

# Unimodular Completion for Computation of Fractionally Flat Outputs for Linear Fractionally Flat Systems

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**Abstract:** The current paper presents a method for the computation of fractionally flat outputs for linear fractionally flat systems based on the notion of unimodular completion. This calculation method, which already exists for integer order non-linear flat systems, is extended for the class of fractionally linear flat systems by employing some fractional calculation properties. Two examples are used to validate the proposed extension.

*Keywords:* Flatness, unimodular completion, fractional calculus, linear systems, flat output.

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## 1. INTRODUCTION

Liouville (1832) and Riemann (1892) introduced fractional calculus as a generalization of the traditionally used ordinary calculus. However, it has been regarded solely as a theoretical notion until the discovery of physical systems that can be modelled by fractional differential equations, such as thermal systems (Battaglia et al. (2000)), nuclear magnetic resonance systems (Magin et al. (2008)) and viscoelastic systems (Moreau et al. (2002)). In this context, studies have shown that this notion of fractional calculation is also useful for a system's robust control, such as in the CRONE command (Oustaloup (1995)).

Additionally, a method based on the flatness property has also proven its efficiency in robust system control. Initially, this property of flatness has been introduced and developed for the class of non-linear integer systems (Martin et al. (1992); Levine (2009)) of the form

$$\dot{x} = f(x, u) \quad (1)$$

where  $x$  is the state variable and  $u$  is the input variable. Later, this has been extended to include the class of fractional linear systems (Melchior et al. (2007)) of the form:

$$x^{(\nu)} = Ax + Bu \quad (2)$$

where  $(\nu)$  is the fractional derivative and  $A$  and  $B$  are constant matrices. Roughly speaking, a system is said to be flat (resp. fractionally flat in the case of linear fractional systems (2)) if and only if all the system's variables can be expressed as functions of a new variable  $y$  and its successive derivatives called flat output (resp. fractionally flat output). A reference trajectory of the system is constructed using  $y$ : the reference trajectories of  $x$  and  $u$  are deduced from the reference trajectory of  $y$  without it being necessary to solve any differential equation.

Therefore, the main purpose of the flatness concept is the calculation of flat outputs. This has been implemented for both non-linear integer systems and linear fractional systems. On one hand, for the class of non-linear integer systems (1), two computational algorithms have been developed to compute flat outputs. The first one is based on the Smith Diagonal Decomposition of  $\frac{d}{dt}$ -polynomial matrices<sup>1</sup>, where  $\frac{d}{dt}$  is the time derivative operator, and for which a formal calculation tool has been developed by (Verhoeven (2016)). The second one is based on the notion of unimodular completion of  $\frac{d}{dt}$ -polynomial matrices (Fritzsche et al. (2016a); Franke and Röbenack (2013)). Both methods have proven to be effective in calculating flat outputs. However, the second provides, in some particular cases, a direct representation of flat outputs (i.e. without the need to make calculations) called direct flat representation (see Fritzsche et al. (2016b)). On the other hand, for the class of fractional linear systems (2), the computation algorithm of the fractionally flat outputs, developed by (Victor et al. (2015)), is also based on the Smith Diagonal Decomposition of  $\mathbf{D}^\gamma$ -polynomial matrices where  $\mathbf{D}^\gamma$  is the fractional operator.

In the present work, we show that the method of calculating flat outputs using unimodular completion can be extended for the class of fractional linear systems, using some properties of fractional calculus. We will also show that such systems admit, in particular cases, a fractionally direct flat representation, which allows calculating fractionally flat outputs without the need for intensive computations.

In this context, the paper is organized as follows: Section 2 presents some fractional calculus basic properties and the notion of linear fractionally flat systems. Section 3 introduces the unimodular completion algorithm for the

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<sup>1</sup> The entries of such matrices are polynomials in the operator  $\frac{d}{dt}$  with coefficients meromorphic functions (Levine (2009)).

computation of fractionally flat output. Afterwards, section 4 represents the particular case of fractionally direct flat representation. Finally, two examples emphasizing potential applications of the proposed method are developed in section 5.

## 2. FRACTIONAL LINEAR FLATNESS

### 2.1 Fractional Calculus

Let  $\gamma \in \mathbb{R}_+$  be a positive real number,  $n = \min\{k \in \mathbb{N} \mid k > \gamma\}$  the smallest integer greater than  $\gamma$  and  $\nu = n - \gamma \in [0, 1[$ . Let  $a \in \mathbb{R}$ , and  $\mathcal{C}^\infty([a, +\infty[)$  the set of infinitely continuously differentiable functions.

