On Lyapunov-Lur'e functional based stability criterion for discrete-time Lur'e systems

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Abstract: In this manuscript, we consider the stability problem of Lur'e systems with slope-restricted nonlinearities. We focus on a specific parametrisation of the Lyapunov-Lur'e functional in the literature, and extend it to a higher order. Meanwhile, we show this Lyapunov-Lur'e functional based stability criterion is equivalent to the search for noncausal FIR multipliers with a restricted form for the SISO case. Finally, we discuss the restrictions of this Lyapunov-Lur'e functional approach with some numerical examples.

Keywords: Nonlinear systems, Absolute stability, Lyapunov-Lur'e functional, Zames-Falb multipliers

1. INTRODUCTION

This manuscript focuses on the stability of a Lur'e system (Lurie and Postnikov, 1944) in discrete-time, as shown in Fig. 1, where the plant G is linear time-invariant (LTI) and stable, and the nonlinearity (uncertainty) operator ϕ is memoryless, sector-bounded in the range $[0, \Psi]$ and slope-restricted in the range $[0, \Gamma]$. A classic problem is to find the largest[1] possible Ψ and Γ that preserve the stability, and this problem is still open due to the lack of necessary and sufficient condition.
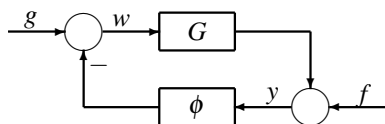

Fig. 1. The Lur'e system.

The stability of Lur'e systems is mainly studied in two frameworks: Lyapunov stability and input-output stability. Asymptotic stability is studied in the Lyapunov framework, which is based on internal state variables of the unforced systems; ℓ_2-stability is studied in the input-output approach, which is based on the input-output mapping of the forced systems. As asymptotic stability and ℓ_2 stability imply each other (Vidyasagar, 2002), it can be instructive to explore the relation between the two approaches. The well-known Popov theorem and small-gain theorem can be constructed in both frameworks (Khaill, 2002; Vidyasagar, 2002), but both of them are conservative in general.

In the Lyapunov approach, the most recent results are provided by Park et al. (2019), where a Lyapunov-Lur'e functional is used for systems with sector and slope restricted nonlinearities. In a Lyapunov-Lur'e functional, a Lur'e term is added to the original quadratic term:

$$\int_{\sigma_1}^{\sigma_2} \phi_j(\sigma)d\sigma \geq 0. \qquad (1)$$

where the operator ϕ_j is the jth element of a MIMO operator ϕ and signals σ_1 and σ_2 are inputs to ϕ_j. Furthermore, the sector and slope conditions can be utilised in the Lur'e term by extending (1) as

$$\int_{\sigma_1}^{\sigma_2} (\psi_j - \phi_j(\sigma))d\sigma \geq 0 \text{ and } \int_{\sigma_1}^{\sigma_2} (\gamma_j - \phi_j(\sigma))d\sigma \geq 0, \quad (2)$$

where ψ_j and γ_j are the jth diagonal elements of Ψ and Γ respectively. With the Lyapunov-Lur'e functional with (1) and (2), three main facts are considered to influence the conservativeness by Park et al. (2019). First, as studied in Park and Kim (1998); Ahmad et al. (2013); Park et al. (2015), a tighter estimation of the upper bound and lower bound of (1) deserves a less-conservative stability condition. Second, it is a natural idea to obtain less-conservative results by including more data points. In other words, more combinations of y_i, y_{i+1}, y_{i+2}, etc. are desired to be included in (1) (2). Finally, some other sector and slope conditions can be added to the final stability LMI. Similar analysis for the continuous-time case is in Turner and Kerr (2014), where the $\mathcal{L}_2$ gain bounds are also provided.

In the input-output approach, recent results are provided by Wang et al. (2014); Fetzer and Scherer (2017); Carrasco et al. (2020), which deals with SISO and MIMO case respectively. Particularly, the search on a wide subclass of LTI Zames-Falb multipliers (Willems and Brockett, 1968), called finite-impulse-response (FIR) multipliers, are proposed by Carrasco et al. (2020). The search on the multiplier M is conducted in frequency domain,

$$Re\{M(z)(1 + \Gamma G(z))\} > 0 \quad \forall |z| = 1. \qquad (3)$$

In contrast to Lyapunov function and continuous-time Zames-Falb multipliers (see the tutorial paper Carrasco et al. (2016)), the class of discrete Zames-Falb multipliers is given as a necessary and sufficient condition (Willems and Brockett, 1968), leading to complete argument which has been used to establish a new conjecture (Wang et al.,

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[1] The definition of largest is not standard for MIMO system.

2018), where the existence of a Zames-Falb multipliers is a necessary and sufficient condition for stability.

In this paper, we provide an insight into the parametrisation of the Lyapunov-Lur'e functional by Park et al. (2019) and its relation to the multiplier search by Carrasco et al. (2020). The main results and contributions are in Section 3. First, we construct a Lyapunov-Lur'e functional with more information based on the analysis by Park et al. (2019). Specifically, we further include the sector condition at y_{i+2}, and the slope conditions between (y_i, y_{i+2}) and (y_{i+1}, y_{i+2}) in the Lur'e terms. Meanwhile, we add the sector condition at y_{i+3}, and the slope conditions between (y_i, y_{i+3}) to construct the LMI. Then, we interpret this parametrisation of Lyapunov-Lur'e functional in frequency domain, and show that it is equivalent to the search on a restricted form of third order FIR multipliers in the SISO case. Finally, some restrictions on the Lyapunov-Lur'e functional stability criterion are discussed. In Section 4, a few examples verify the relations between the two criteria.

