

Distributionally Robust Stochastic Optimal Water Flow and Risk Management ^{*}

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Abstract: We present a computationally efficient framework to solve a multi-stage optimal water flow (OWF) problem with stochastic water demands. The proposed framework explicitly considers the feedback control policies and adjustable water flow with forecast errors over a planning horizon. The objective is to find an optimal operation schedule of controllable devices (e.g., pumps and valves) to trade off operational performance, such as economic efficiency, safety, and smoothness, and risk of constraint violations. We compute feedback policies that are robust to forecast errors in order to accommodate the fluctuating water demands. Given a probabilistic description of forecast errors, our formulations provide two broad approaches based on Conditional Value-at-Risk (CVaR) and distributionally robust optimization (DRO) that offer alternatives to the existing stochastic OWF formulations based on chance-constrained and robust optimization. Numerical case studies on a three-tank water network demonstrate that the proposed framework achieves effective and explicitly adjustable trade offs between operational efficiency and constraint violation risk.

Keywords: optimal water flow, flow regulation, water flow control-oriented model, stochastic optimization, water networks, risk management

1. INTRODUCTION

Due to a broad range of future energy and environmental issues Lavelle and Grose (2013), water distribution network operators are seeking improved strategies to deliver energy-efficient, reliable, and high quality service to consumers Ocampo-Martinez et al. (2013). However, the increasing complexity (e.g., due to high dimensionality, nonlinearities, operation constraints, and uncertainties) in municipal water supply network operation is challenging the current management and control strategies and may threaten the security of this vital infrastructure. Future urban water supply systems will require more sophisticated methods to function robustly and efficiently in the presence of this increasing complexity.

The flexibility of water flow manipulators (pumps and valves) in water networks has been utilized to optimize various objectives, including production and transportation costs, water quality, safe storage, smoothness of control actions, etc. Fooladivanda and Taylor (2018); D'Ambrosio et al. (2015); Jowitt and Germanopoulos (1992); Yu et al. (1994); Verleye and Aghezzaf (2013); Cohen et al. (2000); Montalvo et al. (2008); Zessler and Shamir (1989). However, most optimal water flow control methods use deterministic point forecasts of exogenous water demands, which neglects their inherent stochasticity. In practice, the variation of water demands in real water distribution networks is high and difficult to predict Goryashko and Nemirovski (2014). Further, as complexity of network topology increases Archibald and Marshall (2018), small perturbations can

cause significant performance decrease and even infeasibility of optimal water flow problems Goryashko and Nemirovski (2014).

Recent research on optimal water network operation has been shifting from deterministic to stochastic models, since uncertainties (e.g., human usage, unexpected component failures, climate change) are increasingly key factors in many sectors of water resource management Archibald and Marshall (2018); Brentan et al. (2018); Goryashko and Nemirovski (2014); Watkins Jr and McKinney (1997); Sampathirao et al. (2018); Grosso et al. (2014b); Ocampo-Martinez et al. (2013); Grosso et al. (2014a); Wang et al. (2016); Castelletti et al. (2012); Sankar et al. (2015); Wang et al. (2020). Most stochastic formulations assume that the uncertain water demands follow a prescribed distribution (e.g., Gaussian Wang et al. (2016); Grosso et al. (2014a)), or enforce constraint for all possible water demand realization by assuming only knowledge of bounds on uncertainties Goryashko and Nemirovski (2014); Castelletti et al. (2012), and then utilize robust optimization. Sampling-based stochastic optimization has also been applied to water flow manipulation problem Sampathirao et al. (2018) to quantify the probability of constraint violation based on an assumed data generating mechanism. However, the underlying assumptions in these approaches can be too strong or overly conservative, which can lead to underestimation or overestimation of the actual risks and therefore to ineffective operation. The methods based on chance-constraints effectively only measure the frequency of constraint violations not the severity, which can underestimate risk. The robust methods can enforce constraints for extreme and highly unlikely uncertainty realizations, effectively overestimating risk. Furthermore,

^{*} This material is based on work supported by the National Science Foundation under grant CMMI-1728605. e-mail: {yi.guo2, tyler.summers}@utdallas.edu.

some sampling-based methods give a heavy computational burden due to their requirement of a large numbers of samples. In practice, forecasts of water demand are never perfect, and their distributions must be estimated from finite data sets.