The fractional derivative, or Riemann-Liouville derivative, of order  $\gamma = n - \nu$  of a function  $f \in \mathcal{C}^\infty([a, +\infty[)$  at time  $t$ , denoted by  $\mathbf{D}_a^\gamma f$ , is defined by (Miller and Ross (1993)):

$$\begin{aligned} \mathbf{D}_a^\gamma f(t) &= \mathbf{D}^n(\mathbf{I}_a^\nu f(t)) \\ &\triangleq \left(\frac{d}{dt}\right)^n \left(\frac{1}{\Gamma(\nu)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\nu}} d\tau\right) \end{aligned} \quad (3)$$

where  $\mathbf{I}_a^\nu f(t)$  is the Cauchy integral of order  $\nu$  of  $f$  and the Euler's function defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad \forall x \in \mathbb{R}^* \setminus \mathbb{N}^- \quad (4)$$

is the generalized factorial ( $\forall n \in \mathbb{N}$ ,  $\Gamma(n+1) = n!$ ).

Note that for all  $\nu \in \mathbb{R}_+$ ,  $\nu\Gamma(\nu) = \Gamma(\nu+1)$ . If  $\gamma = n \in \mathbb{N}$ , the fractional derivative coincides with the ordinary derivative ( $\mathbf{D}_a^\gamma f(t) = \mathbf{D}_a^n f(t)$ ). If  $\gamma < 0$ , the fractional derivative is in fact the fractional integral ( $\mathbf{D}_a^\gamma f(t) = \mathbf{I}_a^{-\gamma} f(t)$ ).

Moreover, it is straightforward to show that the operator  $\mathbf{D}_a^\gamma$  is linear as follows:

*Proposition 1.* Let  $f$  and  $g \in \mathcal{C}^\infty([a, +\infty[)$  and  $\alpha$  and  $\beta \in \mathbb{R}$ , we have:

$$\mathbf{D}_a^\gamma(\alpha f(t) + \beta g(t)) = \alpha \mathbf{D}_a^\gamma f(t) + \beta \mathbf{D}_a^\gamma g(t). \quad (5)$$

In systems theory, the signal space is defined as the space of causal functions  $\mathfrak{H}_a$  given by:

$$\mathfrak{H}_a \triangleq \{f : \mathbb{R} \mapsto \mathbb{R} \mid f \in \mathcal{C}^\infty([a, +\infty[), f(t) = 0, \forall t \leq a\}. \quad (6)$$

The operator  $\mathbf{D}_a^\gamma$  is an endomorphism from  $\mathfrak{H}_a$  to  $\mathfrak{H}_a$  (see Podlubny (1999)).

For more properties on fractional derivatives see (Victor and Melchior (2015)).

Let  $\mathbb{R}[\mathbf{D}_a^\gamma]$  be the set of  $\mathbf{D}_a^\gamma$ -polynomials with real coefficients of the form  $\sum_{k=0}^K c_k \mathbf{D}_a^{k\gamma}$ . One can easily verify that such set, endowed with the usual addition and multiplication of polynomials ( $\mathbb{R}[\mathbf{D}_a^\gamma], +, \times$ ), is a commutative principal ideal domain.

Let  $p$  and  $q \in \mathbb{N}$ , we denote by  $\mathbb{R}[\mathbf{D}_a^\gamma]^{p \times q}$  the set of  $\mathbf{D}_a^\gamma$ -polynomial matrices of size  $p \times q$ . An invertible square matrix of  $\mathbb{R}[\mathbf{D}_a^\gamma]^{p \times p}$  whose inverse is also in  $\mathbb{R}[\mathbf{D}_a^\gamma]^{p \times p}$  is called unimodular matrix. The set of unimodular matrices is denoted by  $GL_p(\mathbb{R}[\mathbf{D}_a^\gamma])$ .  $\mathbf{D}_a^\gamma$ -polynomial matrices admit the following property (Antritter et al. (2014)):

*Theorem 1.* A matrix  $M \in \mathbb{R}[\mathbf{D}_a^\gamma]^{p \times q}$  with  $p \leq q$  (resp.  $p \geq q$ ) is said to be hyper-regular if, and only if, there exists a matrix  $U \in GL_p(\mathbb{R}[\mathbf{D}_a^\gamma])$  (resp.  $V \in GL_q(\mathbb{R}[\mathbf{D}_a^\gamma])$ ) such that:

$$MU = (I_p \ 0_{p \times (q-p)}) \quad \left(\text{resp. } VM = \begin{pmatrix} I_q \\ 0_{(p-q) \times q} \end{pmatrix}\right). \quad (7)$$

The decomposition (7) of the matrix  $M$  is called Smith Diagonal Decomposition.

### 2.2 Linear Fractionally Flat System

The linear fractional systems are defined, in an algebraic framework of module theory, as a non-integer finite  $\mathbb{R}$ -module (see Victor (2010)). The controllability and observability properties of such systems can be found in (Fliess and Hotzel (1997)).