2. PRELIMINARIES

Consider the feedback interconnection in Fig. 1, the LTI system G has the state-space representation

$$G(z) \sim \begin{cases} x_{i+1} = Ax_i - B\phi(y_i) \\ y_i = Cx_i \end{cases}, \qquad (4)$$

where $x_i \in \mathbb{R}^n$ and $y_i \in \mathbb{R}^m$ are the state and output of G at time instant i respectively; the matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, and A is Schur. With some abuse of notation, the memoryless nonlinearity operator ϕ is defined by a static multivariable function $\phi : \mathbb{R}^m \mapsto \mathbb{R}^m$ is sector bounded and slope restricted, i.e.

$$\phi(y_i) \equiv \begin{bmatrix} \phi_1(y_i^1) & \phi_2(y_i^2) & \cdots & \phi_m(y_i^m) \end{bmatrix}^T, \qquad (5)$$

$$0 \leq \frac{\phi_j(\sigma)}{\sigma} \leq \psi_j \quad \forall \sigma \neq 0, \qquad (6a)$$

$$0 \leq \frac{\phi_j(\sigma_2) - \phi_j(\sigma_1)}{\sigma_2 - \sigma_1} \leq \gamma_j \quad \forall \sigma_1 \neq \sigma_2, \qquad (6b)$$

where ψ_j and γ_j $(j = 1, 2, \cdots, m)$ are the jth elements in the positive diagonal matrices Ψ and Γ respectively. We denote nonlinearities that satisfy (6) as $\phi \in [0, \Psi]$ and $\phi \in S[0, \Gamma]$.

The expression $G^*(z)$ denotes the complex conjugate transpose of $G(z)$ on $|z| = 1$, i.e. $G^*(z) = G^T\left(\frac{1}{z}\right)$, where the superscript T indicates the transpose. For a matrix M, $He\{M\} = M + M^T$; $M > 0$ means M is positive definite; $\star$ represents terms of a symmetric matrix which can be inferred by symmetry.

Some preliminaries in Park et al. (2019) are repeated here for completeness, and see details there and the references therein.

Lemma 1. (Park et al. (2015)). For the a nonlinearity ϕ in (5) and satisfying (6), lower and upper bounds of the Lur'e term are given by $L \leq \int_{\sigma_1}^{\sigma_2} \phi_j(\sigma) d\sigma \leq U$, where

$$L \equiv \phi_j(\sigma_1)(\sigma_2 - \sigma_1) + \frac{1}{2\gamma_j}\left\{\phi_j(\sigma_2) - \phi_j(\sigma_1)\right\}^2, \qquad (7a)$$

$$U \equiv \phi_j(\sigma_2)(\sigma_2 - \sigma_1) - \frac{1}{2\gamma_j}\left\{\phi_j(\sigma_2) - \phi_j(\sigma_1)\right\}^2. \qquad (7b)$$

Moreover, following Park et al. (2019) conditions in (6) can be rewritten as follows

$$\phi_j(\sigma)\left\{\sigma - \frac{1}{\psi_i}\phi_j(\sigma)\right\} \geq 0, \qquad (8a)$$

$$[\phi_j(\sigma_2) - \phi_j(\sigma_1)]\left\{(\sigma_2 - \sigma_1) - \frac{1}{\gamma_i}[\phi_j(\sigma_2) - \phi_j(\sigma_1)]\right\} \geq 0, \qquad (8b)$$

for any σ, σ_1, and σ_2. In particular, if setting $\sigma_2 = y_{i+n}$ and $\sigma_1 = y_i$ in (8b), this inequality implies a nth step (order) relation.

The details to parametrise a Lyapunov-Lur'e functional candidate will be provided as the main results in Section 3.

We consider a subclass of discrete-time LTI Zames-Falb multipliers (see definition of Zames-Falb multipliers in Willems and Brockett (1968)).

Definition 1. (FIR multipliers (Carrasco et al., 2020)). The convolution operator M is a noncausal FIR Zames-Falb multipliers if

$$M(z) = -h_{n_b}z^{-n_b} \cdots - h_1 z^{-1} + h_0 - h_{-1}z \cdots - h_{-n_f}z^{n_f}, \quad (9)$$

where the causal part is with the backward-shift operator z^{-i_b} $(i_b = 1, 2, \cdots, n_b)$, and the anticausal part is with the forward-shift operator z^{i_f} $(i_f = 1, 2, \cdots, n_f)$. The coefficients $h_{-i_f} > 0$, $h_{i_b} > 0$, and satisfy $\sum_{i_b=1}^{n_b} h_{i_b} + \sum_{i_f=1}^{n_f} h_{-i_f} \leq h_0$, where we can set $h_0 = 1$ without loss of generality.

3. MAIN RESULTS

3.1 Lyapunov-Lur'e stability criterion

In spirit of Park et al. (2019), the theorem below guarantees the stability of Lur'e systems with slope-restricted nonlinearities.

Theorem 1. Consider the feedback system in Fig. 1, with G in (4), and ϕ in (5) satisfies (6), i.e. G is stable and $G \sim [A \ B; C \ 0]$, $\phi \in [0, \Psi]$ and $\phi \in S[0, \Gamma]$. The feedback system is stable if there exist a symmetric matrix $P \in \mathbb{R}^{(3n+3m) \times (3n+3m)}$, positive diagonal matrices $M_{11}, M_{12}, M_{21}, M_{22}, M_{31}, M_{32}, N_{11}, N_{12}, N_{21}, N_{22}. N_{31}, N_{32}, L_0, L_1, L_2, L_3, S_1, S_2, S_3 \in \mathbb{R}^{m \times m}$, such that

$$\widehat{P} = P + \Xi > 0, \qquad (10)$$

$$\widehat{\Omega} = \Omega_1 + \Omega_2 + \Omega_3 < 0, \qquad (11)$$

where $\Xi = [\Xi]_{6 \times 6}$ is defined in (12), and Ω_1, Ω_2, Ω_3 are defined in (13), (14), (15) respectively.

