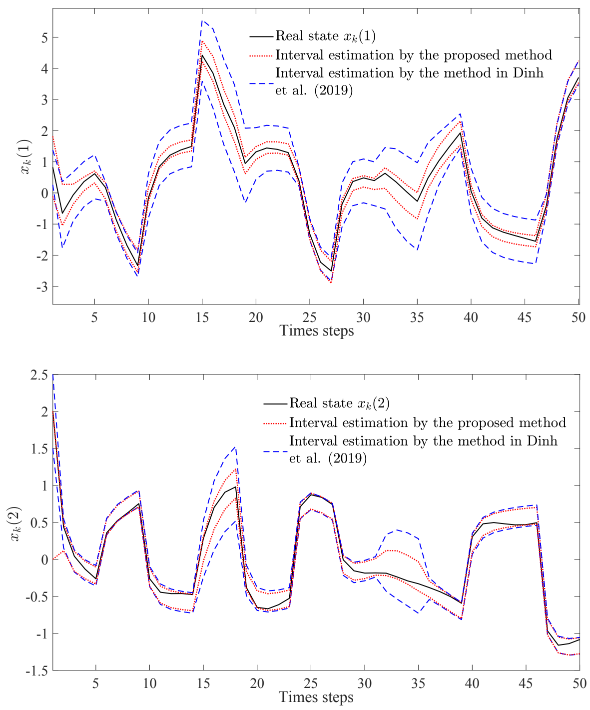

Fig. 3. States and their interval estimations by the proposed approach and by the method in Dinh et al. (2019)

$$L_1 = \begin{bmatrix} -0.2333 \\ 0.1902 \end{bmatrix}, L_2 = \begin{bmatrix} 0.6087 \\ -0.0929 \end{bmatrix}, L_3 = \begin{bmatrix} -1.4913 \\ 0.1845 \end{bmatrix}.$$

The simulation results in Fig 3 show that the proposed approach can provide more accurate interval estimation than optimal interval observers. Consequently, the results all show the feasibility and effectiveness of our approach in state interval estimation.

7. CONCLUSIONS

This paper studies interval estimation for discrete-time linear switched systems with unknown but bounded inputs. A novel interval estimation approach is proposed via the robust observer design and zonotopic techniques. Compared with interval observers, the proposed approach overcomes the cooperativity constraints and avoids the additional conservatism caused by coordinate transformation. Numerical simulations have demonstrated the viability and validity of the proposed interval estimation approach. In the future, we will focus on using the multiple Lyapunov functions to further improve the estimation accuracy of the proposed method and this will be our next research work.

REFERENCES

Alamo, T., Bravo, J., and Camacho, E. (2005). Guaranteed state estimation by zonotopes. *Automatica*, 41(6), 1035–1043.

Aslam, M., Shi, P., and Lim, C. (2019). Robust active noise control design by optimal weighted least squares approach. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 66(10), 3955–3967.

Bejarano, F. and Pisano, A. (2011). Switched observers for switched linear systems with unknown inputs. *IEEE Transactions on Automatic Control*, 56(3), 681–686.

Boyd, S., Ghaoui, L.E., Feron, E., and Balakrishnan, V. (1994). *Linear Matrix Inequalities in System and Control Theory.* SIAM, Philadelphia, Pennsylvania.

Combastel, C. (2005). A state bounding observer for uncertain non-linear continuous-time systems based on zonotopes. In *the 44th IEEE Conference on Decision and Control*, 7228–7234, Seville, Spain. IEEE.

Combastel, C. (2015). Zonotopes and Kalman observers: gain optimality under distinct uncertainty paradigms and robust convergence. *Automatica*, 55, 265–273.

Delshada, I., Hespanha, J., Sadabadi, M., and Karimi, H. (2018). Robust estimator design for switched systems with unknown switching time: An LMI-based approach. *European Journal of Control*, 44, 58–64.

Dinh, T., Marouani, G., Raïssi, T., Wang, Z., and Messaoud, H. (2019). Optimal interval observers for discrete-time linear switched systems. *International Journal of Control*, DOI: 10.1080/00207179.2019.1575518.