Consider the following pseudo-representation<sup>2</sup> of a linear fractional system

$$Ax = Bu \quad (8)$$

where  $x \in (\mathfrak{H}_a)^n$  represents the  $n$ -dimensional pseudo-state vector,  $u \in (\mathfrak{H}_a)^m$  is the  $m$ -dimensional control vector,  $A \in \mathbb{R}[\mathbf{D}_a^\gamma]^{n \times n}$  and  $B \in \mathbb{R}[\mathbf{D}_a^\gamma]^{n \times m}$ . The matrix  $B$  is supposed to be of rank  $m$  and  $m \leq n$ . The system (8) can be written in the form:

$$F \begin{pmatrix} x \\ u \end{pmatrix} = 0 \quad (9)$$

where  $F \triangleq (A \ -B) \in \mathbb{R}[\mathbf{D}_a^\gamma]^{n \times (n+m)}$  is assumed to be of full row rank.

Inspired by the work of (Antritter et al. (2014)) on the flatness of integer linear systems, the definition of fractional linear flatness is then introduced by (Victor et al. (2015)) as follows:

*Definition 1.* The system (9) is said to be fractionally flat if, and only if, there exist two matrices  $P \in \mathbb{R}[\mathbf{D}_a^\gamma]^{m \times (n+m)}$  and  $Q \in \mathbb{R}[\mathbf{D}_a^\gamma]^{(n+m) \times m}$  and a variable  $y \in (\mathfrak{H}_a)^m$  such that

- (1)  $PQ = I_m$ ;
- (2) For all  $(x, u)^T$  satisfying (9), we have  $y = P \begin{pmatrix} x \\ u \end{pmatrix}$  and conversely  $\begin{pmatrix} x \\ u \end{pmatrix} = Qy$ .

The variable  $y$  is called fractionally flat output and the matrices  $P$  and  $Q$  are called defining matrices.

The main property of fractional linear flatness is given by the following theorem (Victor et al. (2015)):

*Theorem 2.* The system (9) is fractionally flat if, and only if, the matrix  $F$  is hyper-regular over  $\mathbb{R}[\mathbf{D}_a^\gamma]$ .

In some cases, the system (9) admits an implicit form as follows:

*Definition 2.* (Implicit Form). If  $B \in \mathbb{R}[\mathbf{D}_a^\gamma]^{n \times m}$  is hyper-regular, i.e. if there exists  $M \in GL_n(\mathbb{R}[\mathbf{D}_a^\gamma])$  such that  $MB = \begin{pmatrix} I_m \\ 0_{(n-m) \times m} \end{pmatrix}$ , then there exist two matrices  $\tilde{F} \in$

<sup>2</sup> This notation is specified for the class of fractional system and it refers to (Oustaloup (1995)).

$\mathbb{R}[\mathbf{D}_a^\gamma]^{(n-m) \times n}$  and  $R \in \mathbb{R}[\mathbf{D}_a^\gamma]^{m \times n}$  such that the system (9) is equivalent to  $Rx = u$ ,  $\tilde{F}x = 0$ .

In practice, the fractionally flat output may depend only on the state variable  $x$ . More precisely, the defining matrix  $P$  of the Definition 1 may be of the form  $P = [P_1 \ 0_m]$  with  $P_1 \in \mathbb{R}[\mathbf{D}_a^\gamma]^{m \times n}$  and the fractionally flat output is then given by  $y = P_1x$ . In this case we say that the system is fractionally (-1)-flat.

*Theorem 3.* Let the matrix  $B$  of the system (9) be hyper-regular, then the system (9) is fractionally (-1)-flat if, and only if, the matrix  $\tilde{F}$  of the implicit form is hyper-regular over  $\mathbb{R}[\mathbf{D}_a^\gamma]$ .

An algorithm of computation of defining matrices  $P$  and  $Q$  and of a fractionally flat output  $y$  is based on the Smith decomposition of the matrix  $F$  (Victor (2010), Victor et al. (2015)): suppose that  $F$  is hyper-regular, then there exists  $W \in GL_{n+m}(\mathbb{R}[\mathbf{D}_a^\gamma])$  such that:

$$FW = (I_n \ 0_{n \times m}). \quad (10)$$

The defining matrices  $Q$  and  $P$  are given by  $Q = W \begin{pmatrix} 0_{n \times m} \\ I_m \end{pmatrix}$  and  $P = (0_{m \times n} \ I_m) W^{-1}$  respectively. Then, a fractionally flat output vector is given by  $y = P \begin{pmatrix} x \\ u \end{pmatrix}$

and conversely we have  $\begin{pmatrix} x \\ u \end{pmatrix} = Qy$ . In the case of a fractionally (-1)-flat system, the same algorithm is applied to the matrix  $\tilde{F}$  and returns  $P_1$  and  $Q_1$  such that  $y = P_1x$  and  $x = Q_1y$ .