$$\Xi_{11} = C^T(M_{12} + M_{32})\Gamma C + C^T N_{12}\Psi C; \quad \Xi_{21} = -C^T M_{12}\Gamma C;$$

$$\Xi_{22} = C^T(M_{12} + M_{22})\Gamma C + C^T N_{22}\Psi C; \quad \Xi_{31} = -C^T M_{32}\Gamma C;$$

$$\Xi_{32} = -C^T M_{22}\Gamma C; \quad \Xi_{33} = C^T(M_{22} + M_{32})\Gamma C + C^T N_{32}\Psi C;$$

$$\Xi_{41} = -(M_{12} + M_{32} + N_{12})C; \quad \Xi_{42} = M_{12}C; \quad \Xi_{43} = M_{32}C;$$

$$\Xi_{44} = (M_{11} + M_{12} + M_{31} + M_{32} + N_{11} + N_{12})\Gamma^{-1}; \Xi_{51} = M_{12}C;$$

$$\Xi_{52} = -(M_{12} + M_{22} + N_{22})C; \quad \Xi_{54} = -(M_{11} + M_{12})\Gamma^{-1};$$

$$\Xi_{53} = M_{22}C; \quad \Xi_{55} = (M_{11} + M_{12} + M_{21} + M_{22} + N_{21} + N_{22})\Gamma^{-1};$$

$$\Xi_{61} = M_{32}C; \quad \Xi_{62} = M_{22}C; \quad \Xi_{63} = -(M_{22} + M_{32} + N_{32})C;$$

$$\Xi_{64} = -(M_{31} + M_{32})\Gamma^{-1}; \quad \Xi_{65} = -(M_{21} + M_{22})\Gamma^{-1};$$

$$\Xi_{66} = (M_{21} + M_{22} + M_{31} + M_{32} + N_{31} + N_{32})\Gamma^{-1}. \quad (12)$$

$$\Omega_1 = \begin{bmatrix} & & & \vdots & \star \\ & & & \vdots & \star \\ & & & \vdots & \star \\ & -P-\Xi & & \vdots & \star \\ & & & \vdots & \star \\ & & & \vdots & \star \\ & & & \vdots & \star \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}. \tag{13}$$

Proof. See the sketch of the proof in Appendix A.

Remark 1. The matrix Ω is expressed as the summation of Ω_1, Ω_2, Ω_3, and some terms are added and subtracted at the same time. This particular expression is not required for Theorem 1, but it will be used in the next part.

Remark 2. Compared to the most recent Lyapunov literature (Park et al. (2019)), we further include conditions involving three future steps in the proof, i.e. y_{i+1}, y_{i+2}, y_{i+3}; hence it implies Theorem 1 is a third order technique.

3.2 Frequency domain interpretation for SISO case

In this part, we consider a special case of Theorem 1 with $m=1$ (SISO) and $\Psi = \Gamma$. Motivated by the analysis in Ahmad et al. (2015), the stability conditions are converted to a frequency domain inequality via the KYP lemma (Rantzer (1996)).

Theorem 2. Consider the feedback system in Fig. 1 in the SISO case. If conditions in Theorem 1 are satisfied with some $\Psi = \Gamma$, then there exists an FIR Zames-Falb multiplier in the form

$$M(z) = -h_3 z^{-3} - h_2 z^{-2} - h_1 z^{-1} + h_0 - h_{-1} z - h_2 z^2 - h_3 z^3, \tag{16}$$

such that $Re\{M(z)(1+\Gamma G(z))\} > 0$, $\forall |z| = 1$.

Proof. In (11), $\Omega_1 + \Omega_2$ can be written as

$$\Omega_1 + \Omega_2 = \begin{bmatrix} \widehat{A}^T \widehat{P} \widehat{A} - \widehat{P} & \widehat{A}^T \widehat{P} \widehat{B} \\ \star & \widehat{B}^T \widehat{P} \widehat{B} \end{bmatrix}, \tag{17}$$

where $\widehat{P}$ is defined in (10), and the state-space matrices are

$$\widehat{A} = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & A & 0 & 0 & -B \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix},$$

which correspond to the augmented state

$$\widehat{x}_i = \begin{bmatrix} x_i^T & x_{i+1}^T & x_{i+2}^T & \phi(y_i)^T & \phi^T(y_{i+1}) & \phi^T(y_{i+2}) \end{bmatrix}^T.$$

Then, applying the KYP lemma, LMI (11) can be converted to the frequency domain inequality

$$\begin{bmatrix} (zI-\widehat{A})^{-1}\widehat{B} \\ I \end{bmatrix}^* \Omega_3 \begin{bmatrix} (zI-\widehat{A})^{-1}\widehat{B} \\ I \end{bmatrix} < 0, \quad \forall |z| = 1. \tag{18}$$

Substituting Ω_3 inside, (18) is equivalent to

$$He\{L_S(G(z)+\Psi^{-1}) + M(z)(G(z)+\Gamma^{-1})\} > 0, \quad \forall |z| = 1, \tag{19}$$

where $L_S = L_0 + L_1 + L_2 + L_3$, and $M(z)$ is in (16) with

$$h_0 = 2(M_{11} + M_{12} + M_{21} + M_{22} + M_{31} + M_{32} + S_1 + S_2 + S_3) + N_{11} + N_{12} + N_{21} + N_{22} + N_{31} + N_{32} > 0;$$
$$h_1 = M_{11} + M_{12} + M_{21} + M_{22} + N_{11} + N_{21} + N_{31} + S_1 > 0;$$
$$h_{-1} = M_{11} + M_{12} + M_{21} + M_{22} + N_{12} + N_{22} + N_{32} + S_1 > 0;$$
$$h_2 = M_{31} + M_{32} + S_2 > 0; \quad h_3 = S_3 > 0. \tag{20}$$

It is clear that $h_0 = h_{-1} + h_1 + 2h_2 + 2h_3$, so $M(z)$ here is a third order noncausal FIR multiplier in (9) with $n_b = n_f = 3$.