In this paper, we investigate a stochastic optimal water flow (OWF) problem using a simple but widely used model, called the control-oriented water flow model, Sampathirao et al. (2018); Wang et al. (2016); Grosso et al. (2014a,b); Ocampo-Martinez et al. (2013). We propose two stochastic formulations that use particular risk quantifications, namely, conditional value-at-risk (CVaR), Rockafellar and Uryasev (2000) and distributionally robust optimization (DRO), Calafiore and El Ghaoui (2006); Delage and Ye (2010); Goh and Sim (2010), to explicitly balance tradeoffs between performance and constraint violation risk in the presence of uncertainty about water demands. To our best knowledge, these two popular stochastic optimization approaches have not been studied in the context of water resource management. The main contributions of this paper are summarized as follows

- 1) We present an approach to solve a multi-stage stochastic optimal water flow (OWF) problem based on convex reformulation of chance constraints. We take a perspective of a water utility to manage the operational risk across the network, where we probabilistically constrain the water flow through the network manipulators (e.g., pumps and valves) and the available volume in water storage tanks.
- 2) Two convex approximations are used to handle the uncertainty in stochastic OWF problems, which effectively interpolate between chance-constrained and robust optimization methods. In the first approach, CVaR OWF penalizes both frequency and severity of constraint violations and uses sample average approximation for computation. In the second approach, DRO OWF ensures robustness to worst-case forecast error distributions within an *ambiguity set* specified by moment constraints, and uses exact second-order cone formulations. Both CVaR OWF and DRO OWF utilize affine disturbance policies to obtain closed-loop feedback control policies, by planning reactions to forecast error realizations.
- 3) We demonstrate the effectiveness of our formulation on a model of part of the Barcelona water distribution system Grosso et al. (2014b). The numerical case studies demonstrate, analyze, and discuss the tradeoffs between the network performance and operational risks. Our proposed approaches offer complementary alternatives to existing stochastic optimal water flow frameworks.

The rest of paper is organized as follows: Section II describes the water flow control model and the general multi-stage stochastic OWF problem. Section III presents the convex reformulations of chance constraints and proposes two tractable stochastic OWF formulations based on conditional value-at-risk (CVaR) and distributionally robust optimization (DRO). Section IV presents the simulation results and compares the proposed approaches with existing stochastic formulations. Section V concludes the paper.

2. PROBLEM STATEMENT

In this section, we pose a general stochastic optimal water flow problem and then discuss a convex reformulation in the following sections. Although there are many complex models for water flow, we utilize a simple but widely used water flow oriented-control model to maintain computational tractability in the presence of uncertainties Ocampo-Martinez et al. (2013); Wang et al. (2016); Grosso et al. (2014b); Goryashko and Nemirovski (2014). Internal dynamics of actuators, head pressure dynamics of pumps, and nonlinearities of pipes and valves and not considered in our model.

2.1 Network Modelling

Consider a water network modeled by an undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, with N_v vertices representing a set of N_s water sources, N_j junction nodes, N_t water tanks and N_d sinks (i.e., demands). These components are connected by a set of edges \mathcal{E} that represent a set of pipes with N_c flow control actuators (pumps and valves) that deliver the water flow from sources to sinks. We define the water volume in the storage tanks as the state vector $x_k \in \mathbf{R}^{N_t}$. The water passing through the control actuators can be regarded as a control input $u_k \in \mathbf{R}^{N_c}$. For simplicity, we assume that the inputs directly manipulate the rate of flow through the corresponding pipes (measured in m^3/s or gallon per minute (GPM)). The water demand at sinks is denoted by $d_k \in \mathbf{R}^{N_d}$. A set of system matrices $\bar{A} \in \mathbf{R}^{N_t \times N_t}$, $\bar{B} \in \mathbf{R}^{N_t \times N_c}$, $\bar{B}_d \in \mathbf{R}^{N_t \times N_d}$, $\bar{E}_u \in \mathbf{R}^{N_j \times N_c}$ and $\bar{E}_d \in \mathbf{R}^{N_j \times N_d}$ describe the network topology and discrete-time dynamics that reflect simple mass balance relationships between nodes and edges, resulting in the model:

$$x_{k+1} = \bar{A}x_k + \bar{B}u_k + \bar{B}_d d_k, \quad (1a)$$

$$0 = \bar{E}_u u_k + \bar{E}_d d_k, \quad (1b)$$

where (1a) captures storage tank level dynamics and (1b) describes mass balance at each junction node. We consider management of tank levels and flows over a planning time horizon T from initial tank levels x_0 . For compact notation, we concatenate the states, inputs and demands over the planning horizon as $\mathbf{x} = [(x_1)^\top, \dots, (x_T)^\top]^\top \in \mathbf{R}^{N_t T}$, $\mathbf{u} = [(u_0)^\top, \dots, (u_{T-1})^\top]^\top \in \mathbf{R}^{N_c T}$ and $\mathbf{d} = [(d_0)^\top, \dots, (d_{T-1})^\top]^\top \in \mathbf{R}^{N_d T}$, yielding