Donkers, M., Heemels, W., Wouw, N., and Hetel, L. (2011). Stability analysis of networked control systems using a switched linear systems approach. *IEEE Transactions on Automatic Control*, 56(9), 2101–2115.

Efimov, D. and Raïssi, T. (2016). Design of interval observers for uncertain dynamical systems. *Automation and Remote Control*, 77(2), 191–225.

Ethabet, H., Raïssi, T., Amairi, M., and Aoun, M. (2017). Interval observers design for continuous-time linear switched systems. In *the 20th World Congress of the International Federation of Automatic Control*, 6259–6264, Toulouse, France. IEEE.

Fei, Z., Shi, S., Zhao, C., and Wu, L. (2017). Asynchronous control for 2-d switched systems with mode-dependent average dwell time. *Automatica*, 79, 198–206.

Guo, S., Zhu, F., and Jiang, B. (2018). Reduced-order switched UIO design for switched discrete-time descriptor systems. *Nonlinear Analysis: Hybrid Systems*, 30, 240–255.

Le, V., Stoica, C., Alamo, T., Camacho, E., and Dumur, D. (2013). Zonotopic guaranteed state estimation for uncertain systems. *Automatica*, 49(11), 3418–3424.

Liberzon, D. (2003). *Switching in systems and control.* Springer Science & Business Media, Berlin Heidelberg, Germany.

Meslem, N., Loukkas, N., and Martinez, J. (2018). Using set invariance to design robust interval observers for discrete-time linear systems. *International Journal of Robust and Nonlinear Control*, 28(11), 3623–3639.

Niu, B. and Zhao, J. (2013). Barrier Lyapunov functions for the output tracking control of constrained nonlinear switched systems. *Systems & Control Letters*, 62(10), 963–971.

Raïssi, T., Efimov, D., and Zolghadri, A. (2012). Interval state estimation for a class of nonlinear systems. *IEEE Transactions on Automatic Control*, 57(1), 260–265.

Ríos, H., Efimov, D., Davila, J., Raïssi, T., Fridman, L., and Zolghadri, A. (2012). State estimation for linear switched systems with unknown inputs. In *the 4th IFAC Conference on Analysis and Design of Hybrid Systems*, 271–276, Eindhoven, The Netherlands. IEEE.

Ríos, H., Mincarelli, D., Efimov, D., Perruquetti, W., and Davila, J. (2015). Continuous and discrete state estimation for switched LPV systems using parameter identification. *Automatica*, 62, 139–147.

Scott, J., Findeisen, R., Braatz, R., and Raimondo, D. (2014). Input design for guaranteed fault diagnosis using zonotopes. *Automatica*, 50(6), 1580–1589.

Shi, P., Su, X., and Li, F. (2015). Dissipativity-based filtering for fuzzy switched systems with stochastic perturbation. *IEEE Transactions on Automatic Control*, 61(6), 1694–1699.

Tang, W., Wang, Z., Wang, Y., Raïssi, T., and Shen, Y. (2019). Interval estimation methods for discrete-time linear time-invariant systems. *IEEE Transactions on Automatic Control*, 64(11), 4717–4724.

Vu, L. and Morgansen, K. (2010). Stability of time-delay feedback switched linear systems. *IEEE Transactions on Automatic Control*, 55(10), 2385–2390.

Wang, Z., Lim, C., Shi, P., and Shen, Y. (2017). H_-/L_∞ fault detection observer design for linear parameter-varying systems. In *the 20th World Congress of the International Federation of Automatic Control*, 5137–5142, Toulouse, France. IEEE.

Yang, J., Chen, Y., Zhu, F., and Wang, F. (2017). Simultaneous state and output disturbance estimations for a class of switched linear systems with unknown inputs. *International Journal of Systems Science*, 48(1), 22–33.

Zhang, W., Wang, Z., Raïssi, T., Wang, Y., and Shen, Y. (2020). A state augmentation approach to interval fault estimation for descriptor systems. *European Journal of Control*, 51, 19–29.

Zhao, X., Zhang, L., Shi, P., and Liu, M. (2012). Stability and stabilization of switched linear systems with mode-dependent average dwell time. *IEEE Transactions on Automatic Control*, 57(7), 1809–1815.