3.3 Discussions

Using insights provided by Theorem 2, the following proposition can be used to simplify Theorem 1 for SISO systems.

Proposition 1. Consider the feedback system in Fig. 1 in the SISO case. If conditions in Theorem 1 are satisfied with some $\Psi = \Gamma$, then it is also satisfied with M_{11}, M_{12}, M_{21}, M_{22}, M_{31}, M_{32}, N_{21}, N_{22}. N_{31}, N_{32}, L_0, L_1, S_1 being zero.

Proof. By Theorem 2, it is without loss of generality to set $h_1 = N_{11}$, $h_{-1} = N_{12}$, $h_2 = S_2$, $h_3 = S_3$ in (9) from the multiplier point of view. Equivalently, other variables in the multiplier can be set as zero. In addition, the sector conditions at y_i and y_{i+1} are included in $V^{3,1}$ and $\Delta V^{3,1}$, and N_{11}, N_{12} are nonzero, then L_0 and L_1 can be set as zero.

Furthermore, in (9), the coefficients of second order terms: h_2 and h_{-2}, and the coefficients of third order terms: h_3 and h_{-3}, can have different values respectively. However, the restricted form $h_2 = h_{-2}$, $h_3 = h_{-3}$, is forced in (16). As a result, the search on second and third order terms is restricted and may fail, then the multiplier would perform as a first order multiplier.

4. NUMERICAL RESULTS

In Park et al. (2019), its results are proved to be less conservative than other Lyapunov literature, so the comparison of different Lyapunov criteria are not repeated. In this manuscript, we focus on Lyapunov-Lur'e functional and its relation to FIR multipliers, so we introduce some special examples to stress the relation, which are not included in other existing literature. We assume that the nonlinearity is repeated, so each ϕ_j has the same bound k, i.e. $\psi_j = \gamma_j = k$, $\forall j$.

Examples

(1) $G_1(z) = \frac{0.1z}{z^2 - 1.8z + 0.81}$

(2) $G_2(z) = \frac{2z + 0.92}{z^2 - 0.5z}$

(3) $G_3(z) = \frac{0.2343z^2 + 0.1224z + 0.04805}{z^3 + 1.611z^2 + 1.065z + 0.08843}$

(4) $G_4(z) = \frac{1.341z^4 - 1.221z^3 + 0.6285z^2 - 0.5618z + 0.1993}{z^5 - 0.935z^4 + 0.7697z^3 - 1.118z^2 + 0.6917z - 0.1352}$

(5) $G_5(z) = \frac{z^4 - 1.5z^3 + 0.5z^2 - 0.5z + 0.5}{4.4z^5 - 8.957z^4 + 9.893z^3 - 5.671z^2 + 2.207z - 0.5}$

(6) $G_6(z) = \frac{2.685z^5 + 1.404z^4 - 1.399z^3 - 0.7952z^2 + 0.4009z + 0.008153}{z^6 + 0.4768z^5 - 0.4619z^4 - 0.383z^3 + 0.1349z^2 + 0.04182z - 0.009093}$

(7) $G_7(z) = \begin{bmatrix} \frac{0.2}{z - 0.98} & \frac{-0.2}{z - 0.92} \\ \frac{0.3}{z - 0.97} & \frac{0.1}{z - 0.91} \end{bmatrix}$

The numerical results are listed in Table 1, which are obtained using SDP toolbox Yalmip (Löfberg, 2004) with

$$\Omega_2 = \begin{bmatrix} 0 & \star & \star & \star & \star & \star & \star \\ 0 & \begin{matrix}C^T(\bar{M}_{12}+\bar{M}_{32})\Gamma C \\ +C^TN_{12}\Psi C+P_{11}\end{matrix} & \star & \star & \star & \star & \star \\ 0 & \begin{matrix}-C^TM_{12}\Gamma C \\ -A^TC^TM_{32}\Gamma C \\ +P_{21}+A^TP_{31}\end{matrix} & \begin{matrix}C^T(\bar{M}_{12}+\bar{M}_{22})\Gamma C \\ +A^TC^T(M_{22}+M_{32})\Gamma CA \\ +C^TN_{22}\Psi C+A^TC^TN_{32}\Psi CA \\ -He\{A^TC^TM_{22}\Gamma C\}+P_{22} \\ +He\{A^TP_{32}\}+A^TP_{33}A\end{matrix} & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star & \star \\ 0 & -\begin{pmatrix}M_{12}+M_{32}\\+N_{12}\\+P_{41}\end{pmatrix}C & \begin{matrix}M_{12}C+M_{32}CA\\+P_{42}+P_{43}A\end{matrix} & 0 & \begin{pmatrix}\bar{M}_{11}+\bar{M}_{12}\\+\bar{M}_{31}+\bar{M}_{32}\\+N_{11}+N_{12}\\+P_{44}\end{pmatrix}\Gamma^{-1} & \star & \star \\ 0 & \begin{matrix}B^TC^TM_{32}\Gamma C\\+M_{12}C+P_{51}-B^TP_{31}\end{matrix} & \begin{matrix}-(M_{12}+M_{22}+N_{22})C\\+M_{22}CA+B^TC^TM_{22}\Gamma C\\-B^TC^T(M_{22}+M_{32})\Gamma CA\\-B^TC^TN_{32}\Psi CA+P_{52}\\-B^TP_{32}-B^TP_{33}A+P_{53}A\end{matrix} & 0 & \begin{matrix}-B^TC^TM_{32}\\-(\bar{M}_{11}+\bar{M}_{12})\Gamma^{-1}\\+P_{54}-B^TP_{43}^T\end{matrix} & \begin{matrix}B^TC^T(\bar{M}_{22}+\bar{M}_{32})\Gamma CB\\+B^TC^TN_{32}\Psi CB\\-He\{M_{22}CB\}\\+\begin{pmatrix}\bar{M}_{11}+\bar{M}_{12}\\+\bar{M}_{21}+\bar{M}_{22}\\+N_{21}+N_{22}\end{pmatrix}\Gamma^{-1}\\+P_{55}-He\{P_{53}B\}\\+B^TP_{33}B\end{matrix} & \star \\ 0 & M_{32}C+P_{61} & \begin{matrix}M_{32}C\\-(M_{22}+M_{32}+N_{32})CA\\+P_{62}+P_{63}A\end{matrix} & 0 & -(\bar{M}_{31}+\bar{M}_{32})\Gamma^{-1}+P_{64} & \begin{matrix}(M_{22}+M_{32}+N_{32})CB\\-(\bar{M}_{21}+\bar{M}_{22})\Gamma^{-1}\\+P_{65}-P_{63}B\end{matrix} & \begin{pmatrix}\bar{M}_{21}+\bar{M}_{22}\\+\bar{M}_{31}+\bar{M}_{32}\\+N_{31}+N_{32}\end{pmatrix}\Gamma^{-1}+P_{66} \end{bmatrix}. \quad (14)$$