$$\mathbf{x} = A\mathbf{x}_0 + B\mathbf{u} + B_d\mathbf{d}, \quad (2a)$$

$$0 = E_u\mathbf{u} + E_d\mathbf{d}, \quad (2b)$$

where

$$E_u = I_T \otimes \bar{E}_u, \quad E_d = I_T \otimes \bar{E}_d,$$

$$A = \begin{bmatrix} \bar{A} \\ \bar{A}^2 \\ \vdots \\ \bar{A}^T \end{bmatrix}, B_d = \begin{bmatrix} \bar{B}_d & 0 & \cdots & 0 \\ \bar{A}\bar{B}_d & \bar{B}_d & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \bar{A}^{T-1}\bar{B}_d & \cdots & \bar{A}\bar{B}_d & \bar{B}_d \end{bmatrix},$$

$$B = \begin{bmatrix} \bar{B} & 0 & \cdots & 0 \\ \bar{A}\bar{B} & \bar{B} & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \bar{A}^{T-1}\bar{B} & \cdots & \bar{A}\bar{B} & \bar{B} \end{bmatrix}.$$

Note that I_T indicates a T -dimension identity matrix and \otimes denotes Kronecker product.

This control-oriented model for water resource management in water networks has been widely studied Sampathirao et al. (2018); Sun et al. (2016); Grosso et al. (2014b) and allows for real-time computational efficiency. Although this simplified model neglects factors such as energy loss, hydraulic dynamics, and pressure constraints, it still captures the main network response of hydraulic variables of network to control actions Ocampo-Martinez et al. (2013). Our ongoing follow-up work is considering nonlinear dynamics and stochastic pressure management, in addition to water flow control.

2.2 Cost Functions and Constraints

The operational cost for the water network is assumed to be convex quadratic in states and inputs in the form

$$J(\mathbf{x}, \mathbf{u}) = f_x^T \mathbf{x} + \frac{1}{2} \mathbf{x}^T H_x \mathbf{x} + f_u^T \mathbf{u} + \frac{1}{2} \mathbf{u}^T H_u \mathbf{u}, \quad (3)$$

where H_x and H_u are positive semidefinite matrices. The parameters f_x , f_u , H_x and H_u can be selected to capture various performance criteria, such as electricity cost for pumping, water production and transportation costs, safe storage levels, and control effort Ocampo-Martinez et al. (2013).

We incorporate state and input constraints on the tank volume and flow control inputs

$$\mathbf{u}_{\min} \leq \mathbf{u} \leq \mathbf{u}_{\max}, \quad (4a)$$

$$\mathbf{x}_{\min} \leq \mathbf{x} \leq \mathbf{x}_{\max}. \quad (4b)$$

Here, \mathbf{x}_{\min} and \mathbf{x}_{\max} denote the minimum and maximum admissible water volume of tanks, respectively. The lower storage limitation \mathbf{x}_{\min} can be zero to represent an empty tank, or can be set to a nonzero value to leave water availability for emergencies. The lower and upper physical limits of actuators are \mathbf{u}_{\min} and \mathbf{u}_{\max} , respectively.

2.3 Stochastic Optimal Water Flow

In a deterministic optimal water flow control problem, water demand uncertainty is not explicitly considered. Since actual water demands can exhibit large variations and unpredictability Brentan et al. (2018), we model demand stochastically as $\mathbf{d} = \bar{\mathbf{d}} + \xi$, with a nominal predicted value $\bar{\mathbf{d}} \in \mathbf{R}^{N_d T}$ and a zero-mean forecast error $\xi = [\xi_0^T, \dots, \xi_{T-1}^T]^T \in \mathbf{R}^{N_d T}$ from a probability space $(\Omega, \mathcal{F}, \mathbb{P}_\xi)$. We assume that information about the joint probability distribution of ξ is known. The distribution captures spatiotemporal variations and dependencies among the demands. In particular, we will assume knowledge of certain moments such as the mean and variance, or a model of ξ from which we can draw samples.