As mentioned in the section 1, a method of calculation of flat outputs, based on the notion of unimodular completion, has been developed in (Fritzsche et al. (2016a)) for the class of non-linear integer flat system. In the next section, we will extend this method to the class of linear fractionally flat system.

### 3. UNIMODULAR COMPLETION ALGORITHM

#### 3.1 Preliminary Definitions

*Definition 3.* Given a hyper-regular matrix  $M \in \mathbb{R}[\mathbf{D}_a^\gamma]^{p \times q}$  with  $p \leq q$ , we say that  $N \in \mathbb{R}[\mathbf{D}_a^\gamma]^{(q-p) \times q}$  is a unimodular completion of  $M$  if and only if

$$\begin{pmatrix} M \\ N \end{pmatrix} \in GL_q(\mathbb{R}[\mathbf{D}_a^\gamma]).$$

*Proposition 2.* Let  $F$  defined by (9) be hyper-regular. Then, the vector  $y$  is a fractionally flat output of (9), if and only if, the matrix  $P \in \mathbb{R}[\mathbf{D}_a^\gamma]^{m \times (n+m)}$  such that  $y = P \begin{pmatrix} x \\ u \end{pmatrix}$ , is a unimodular completion of  $F$ .

**Proof.** The matrix  $F \in \mathbb{R}[\mathbf{D}_a^\gamma]^{n \times (n+m)}$  is hyper-regular, then, by the Smith decomposition of  $F$ , there exists a matrix  $W \in GL_{n+m}(\mathbb{R}[\mathbf{D}_a^\gamma])$  such that:

$$FW = (I_n \ 0_{n \times m}) \quad (11)$$

which implies that  $F = (I_n \ 0_{n \times m}) W^{-1}$ , and  $F$  constitutes the first  $n$  rows of  $W^{-1}$ .

In addition, the defining matrix  $P$  given by  $P = (0_{m \times n} \ I_m) W^{-1}$  constitutes the last  $m$  rows of  $W^{-1}$ , then we get:

$$W^{-1} = \begin{pmatrix} F \\ P \end{pmatrix} \in GL_{n+m}(\mathbb{R}[\mathbf{D}_a^\gamma])$$

and  $P$  is a unimodular completion of  $F$ .  $\square$

#### 3.2 The Computation Procedure

The notations used in the following are the same adapted in (Fritzsche et al. (2016a)). The algorithm is iterative and consists of three steps: Reduction, Zero-space Decomposition and Elimination. The starting point is the system (9) which can be decomposed into the form

$$(F_{0,[0]} + F_{1,[0]} \mathbf{D}_a^\gamma) v_{[0]} = 0 \quad (12)$$

where  $F_{0,[0]}$  and  $F_{1,[0]}$  in  $\mathbb{R}^{n \times (n+m)}$  are two coefficient matrices and  $v_{[0]} = (x, u)^T$ . The index in brackets indicates the iteration number.

*Reduction:* Starting from

$$(F_{0,[i]} + F_{1,[i]} \mathbf{D}_a^\gamma) v_{[i]} = 0, \quad (13)$$

we consider the change of coordinates

$$v_{[i]} = F_{1,[i]}^{\dagger R} v_{[i+1]} + F_{1,[i]}^{\perp R} w_{[i+1]} \quad (14)$$

with  $F_{1,[i]}^{\dagger R}$  the right pseudo-inverse (i.e.  $F_{1,[i]} F_{1,[i]}^{\dagger R} = I_n$ ) and  $F_{1,[i]}^{\perp R}$  is such that  $F_{1,[i]} F_{1,[i]}^{\perp R} = 0$ .

By injecting equation (14) in (13) and using the property (5), we get

$$v_{[i+1]}^{(\gamma)} + A_{[i]} v_{[i+1]} + B_{[i]} w_{[i+1]} = 0 \quad (15)$$

with

$$A_{[i]} = F_{0,[i]} F_{1,[i]}^{\dagger R} \quad (15a)$$

$$B_{[i]} = F_{0,[i]} F_{1,[i]}^{\perp R}. \quad (15b)$$

The matrix  $B_{[i]}$ , being in  $\mathbb{R}^{n_i \times m_i}$ , two cases can be distinguished: if  $\text{rank}(B_{[i]}) = r_i < m_i$  then a zero-space decomposition is needed to reduce the dimension. If  $\text{rank}(B_{[i]}) = m_i$ , i.e.  $B_{[i]}$  is of full column rank, we move on to the Elimination step.

*Remark 1.* In (15b) the case where  $B_{[i]} \equiv 0$  is not considered because it contradicts the controllability condition and consequently the system is not flat (Franke and Röbenack (2013)).