$$\Omega_3 = \begin{bmatrix} 0 & \star & \star & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star & \star \\ \begin{pmatrix}L_0+S_1\\+S_2+S_3\end{pmatrix}C & -N_{12}C-S_1C & -S_2C-S_3CA & \begin{matrix}-2\begin{pmatrix}S_1+S_2\\+S_3\end{pmatrix}\Gamma^{-1}\\-2L_0\Psi^{-1}\end{matrix} & \star & \star & \star \\ -N_{11}C-S_1C & \begin{pmatrix}\bar{M}_{11}+\bar{M}_{12}\\+\bar{M}_{31}+\bar{M}_{32}\\+N_{11}+N_{22}\\+(L_1+S_1)\end{pmatrix}C & -\begin{pmatrix}\bar{M}_{11}+\bar{M}_{12}\\+N_{22}\end{pmatrix}C-(\bar{M}_{31}+\bar{M}_{32})CA & \begin{matrix}(N_{11}+N_{12})\Gamma^{-1}\\+2S_1\Gamma^{-1}\end{matrix} & \begin{matrix}-2\begin{pmatrix}\bar{M}_{11}+\bar{M}_{12}\\+\bar{M}_{31}+\bar{M}_{32}\\+N_{11}+N_{12}\end{pmatrix}\Gamma^{-1}\\-2S_1\Gamma^{-1}-2L_1\Psi^{-1}\end{matrix} & \star & \star \\ -S_2C & -\begin{pmatrix}\bar{M}_{11}+\bar{M}_{12}\\+N_{21}\end{pmatrix}C & \begin{matrix}\begin{pmatrix}\bar{M}_{11}+\bar{M}_{12}\\+\bar{M}_{21}+\bar{M}_{22}\\+N_{21}+N_{22}\end{pmatrix}C\\-\begin{pmatrix}\bar{M}_{21}+\bar{M}_{22}\\+N_{32}\end{pmatrix}CA\\+(L_2+S_2)C\end{matrix} & 2S_2\Gamma^{-1}+S_3CB & \begin{matrix}B^TC^T(\bar{M}_{31}+\bar{M}_{32})\\\begin{pmatrix}2\bar{M}_{11}+2\bar{M}_{12}\\+N_{21}+N_{22}\end{pmatrix}\Gamma^{-1}\end{matrix} & \begin{matrix}He\left\{\begin{pmatrix}\bar{M}_{21}+\bar{M}_{22}\\+N_{32}\end{pmatrix}CB\right\}\\-2\begin{pmatrix}\bar{M}_{11}+\bar{M}_{12}\\+\bar{M}_{21}+\bar{M}_{22}\\+N_{21}+N_{22}\end{pmatrix}\Gamma^{-1}\\-2S_2\Gamma^{-1}-2L_2\Psi^{-1}\end{matrix} & \star \\ -S_3C & -(\bar{M}_{31}+\bar{M}_{32})C & \begin{matrix}-\begin{pmatrix}\bar{M}_{21}+\bar{M}_{22}\\+N_{31}\end{pmatrix}C\\+\begin{pmatrix}\bar{M}_{21}+\bar{M}_{22}\\+\bar{M}_{31}+\bar{M}_{32}\\+N_{31}+N_{32}\end{pmatrix}CA\\+(L_3+S_3)C\end{matrix} & 2S_3\Gamma^{-1} & 2(\bar{M}_{31}+\bar{M}_{32})\Gamma^{-1} & \begin{matrix}-\begin{pmatrix}\bar{M}_{21}+\bar{M}_{22}\\+\bar{M}_{31}+\bar{M}_{32}\\+N_{31}+N_{32}\end{pmatrix}CB\\-(L_3+S_3)CB\\\begin{pmatrix}2\bar{M}_{21}+2\bar{M}_{22}\\+N_{31}+N_{32}\end{pmatrix}\Gamma^{-1}\end{matrix} & \begin{matrix}-2\begin{pmatrix}\bar{M}_{21}+\bar{M}_{22}\\+\bar{M}_{31}+\bar{M}_{32}\\+N_{31}+N_{32}\end{pmatrix}\Gamma^{-1}\\-2S_3\Gamma^{-1}-2L_3\Psi^{-1}\end{matrix} \end{bmatrix}. \quad (15)$$