To explicitly account for this stochasticity of water demands, we formulate the following general stochastic optimal water flow problem to find an optimal strategy for responding to forecast errors via an optimal control policy for the flow actuators $\mathbf{u} = \pi(\xi)$, where $\pi: \mathbf{R}^{N_d T} \rightarrow \mathbf{R}^{N_c T}$ is a function from a set Π_c of causal policies. Specifically, we consider

$$\underset{\pi \in \Pi_c}{\text{minimize}} \quad \mathbb{E}_\xi [J(Ax_0 + B\pi(\xi) + B_d \mathbf{d}, \pi(\xi))], \quad (5a)$$

$$\text{subject to} \quad 0 = E_u \pi(\xi) + E_d \mathbf{d}, \quad \forall \xi \quad (5b)$$

$$\mathcal{R}(\mathbf{u}_{\min} - \pi(\xi)) \leq 0, \quad (5c)$$

$$\mathcal{R}(\pi(\xi) - \mathbf{u}_{\max}) \leq 0, \quad (5d)$$

$$\mathcal{R}(\mathbf{x}_{\min} - \mathbf{x}) \leq 0, \quad (5e)$$

$$\mathcal{R}(\mathbf{x} - \mathbf{x}_{\max}) \leq 0, \quad (5f)$$

where \mathcal{R} donates a transformation of the inequality constraint functions into a risk metric, which will be described in more detail shortly. Note that this transformation can be different in general for each constraint. In general, some of these constraints on real device cannot be physically violated. Here, we consider non-physical upper and lower boundaries of states and inputs from a pre-specified safe operation zone, which can be violated probabilistically but results in safety or operational risks Grosso et al. (2017). We will introduce two popular risk metrics for (5c)-(5f) in the next section.

Since optimizing over general policies makes problem (5) infinite dimensional, we optimize instead over a set of affine control policies

$$\mathbf{u} = D\xi + e,$$

where

$$D = \begin{bmatrix} [D]_{(0,0)} & 0 & \cdots & 0 \\ [D]_{(1,0)} & [D]_{(1,1)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ [D]_{(T-1,0)} & \cdots & [D]_{(T-1,T-2)} & [D]_{(T-1,T-1)} \end{bmatrix}.$$

Here, $e \in \mathbf{R}^{N_c T}$ represents a nominal plan for actuators, and $[D]_{(i,i)} \in \mathbf{R}^{N_c \times N_d}$ can be understood as a feedback control matrix, which reacts to forecast errors as they are revealed in time. The block lower-triangular matrix $D \in \mathbf{R}^{N_c T \times N_d T}$ ensures that the controller is causal.

Substituting the affine control policies into (5), the objective function (5a) becomes convex quadratic in D and e and depends on the first and second moments information of ξ . Since the policy is affine, the robust equality constraint (5b) is equivalent to

$$E_u D + E_d = 0, \quad E_u e + E_d \bar{\mathbf{d}} = 0. \quad (6)$$

With affine policies, (5c)-(5f) become

$$\mathcal{R}([D]_i \xi + e(i) - \mathbf{u}_{\max}(i)) \leq 0, \quad (7a)$$

$$\mathcal{R}(\mathbf{u}_{\min}(i) - [D]_i \xi - e(i)) \leq 0, \quad (7b)$$

$$\mathcal{R}([BD]_j \xi + [Ax_0 + Be + B_d \mathbf{d}]_j - \mathbf{x}_{\max}(j)) \leq 0, \quad (7c)$$

$$\mathcal{R}(\mathbf{x}_{\min}(j) - [BD]_j \xi - [Ax_0 + Be + B_d \mathbf{d}]_j) \leq 0, \quad (7d)$$

$$i = 1, \dots, N_c T, j = 1, \dots, N_t T.$$

where the operator $[\cdot]_i$ denotes the i -th row of a matrix or the i -th element of a vector. We collect all above constraints (7a)-(7d) into a set \mathcal{V} of $N_t T + N_c T$ constraints, and the expressions inside the brackets can be written in a general linear form $a_i(D)^T \xi + b_i(e)$, $\forall i \in \mathcal{V}$. Additionally, we denote $\phi_j(e) := [Ax_0 + Be + B_d \mathbf{d}]_j$, $j = 1, \dots, N_t T$ in the rest of this paper for notational convenience.

3. CONVEX APPROXIMATION OF STOCHASTIC OPTIMAL WATER FLOW

Following our proposed formulation above, we begin this section by introducing chance constraints and propose two convex reformulations of the stochastic optimal water flow problem based on conditional value-at-risk and distributionally robust optimization.