*Zero-space Decomposition:* As mentioned above, if  $\text{rank}(B_{[i]}) = r_i < m_i$ , it is necessary to decompose the matrix  $F_{1,[i]}^{\perp R}$  into the form

$$F_{1,[i]}^{\perp R} = \begin{pmatrix} \tilde{F}_{1,[i]}^{\perp R} & Z_{[i]} \end{pmatrix} \quad (16)$$

such that

$$B_{[i]} = F_{0,[i]} \begin{pmatrix} \tilde{F}_{1,[i]}^{\perp R} & Z_{[i]} \end{pmatrix} = \begin{pmatrix} \tilde{B}_{[i]} & 0 \end{pmatrix} \quad (17)$$

with  $\text{rank}(\tilde{B}_{[i]}) = r_i$ . To do this, we introduce the matrices:

$$Z_{[i]} := F_{1,[i]}^{\perp R} B_{[i]}^{\perp R} \quad (18)$$

$$\tilde{F}_{1,[i]}^{\perp R} := F_{1,[i]}^{\perp R} \left( (B_{[i]}^{\perp R})^{\perp L} \right)^T \quad (19)$$

In this case the change of coordinates (14) is replaced by

$$v_{[i]} = F_{1,[i]}^{\dagger R} v_{[i+1]} + \tilde{F}_{1,[i]}^{\perp L R} w_{[i+1]} + Z_{[i]} z_{[i+1]} \quad (20)$$

and the equation (15) becomes:

$$v_{[i+1]}^{(\gamma)} + A_{[i]} v_{[i+1]} + \tilde{B}_{[i]} w_{[i+1]} = 0. \quad (21)$$

*Elimination:* Returning to the Reduction step, if the matrix  $B_{[i]}$  is not of full row rank, i.e.  $\text{rank}(B_{[i]}) < n_i$  then the dimension of the system (13) must be reduced. For this purpose, the variables  $w_{[i+1]}$  are eliminated from equation (15) by multiplying it by  $B_{[i]}^{\perp L}$ , which leads to:

$$\left( F_{0,[i+1]} + \mathbf{D}_a^\gamma F_{1,[i+1]} \right) v_{[i+1]} = 0, \quad (22)$$

with  $F_{0,[i+1]} = B_{[i]}^{\perp L} A_{[i]}$  and  $F_{1,[i+1]} = B_{[i]}^{\perp L}$ . Here the system (22) is a reduced dimension of the system (13) and then the same procedure is repeated for the iteration  $i + 1$  on (22).

The calculations stop at iteration  $k$  when a full row rank of  $B_{[k]}$  is reached.

*Remark 2.* In the case where the zero-space decomposition is considered, the process of the Elimination step is applied to the equation (21) by replacing  $B_{[i]}^{\perp L}$  by  $\tilde{B}_{[i]}^{\perp L}$ .

*Construction of the unimodular completion:* In each iteration  $i$ , a relation between  $v_{[i]}$  and  $v_{[i+1]}$  can be deduced from (14) by left multiplying it by  $F_{1,[i]}$ :

$$v_{[i+1]} = F_{1,[i]} v_{[i]}. \quad (23)$$

After a finite number  $k + 1$  of iterations, a relation between  $v_{[k+1]}$  and  $v_{[0]}$  is determined as follows:

$$v_{[k+1]} = F_{1,[k]} F_{1,[k-1]} \dots F_{1,[0]} v_{[0]} := P v_{[0]} \quad (24)$$

and the matrix  $P$  is then a unimodular completion of the matrix  $F$  of the system (9).

*Remark 3.* If the case where the zero-space decomposition is considered, the inverse of the equation (20) is given by:

$$z_{[i+1]} = Z_{[i]}^{\dagger L} F_{1,[i-1]} \dots F_{1,[0]} v_{[0]} = \hat{P}_{[i]} v_{[0]} \quad (25)$$

where  $Z_{[i]}^{\dagger L}$  is obtained by the unique following conditions:

$$Z_{[i]}^{\dagger L} Z_{[i]} = I, \quad Z_{[i]}^{\dagger L} F_{1,[i]}^{\perp R} = 0 \quad \text{and} \quad Z_{[i]}^{\dagger L} \tilde{F}_{1,[i]}^{\perp R} = 0. \quad (26)$$

Finally, equations (24) and (25) constitute the fractionally flat output. Calculations in this algorithm can be performed using Maple's LinearAlgebra package.