Table 1. Maximum slope bound by multiplier and Lyapunov approaches

	Ex (1)	Ex (2)	Ex (3)	Ex (4)	Ex (5)	Ex (6)	Ex (7)
Carrasco et al. (2020) ($n_f = n_b = 1$)	12.9960	0.9108	0.6659	0.1695	2.5904	0.3468	N/A
Carrasco et al. (2020) ($n_f = n_b = 3$)	12.9960	0.9115	1.3651	0.4347	3.2253	0.4088	N/A
Park et al. (2019)	12.9960	0.9108	0.6659	0.1695	2.5904	0.3468	3.8091
Theorem 1	12.9960	0.9108	0.6659	0.1695	2.5904	0.3468	3.8091
Proposition 1	12.9960	0.9108	0.6659	0.1695	2.5904	0.3468	3.8052
Nyquist Value	36.1000	1.0870	2.2855	1.1766	7.9070	0.4938	3.8500

solver sdpt3 (Tütüncü et al., 2003). The multiplier results are obtained by the search of FIR multipliers in (Carrasco et al., 2020). In particular, Example 1 has been used to show that Kalman conjecture is false for second order plants (Heath et al., 2015). The results by Park et al. (2019) are included for comparison.

As shown in the table, for SISO systems (Ex.(1)-(6)), Theorem 1 can be simplified as Proposition 1 without increas-

ing the conservativeness, but not in the MIMO case (Ex (7)). However, the stability criteria based on Lyapunov-Lur'e functionals (Park et al. (2019) and Theorem 1) are possible to deteriorate as first order FIR multipliers, although Theorem 1 is proved equivalent to a restricted form of third order multipliers.

In particular, we take Ex.(2) as an example to show the difference on resultant multipliers. By Carrasco et al.

(2020), the maximum slope is 0.9115, where the third order multiplier is

$$M(z) = -0.8325z^{-3} - 0.0155z^{-1} + 1 - 0.1232z^2 - 0.0287z^3,$$

whose parameters are asymmetric, and the third order causal term is dominant. In contrast, the equivalent third order multiplier obtained by Theorem 1 is

$$M(z) = -0.11424z^{-3} - 0.11424z^{-2} - 0.4288z^{-1}$$
$$+ 1 - 0.11424z - 0.11424z^2 - 0.11424z^3,$$

where the coefficients are scaled to make $h_0 = 1$. Due to the symmetry on second and third order coefficients, the first order causal term becomes dominant. Hence, this multiplier would be considered as first order, with the maximum slope 0.9108.

Similarly, the additional information with third order relations, such as the slope condition between y_i and y_{i+3} in Theorem 1, compared to Park et al. (2019), did not reduce the conservativeness, because the equivalent multipliers are forced to be first order.

In conclusion, it is important to improve the parametrisation of Lyapunov-Lur'e functionals to break the restriction on its equivalent FIR multipliers. On the other hand, the relations for the MIMO case is still open.

5. CONCLUSION

In this manuscript, we extended the Lyapunov-Lur'e functional in Park et al. (2019), and demonstrate it is equivalent to a restricted form of third order noncausal FIR multipliers for SISO systems. However, the Lyapunov-Lur'e functional performed as first order noncausal FIR multipliers in the numerical examples, which implies the necessity to break the symmetry in the parametrisation in future work.

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Appendix A. PROOF OF THEOREM 1

Without loss of generality, we assume the matrices $M_a \equiv \text{diag}(m_{1,a}, m_{2,a}, \cdots, m_{m,a})$, $N_b \equiv \text{diag}(n_{1,b}, n_{2,b}, \cdots, n_{m,b})$, where a and b are $11, 12, 21, 22, 31, 32$, and all entries are positive. In addition, we assume the matrix P is comprised of submatrices $P_{p,q}$ with appropriate dimensions, where p and q are $1, 2, \ldots, 6$.

First, we consider the Lyapunov-Lur'e functional candidate,

$$V_i = V_i^1 + V_i^{2,1} + V_i^{2,2} + V_i^{2,3} + V_i^{3,1} + V_i^{3,2} + V_i^{3,3}, \quad (A.1)$$

where $V_i^1 = \eta_i^T P \eta_i$;

$$V_i^{2,1} = 2 \sum_{j=1}^m m_{j,11} \int_{y_{j,i}}^{y_{j,i+1}} \left\{ \phi_j(\sigma) - \phi_j(y_{j,i}) \right\}^T d\sigma$$

$$+ 2 \sum_{j=1}^m m_{j,12} \int_{y_{j,i}}^{y_{j,i+1}} \left\{ \gamma_j(\sigma - y_{j,i}) - (\phi_j(\sigma) - \phi_j(y_{j,i})) \right\}^T d\sigma;$$

$$V_i^{2,2} = 2 \sum_{j=1}^m m_{j,21} \int_{y_{j,i+1}}^{y_{j,i+2}} \left\{ \phi_j(\sigma) - \phi_j(y_{j,i+1}) \right\}^T d\sigma$$

$$+ 2 \sum_{j=1}^m m_{j,22} \int_{y_{j,i+1}}^{y_{j,i+2}} \left\{ \gamma_j(\sigma - y_{j,i+1}) - (\phi_j(\sigma) - \phi_j(y_{j,i+1})) \right\}^T d\sigma;$$

$$V_i^{2,3} = 2\sum_{j=1}^{m} m_{j,31} \int_{y_{j,i}}^{y_{j,i+2}} \left\{ \phi_j(\sigma) - \phi_j(y_{j,i}) \right\}^T d\sigma$$

$$+ 2\sum_{j=1}^{m} m_{j,32} \int_{y_{j,i}}^{y_{j,i+2}} \left\{ \gamma_j(\sigma - y_{j,i}) - (\phi_j(\sigma) - \phi_j(y_{j,i})) \right\}^T d\sigma;$$

$$V_i^{3,1} = 2\sum_{j=1}^{m} n_{j,11} \int_0^{y_{j,i}} \left\{ \phi_j(\sigma) \right\}^T d\sigma + 2\sum_{j=1}^{m} n_{j,12} \int_0^{y_{j,i}} \left\{ \psi_j\sigma - \phi_j(\sigma) \right\}^T d\sigma;$$