3.1 Chance Constraints

Consider the stochastic OWF problem as a chance-constrained optimization problem

$$\underset{D, e}{\text{minimize}} \quad J(\mathbf{x}, \mathbf{u}), \quad (8a)$$

$$\text{subject to} \quad \mathbb{P}_\xi(a_i(D)^\top \xi + b_i(e) \leq 0) \geq 1 - \alpha_i, \quad (8b)$$

and (6), $\forall i \in \mathcal{V}$,

where $\alpha_i \in \mathbf{R}$ is the prescribed safety parameter or ‘‘risk budget’’ for i -th linear constraint within set \mathcal{V} . If ξ is normally distributed, then it is known that the chance constraint can be written as a second-order cone constraint Delage and Ye (2010); Calafiore and El Ghaoui (2006),

$$a_i(D)^\top \mu + b_i(e) + \Phi^{-1}(1 - \alpha_i) \|\Sigma^{\frac{1}{2}} a_i(D)\|_2 \leq 0, \quad (9)$$

where $\mathbb{E}[\xi] = \mu$, $\mathbb{E}[\xi\xi^\top] = \Sigma$ are the first and second moments of ξ and Φ^{-1} denotes the Gaussian quantile function. This group of constraints (9) is closely related to value-at-risk (VaR), a well known risk quantification in finance applications. However, the Gaussian chance constraints (9) only restrict the frequency of constraint violations, not the severity. Since distributions in practice may not be Gaussian, this approach can lead to underestimation of actual risks. In this paper, we leverage CVaR function to penalize both frequency and severity of constraint violation and use moment-based distributionally robust optimization to make decisions less sensitive to assumptions about the distribution of ξ .

3.2 Stochastic OWF with Conditional Value-at-Risk (CVaR)

To strengthen the chance constraint (9), we leverage CVaR to limit both frequency and expected severity of constraint violation Rockafellar and Uryasev (2000), which is arguably more appropriate for many types of constraints in stochastic optimal water flow problems to manage constraints on tanks (5e)-(5f) and actuators (e.g., pumps and valves (5c)-(5d)). Using a CVaR metric, the constraints (7a)-(7d) can be expressed as Rockafellar and Uryasev (2000)

$$\begin{aligned} \mathbb{E}_\xi \left[[D]_i \xi + e(i) - \mathbf{u}_{\max}(i) + \bar{t}_i \right]_+ &\leq \bar{t}_i \alpha_i, \\ \mathbb{E}_\xi \left[\mathbf{u}_{\min}(i) - [D]_i \xi - e(i) + \bar{t}_i \right]_+ &\leq \bar{t}_i \alpha_i, \\ \mathbb{E}_\xi \left[[BD]_j \xi + \phi_j(e) - \mathbf{x}_{\max}(j) + \bar{t}_j \right]_+ &\leq \bar{t}_j \alpha_j, \\ \mathbb{E}_\xi \left[\mathbf{x}_{\min}(j) - [BD]_j \xi - \phi_j(e) + \bar{t}_j \right]_+ &\leq \bar{t}_j \alpha_j, \\ i &= 1, \dots, N_c T, j = 1, \dots, N_t T, \end{aligned}$$

where $\bar{t}_i, \bar{t}_i, \bar{t}_j$ and \bar{t}_j are auxiliary optimization variables and $[\cdot]_+ = \max\{0, \cdot\}$. Since an exact evaluation of the expectation is difficult due to the piece-wise linearity of $[\cdot]_+$, we approximate these constraints using sample average approximation methods based on a finite set of scenario data $\hat{\xi}_s \in \hat{\Xi}$, which is a widely used and highly effective approach for handling CVaR constraints in practice Bertsimas et al. (2018).

Using the CVaR risk metric, we come to the following stochastic optimal water flow problem

$$\underset{D, e, t}{\text{minimize}} \quad \mathbb{E}_\xi [\hat{J}(Ax_0 + B(D\xi + e) + B_d \mathbf{d}, D\xi + e)], \quad (11a)$$

$$\text{subject to} \quad E_u D + E_d = 0, \quad E_u e + E_d \bar{\mathbf{d}} = 0,$$

$$\frac{1}{N} \sum_{s=1}^N \left[[D]_i \hat{\xi}_s + e(i) - \mathbf{u}_{\max}(i) + \bar{t}_i \right]_+ - \bar{t}_i \alpha_i \leq 0,$$

$$\frac{1}{N} \sum_{s=1}^N \left[\mathbf{u}_{\min}(i) - [D]_i \hat{\xi}_s - e(i) + \bar{t}_i \right]_+ - \bar{t}_i \alpha_i \leq 0,$$