#### 4. FRACTIONALLY DIRECT FLAT REPRESENTATION

Inspired by the work presented in (Fritzsche et al. (2016b)), the algorithm for calculating the unimodular completion can be reduced under the following condition:

*Proposition 3.* Let  $F \in \mathbb{R}[\mathbf{D}_a^\gamma]^{n \times (n+m)}$  of the system (9) be hyper-regular. If there exists a column permutation matrix  $\Pi$  such that

$$\hat{F} \triangleq F\Pi = \left( S[\mathbf{D}_a^\gamma] \quad T[\mathbf{D}_a^\gamma] \right) \quad (27)$$

with  $S[\mathbf{D}_a^\gamma] \in GL_n(\mathbb{R}[\mathbf{D}_a^\gamma])$  is unimodular, then a unimodular completion of  $\hat{F}$  is given by  $\hat{P} = \begin{pmatrix} 0_{m \times (n-m)} & I_m \end{pmatrix}$  and a unimodular completion of  $F$  is given by

$$P = \hat{P} \Pi^T \quad (28)$$

By this way, the vector  $y$  such that  $y = P \begin{pmatrix} x \\ u \end{pmatrix}$  is a fractionally flat output.

From the expression of  $P$  in (28), we can see that the  $m$  components of  $y$  are simply a permutation of  $m$  elements of the state and the input vectors. From here, expression (28) is called fractionally direct flat representation and  $y$  is a fractionally direct flat output.

*Remark 4.* Proposition 3 is also applicable on the matrix  $\tilde{F}$  in the case of fractionally (-1)-flat system.

## 5. APPLICATIONS

### 5.1 Academic Example

Consider the following system

$$\begin{cases} x_1^{(2\nu)} + x_1 - x_2 = u \\ x_2^{(2\nu)} + x_2 - x_1 = 0 \end{cases} \quad (29)$$

where  $x = (x_1, x_2)^T$  is the pseudo-state vector and  $u$  the input vector. The system (29) can be represented by

$$F \begin{pmatrix} x \\ u \end{pmatrix} = 0 \quad \text{with}$$

$$F = \begin{pmatrix} \mathbf{D}^{(2\nu)} + 1 & -1 & -1 \\ -1 & \mathbf{D}^{(2\nu)} + 1 & 0 \end{pmatrix} \in \mathbb{R}[\mathbf{D}_a^{2\nu}]^{2 \times 3} \quad (30)$$

Using (13), the matrix  $F$  is decomposed into the form

$$F = F_{0,[0]} + F_{1,[0]} \mathbf{D}^{(2\nu)} \quad (31)$$

with  $F_{0,[0]} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix}$  and  $F_{1,[0]} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ .

By following the algorithm, first we calculate

$$F_{1,[0]}^{\dagger R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad F_{1,[0]}^{\perp R} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (32)$$

which leads to (see (15a) and (15b))

$$A_{[0]} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad B_{[0]} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (33)$$

The matrix  $B_{[0]}$  is of full column rank but not of full row rank, then we need to reduce the dimension by the elimination step, so we calculate

$$F_{1,[1]} = B_{[0]}^{\perp L} = (0 \ 1) \quad \text{and} \quad F_{0,[1]} = B_{[0]}^{\perp L} A_{[0]} = (-1 \ 1). \quad (34)$$

We continue to the step  $i = 1$ :

$$F_{1,[1]}^{\dagger R} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad F_{1,[1]}^{\perp R} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (35)$$

which gives  $A_{[1]} = 1$  and  $B_{[1]} = -1$ . Here  $B_{[1]}$  reaches a full row rank and the algorithm ends at this step.

Using (24), the unimodular completion of the matrix  $F$  is then given by

$$P = F_{1,[1]} F_{1,[0]} = (0 \ 1 \ 0) \quad (36)$$

and the system (29) is fractionally flat with  $y = P \begin{pmatrix} x \\ u \end{pmatrix} = x_2$  is a fractionally flat output. Then, we can found  $x_1 = y^{(2\nu)} + y$  and  $u = y^{(4\nu)} + 2y^{(2\nu)}$ .

*Remark 5.* The same algorithm can be applied on the implicit form of (29), because of the hyper-regularity of the matrix  $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .

### 5.2 Thermal Bi-dimensional System

The following example has already been processed in (Victor et al. (2015)). The defining matrices  $P$  and  $Q$  are calculated using the Smith decomposition of the implicit system's matrix  $\tilde{F}$ . Here, we will show that this system has a fractionally direct flat representation, which allow us to compute the fractionally flat output without the need to make calculations.

The thermal bi-dimensional system is about a 2D metallic sheet which is isolated and without heat losses (see Fig. 1). The variable  $T(x_0, y_0, t)$  represents the temperature at a point  $(x_0, y_0)$  at time  $t$  and it is controlled by the heat flux  $\varphi(y, t)$  for  $y \geq 0$ .