$$V_i^{3,2} = 2\sum_{j=1}^{m} n_{j,21} \int_0^{y_{j,i+1}} \left\{ \phi_j(\sigma) \right\}^T d\sigma + 2\sum_{j=1}^{m} n_{j,22} \int_0^{y_{j,i+1}} \left\{ \psi_j\sigma - \phi_j(\sigma) \right\}^T d\sigma;$$

$$V_i^{3,3} = 2\sum_{j=1}^{m} n_{j,31} \int_0^{y_{j,i+2}} \left\{ \phi_j(\sigma) \right\}^T d\sigma + 2\sum_{j=1}^{m} n_{j,32} \int_0^{y_{j,i+2}} \left\{ \psi_j\sigma - \phi_j(\sigma) \right\}^T d\sigma;$$

with $\eta_i = \left[x_i^T\ x_{i+1}^T\ x_{i+2}^T\ \phi(y_i)^T\ \phi^T(y_{i+1})\ \phi^T(y_{i+2}) \right]^T$.

By Lemma 1, the following inequalities hold.

$$V_i^{2,1} \geq (y_{i+1} - y_i)M_{12}\Gamma(y_{i+1} - y_i)$$
$$+ He\left\{ \phi^T(y_i)M_{12}(y_{i+1} - y_i) - \phi^T(y_{i+1})M_{12}(y_{i+1} - y_i) \right\}$$
$$+ [\phi(y_{i+1}) - \phi(y_i)]^T (M_{11} + M_{12})\Gamma^{-1}[\phi(y_{i+1}) - \phi(y_i)];$$

$$V_i^{2,2} \geq (y_{i+2} - y_{i+1})M_{22}\Gamma(y_{i+2} - y_{i+1})$$
$$+ He\left\{ \phi^T(y_{i+1})M_{22}(y_{i+2} - y_{i+1}) - \phi^T(y_{i+2})M_{22}(y_{i+2} - y_{i+1}) \right\}$$
$$+ [\phi(y_{i+2}) - \phi(y_{i+1})]^T (M_{21} + M_{22})\Gamma^{-1}[\phi(y_{i+2}) - \phi(y_{i+1})];$$

$$V_i^{2,3} \geq (y_{i+2} - y_i)M_{32}\Gamma(y_{i+2} - y_i)$$
$$+ He\left\{ \phi^T(y_i)M_{32}(y_{i+2} - y_i) - \phi^T(y_{i+2})M_{32}(y_{i+2} - y_i) \right\}$$
$$+ [\phi(y_{i+2}) - \phi(y_i)]^T (M_{31} + M_{32})\Gamma^{-1}[\phi(y_{i+2}) - \phi(y_i)];$$

$$V_i^{3,1} \geq y_i^T N_{12}\Psi y_i - He\left\{ \phi^T(y_i)N_{12}y_i \right\}$$
$$+ \phi^T(y_i)(N_{11} + N_{12})\Gamma^{-1}\phi(y_i);$$

$$V_i^{3,2} \geq y_{i+1}^T N_{22}\Psi y_{i+1} - He\left\{ \phi^T(y_{i+1})N_{22}y_{i+1} \right\}$$
$$+ \phi^T(y_{i+1})(N_{21} + N_{22})\Gamma^{-1}\phi(y_{i+1});$$

$$V_i^{3,3} \geq y_{i+2}^T N_{32}\Psi y_{i+2} - He\left\{ \phi^T(y_{i+2})N_{32}y_{i+2} \right\}$$
$$+ \phi^T(y_{i+2})(N_{31} + N_{32})\Gamma^{-1}\phi(y_{i+2}).$$

The above inequalities can be written as

$$V_i^{2,1} + V_i^{2,2} + V_i^{2,3} + V_i^{3,1} + V_i^{3,2} + V_i^{3,3} \geq \eta_i^T \Xi \eta_i, \quad (A.2)$$

where Ξ is given in (12). Therefore, the lower bound of the Lyapunov-Lur'e functional (A.1) can be expressed as $\eta_i^T(P + \Xi)\eta_i$, and it is positive for any non-zero η_i if (10) holds.

Second, we consider the difference of (A.1)

$$\Delta V_i \equiv V_{i+1} - V_i = \Delta V_i^1 + \Delta V_i^{2,1} + \Delta V_i^{2,2} + \Delta V_i^{2,3}$$
$$+ \Delta V_i^{3,1} + \Delta V_i^{3,2} + \Delta V_i^{3,3}. \quad (A.3)$$

Similarly, the following inequalities hold by Lemma 1:

$$\Delta V_i^{2,1} \leq (y_{i+2} - y_{i+1})^T M_{12}\Gamma(y_{i+2} - y_{i+1})$$
$$- (y_{i+1} - y_i)^T M_{12}\Gamma(y_{i+1} - y_i)$$
$$+ 2\phi^T(y_{i+2})M_{11}(y_{i+2} - y_{i+1}) - 2\phi^T(y_{i+1})M_{11}(y_{i+2} - y_{i+1})$$
$$+ 2\phi^T(y_{i+1})M_{12}(y_{i+1} - y_i) - 2\phi^T(y_i)M_{12}(y_{i+1} - y_i)$$
$$- [\phi(y_{i+2}) - \phi(y_{i+1})]^T (M_{11} + M_{12})\Gamma^{-1}[\phi(y_{i+2}) - \phi(y_{i+1})]$$
$$- [\phi(y_{i+1}) - \phi(y_i)]^T (M_{11} + M_{12})\Gamma^{-1}[\phi(y_{i+1}) - \phi(y_i)];$$

$$\Delta V_i^{2,2} \leq (y_{i+3} - y_{i+2})^T M_{22}\Gamma(y_{i+3} - y_{i+2})$$
$$- (y_{i+2} - y_{i+1})^T M_{22}\Gamma(y_{i+2} - y_{i+1})$$
$$+ 2\phi^T(y_{i+3})M_{21}(y_{i+3} - y_{i+2}) - 2\phi^T(y_{i+2})M_{21}(y_{i+3} - y_{i+2})$$
$$+ 2\phi^T(y_{i+2})M_{22}(y_{i+2} - y_{i+1}) - 2\phi^T(y_{i+1})M_{22}(y_{i+2} - y_{i+1})$$
$$- [\phi(y_{i+3}) - \phi(y_{i+2})]^T (M_{21} + M_{22})\Gamma^{-1}[\phi(y_{i+3}) - \phi(y_{i+2})]$$
$$- [\phi(y_{i+2}) - \phi(y_{i+1})]^T (M_{21} + M_{22})\Gamma^{-1}[\phi(y_{i+2}) - \phi(y_{i+1})];$$

$$\Delta V_i^{2,3} \leq (y_{i+3} - y_{i+1})^T M_{32}\Gamma(y_{i+3} - y_{i+1})$$
$$- (y_{i+2} - y_i)^T M_{32}\Gamma(y_{i+2} - y_i)$$
$$+ 2\phi^T(y_{i+3})M_{31}(y_{i+3} - y_{i+1}) - 2\phi^T(y_{i+1})M_{31}(y_{i+3} - y_{i+1})$$
$$+ 2\phi^T(y_{i+2})M_{32}(y_{i+2} - y_i) - 2\phi^T(y_i)M_{32}(y_{i+2} - y_i)$$
$$- [\phi(y_{i+3}) - \phi(y_{i+1})]^T (M_{31} + M_{32})\Gamma^{-1}[\phi(y_{i+3}) - \phi(y_{i+1})]$$
$$- [\phi(y_{i+2}) - \phi(y_i)]^T (M_{31} + M_{32})\Gamma^{-1}[\phi(y_{i+2}) - \phi(y_i)];$$

$$\Delta V_i^{3,1} \leq y_{i+1}^T N_{12}\Psi y_{i+1} - y_i^T N_{12}\Psi y_i$$
$$+ 2\phi^T(y_{i+1})N_{11}(y_{i+1} - y_i) - 2\phi^T(y_i)N_{12}(y_{i+1} - y_i)$$
$$- [\phi(y_{i+1}) - \phi(y_i)]^T (N_{11} + N_{12})\Gamma^{-1}[\phi(y_{i+1}) - \phi(y_i)];$$

$$\Delta V_i^{3,2} \leq y_{i+2}^T N_{22}\Psi y_{i+2} - y_{i+1}^T N_{22}\Psi y_{i+1}$$
$$+ 2\phi^T(y_{i+2})N_{21}(y_{i+2} - y_{i+1}) - 2\phi^T(y_{i+1})N_{22}(y_{i+2} - y_{i+1})$$
$$- [\phi(y_{i+2}) - \phi(y_{i+1})]^T (N_{21} + N_{22})\Gamma^{-1}[\phi(y_{i+2}) - \phi(y_{i+1})];$$

$$\Delta V_i^{3,3} \leq y_{i+3}^T N_{32}\Psi y_{i+3} - y_{i+2}^T N_{32}\Psi y_{i+2}$$
$$+ 2\phi^T(y_{i+3})N_{31}(y_{i+3} - y_{i+2}) - 2\phi^T(y_{i+2})N_{32}(y_{i+3} - y_{i+2})$$
$$- [\phi(y_{i+3}) - \phi(y_{i+2})]^T (N_{31} + N_{32})\Gamma^{-1}[\phi(y_{i+3}) - \phi(y_{i+2})].$$

In addition, by relations in (8), it follows

$$0 \leq 2\phi^T(y_i)L_0 \left[y_i - \Psi^{-1}\phi(y_i) \right];$$
$$0 \leq 2\phi^T(y_{i+1})L_1 \left[y_{i+1} - \Psi^{-1}\phi(y_{i+1}) \right];$$
$$0 \leq 2\phi^T(y_{i+2})L_2 \left[y_{i+2} - \Psi^{-1}\phi(y_{i+2}) \right];$$
$$0 \leq 2\phi^T(y_{i+3})L_3 \left[y_{i+3} - \Psi^{-1}\phi(y_{i+3}) \right];$$
$$0 \leq 2(\phi^T(y_{i+1}) - \phi^T(y_i))S_1 \left[(y_{i+1} - y_i) - \Gamma^{-1}(\phi(y_{i+1}) - \phi(y_i)) \right];$$
$$0 \leq 2(\phi^T(y_{i+2}) - \phi^T(y_i))S_2 \left[(y_{i+2} - y_i) - \Gamma^{-1}(\phi(y_{i+2}) - \phi(y_i)) \right];$$
$$0 \leq 2(\phi^T(y_{i+3}) - \phi^T(y_i))S_3 \left[(y_{i+3} - y_i) - \Gamma^{-1}(\phi(y_{i+3}) - \phi(y_i)) \right].$$

Invoking inequalities above, the upper bound of ΔV_i in (A.3) is

$$\Delta V_i \leq \zeta_i^T \widehat{\Omega} \zeta_i, \quad (A.4)$$

where $\zeta_i = \left[x_i^T\ x_{i+1}^T\ x_{i+2}^T\ \phi^T(y_i)\ \phi^T(y_{i+1})\ \phi^T(y_{i+2})\ \phi^T(y_{i+3}) \right]^T$, and $\widehat{\Omega}$ is equivalent to the summation of $\Omega_1, \Omega_2, \Omega_3$ in (13), (14), (15) respectively. Therefore, ΔV_i is negative for any non-zero ζ_i if (11) holds.

Finally, it is clear that (10) and (11) is sufficient for the stability.