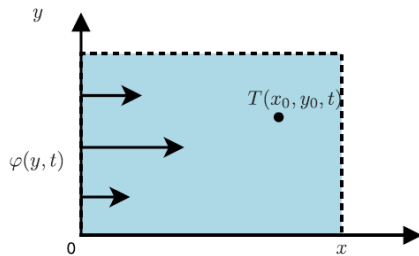


Fig. 1. *Thermal bi-dimensional System*, Source: Victor et al. (2015)

The heated metallic model is represented by the following system:

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{\alpha} \frac{\partial}{\partial t} \right) T(x, y, t) = 0 \quad (37)$$

$$-\lambda \frac{\partial T(x, y, t)}{\partial x} \Big|_{x=0} = \varphi(y, t) \quad \forall y > 0, \forall t > 0 \quad (38)$$

$$\lim_{x \rightarrow +\infty} T(x, y, t) = 0, \quad \forall y > 0, \forall t > 0 \quad (39)$$

$$\lim_{y \rightarrow +\infty} T(x, y, t) = 0, \quad \forall x > 0, \forall t > 0 \quad (40)$$

$$T(x, y, 0) = 0 \quad \forall x > 0, \forall y > 0 \quad (41)$$

where  $\alpha$  and  $\lambda$  are the coefficients of diffusivity and conductivity respectively and  $\varphi$  is the control variable. Equation (37) is the heat equation, (38) is the boundary condition, (39) and (40) represent the limit conditions and (41) is the initial condition known as Cauchy condition.

The Laplace transformation of the equation (37) is given by:

$$\frac{s}{\alpha} \hat{T}(x, y, s) = \frac{\partial^2 \hat{T}(x, y, s)}{\partial x^2} + \frac{\partial^2 \hat{T}(x, y, s)}{\partial y^2} \quad (42)$$

where  $s \in \mathbb{C}$  is the Laplace variable and  $\hat{T}(x, y, s) = \int_0^{+\infty} T(x, y, t) e^{-st} dt$ .

Let  $\hat{T}(x, y, s) = \sum_{i=0}^{\infty} \hat{T}_i(x, y, s)$  with

$$\hat{T}_i(x, y, s) = H_i(x, y, s) \hat{\varphi}_i(s), \quad (43)$$

where

$$H_i(x, y, s) \triangleq \frac{(i+1) e^{-\sqrt{s} \left( \frac{x}{i+1} + y \sqrt{\frac{1}{\alpha} - \frac{1}{(i+1)^2}} \right)}}{\lambda \sqrt{s}} \quad (44)$$

is thermal impedance and

$$\hat{\varphi}_i(s) \triangleq \frac{\lambda \sqrt{s}}{i+1} L_{x,i}(s) L_{y,i}(s). \quad (45)$$

$L_{x,i}(s)$  and  $L_{y,i}(s)$  are arbitrary functions of the complex variable  $s$ .

It may be seen that  $\hat{T}(x, y, s)$  verifies the equation (42). For more details see Victor et al. (2015).

Applying the Padé approximation of  $e^{-x}$  at the order  $\mathbf{K}$  at a point  $(x_0, y_0)$  gives

$$H_i(x_0, y_0, s) \approx \frac{\sum_{k=0}^{\mathbf{K}} \frac{(i+1)}{\lambda} a_{i,k} s^{\frac{k}{2}}}{\sum_{k=0}^{\mathbf{K}} |a_{i,k}| s^{\frac{(k+1)}{2}}} \triangleq H_{i,\mathbf{K}}(x_0, y_0, s) \quad (46)$$

with  $a_{i,k} = \frac{(-1)^k (2\mathbf{K}-k)! \mathbf{K}!}{2\mathbf{K}! k! (\mathbf{K}-k)!} \left( \frac{x_0}{i+1} + y_0 \sqrt{\frac{1}{\alpha} - \frac{1}{(i+1)^2}} \right)^k$ .

$H_{i,\mathbf{K}}$  converges to  $H_i$  when  $\mathbf{K}$  tends to infinity. Similarly, for a finite number  $\mathbf{I}$ , we truncate  $\hat{T}(x_0, y_0, s)$  into

$$\hat{T}_{\mathbf{I},\mathbf{K}}(x_0, y_0, s) \triangleq \sum_{i=0}^{\mathbf{I}} H_{i,\mathbf{K}}(x_0, y_0, s) \hat{\varphi}_i(s) \quad (47)$$

and  $\hat{T}_{\mathbf{I},\mathbf{K}}(x_0, y_0, s)$  converges to  $\hat{T}(x_0, y_0, s)$  as  $\mathbf{I}$  tends to infinity.

From here, the fractional linear system is given by:

$$AX = BU \quad (48)$$

where  $A = \text{diag}\{A_i\}$ ,  $B = \text{diag}\{B_i\}$ ,  $X = \begin{pmatrix} X_0 \\ \vdots \\ X_{\mathbf{I}} \end{pmatrix}$  and

$U = \begin{pmatrix} \varphi_0 \\ \vdots \\ \varphi_{\mathbf{I}} \end{pmatrix}$ , with for  $i = 0, \dots, \mathbf{I}$  we have

$$A_i = \begin{pmatrix} \mathbf{D}_a^{\frac{1}{2}} + |a'_{i,\mathbf{K}-1}| & |a'_{i,\mathbf{K}-2}| & \cdots & |a'_{i,0}| & 0 \\ -1 & \mathbf{D}_a^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & \mathbf{D}_a^{\frac{1}{2}} \end{pmatrix},$$

$X_i = \begin{pmatrix} X_{i,\mathbf{K}} \\ \vdots \\ X_{i,0} \end{pmatrix}$ ,  $B_i = \begin{pmatrix} 1 \\ 0_{\mathbf{K} \times 1} \end{pmatrix}$  and for all  $k = 0, \dots, \mathbf{K}$

and  $i = 0, \dots, \mathbf{I}$  we have  $a'_{i,k} = \frac{a_{i,k}}{|a_{i,k}|}$ . The state vector  $X$  is of dimension  $n \times 1$  with  $n = (\mathbf{I}+1)(\mathbf{K}+1)$  and the control vector  $U$  is of dimension  $m \times 1$  with  $m = \mathbf{I}+1$ .

Using the unimodular completion algorithm, we can prove that the system (48) is fractionally flat. For  $i = 0, \dots, \mathbf{I}$ , the matrix  $B_i$ , being in its Smith form, is hyper-regular and according to *Definition 2*, there exist two matrices  $\tilde{F}_i$  and  $R_i$  such that  $A_i = \begin{pmatrix} R_i \\ \tilde{F}_i \end{pmatrix}$  with

$$R_i = \left( \mathbf{D}_a^{\frac{1}{2}} + |a'_{i,\mathbf{K}-1}| |a'_{i,\mathbf{K}-2}| \cdots |a'_{i,0}| 0 \right) \quad (49)$$

$$\text{and } \tilde{F}_i = \begin{pmatrix} -1 \mathbf{D}_a^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 \mathbf{D}_a^{\frac{1}{2}} \end{pmatrix}. \quad (50)$$

The matrix  $\tilde{F} = \text{diag}\{\tilde{F}_i, i = 0, \dots, \mathbf{I}\}$  of the system (48) is hyper-regular, then the system is fractionally (-1)-flat.

To calculate the fractionally flat output, one can easily follow the steps of the algorithm and compute the unimodular completion matrix. But here, it is clear that the matrix  $\tilde{F}_i \in \mathbb{R}[\mathbf{D}_a^{\frac{1}{2}}]^{\mathbf{K} \times (\mathbf{K}+1)}$  is of the form  $\tilde{F}_i = (S_i \ T_i)$  with

$$S_i = \begin{pmatrix} -1 \mathbf{D}_a^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \mathbf{D}_a^{\frac{1}{2}} \\ 0 & \cdots & \cdots & 0 & -1 \end{pmatrix} \in GL_{\mathbf{K}}(\mathbb{R}[\mathbf{D}_a^{\frac{1}{2}}])$$

is unimodular and  $T_i = \begin{pmatrix} O_{\mathbf{K}-1} \\ \mathbf{D}_a^{\frac{1}{2}} \end{pmatrix} \in \mathbb{R}[\mathbf{D}_a^{\frac{1}{2}}]^{\mathbf{K} \times 1}$ , then, according to Proposition 3, the system admits a fractionally direct flat representation and a unimodular completion of  $\tilde{F}_i$  is given by

$$P_i = (0_{1 \times \mathbf{K}} \ 1). \quad (51)$$

Finally, a unimodular completion of  $\tilde{F}$  is  $P = \text{diag}\{P_i, i = 0, \dots, \mathbf{I}\}$  and a fractionally flat output is then given by

$$Y = PX = \begin{pmatrix} X_{0,0} \\ \vdots \\ X_{\mathbf{I},0} \end{pmatrix}. \quad (52)$$

## 6. CONCLUSION

The current work extends the method for calculating flat outputs by unimodular completion to the class of fractional linear systems. Particularly, the coefficient matrices are used instead of the  $\mathbf{D}_a^\alpha$ -polynomial matrices which makes the calculation easier. In addition, this algorithm provides a fractionally direct flat representation. The method was then validated using two examples. Actually, the flatness property was introduced to deal with the control of nonlinear systems. However, for the class of fractional nonlinear systems, to the knowledge of the authors, mathematical tools must be developed first in order to show the flatness of these systems, which can be an interesting perspective for future works.

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