$$\frac{1}{N} \sum_{s=1}^N \left[[BD]_j \hat{\xi}_s + \phi_j(e) - \mathbf{x}_{\max}(j) + \bar{t}_j \right]_+ - \bar{t}_j \alpha_j \leq 0,$$

$$\frac{1}{N} \sum_{s=1}^N \left[\mathbf{x}_{\min}(j) - [BD]_j \hat{\xi}_s - \phi_j(e) + \bar{t}_j \right]_+ - \bar{t}_j \alpha_j \leq 0,$$

$$i = 1, \dots, N_c T, j = 1, \dots, N_t T, \forall s \in \hat{\Xi}, \quad (\text{CVaR OWF})$$

where $\hat{\Xi}$ contains sample data of water demand forecast errors. Since the objective function J is quadratic, the expectation can be explicitly evaluated using first and second moment information, which can be estimated from the finite sampling data set $\hat{\Xi}$. Note that the superscript $\hat{\cdot}$ is reserved for the objects depend on the training data set $\hat{\Xi}$. Note also that we treat all chance constraints in set \mathcal{V} independently. It is also interesting to discuss how a group of chance constraints can be considered jointly by appropriately selecting the ‘‘risk budgets’’ α_i , which we leave as future work.

3.3 Distributionally Robust Stochastic OWF

As another significant strengthening of the chance constraints (8b), we propose an alternative using moment-based distributionally robust constraints. In particular, we suppose that the actual distribution \mathbb{P}_ξ of ξ is unknown and assumed to lie in an *ambiguity set* \mathcal{P} , and enforce the constraints for all possible distributions within this set:

$$\mathbb{P}_\xi(a_i(D)^\top \xi + b_i(e) \leq 0) \geq 1 - \alpha_i, \quad \forall i \in \mathcal{V}, \forall \mathbb{P}_\xi \in \mathcal{P}, \quad (12)$$

We utilize moment-based ambiguity sets, which contain all possible distributions that share first and second moment information

$$\mathcal{P} := \left\{ \mathbb{P}_\xi \mid \mathbb{E}^{\mathbb{P}_\xi}[\xi] = \mu, \mathbb{E}^{\mathbb{P}_\xi}[\xi\xi^\top] = \Sigma \right\}.$$

It can be shown Calafiore and El Ghaoui (2006) that (12) can also be exactly reformulated as a second-order cone constraint

$$a_i(D)^\top \mu + b_i(e) + \sqrt{\frac{1 - \alpha_i}{\alpha_i}} \|\Sigma^{\frac{1}{2}} a_i(D)\|_2 \leq 0, \quad (13)$$

which only depends on the mean μ and covariance Σ of ξ . Note that (13) is similar to (9) but tighter, and leads to more conservative but more robust decisions.

Using a distributionally robust risk metric, we obtain the distributionally robust stochastic OWF problem

$$\underset{D, e}{\text{minimize}} \quad \mathbb{E}_\xi [\hat{J}(Ax_0 + B(D\xi + e) + B_d \mathbf{d}, D\xi + e)], \quad (14a)$$

$$\text{subject to} \quad E_u D + E_d = 0, \quad E_u e + E_d \bar{\mathbf{d}} = 0,$$

$$\begin{aligned}
 [D]_i \mu + e(i) - \mathbf{u}_{\max}(i) + \sqrt{\frac{1-\alpha_i}{\alpha_i}} \|\Sigma^{\frac{1}{2}} [D]_i^T\|_2 &\leq 0, \\
 \mathbf{u}_{\min}(i) - [D]_i \mu - e(i) + \sqrt{\frac{1-\alpha_i}{\alpha_i}} \|\Sigma^{\frac{1}{2}} [D]_i^T\|_2 &\leq 0, \\
 [BD]_j \mu + \phi_j(e) - \mathbf{x}_{\max}(j) + \sqrt{\frac{1-\alpha_j}{\alpha_j}} \|\Sigma^{\frac{1}{2}} [BD]_j^T\|_2 &\leq 0, \\
 \mathbf{x}_{\min}(j) - [BD]_j \mu - \phi_j(e) + \sqrt{\frac{1-\alpha_j}{\alpha_j}} \|\Sigma^{\frac{1}{2}} [BD]_j^T\|_2 &\leq 0, \\
 i = 1, \dots, N_c T, j = 1, \dots, N_t T. & \quad (\text{DRO OWF})
 \end{aligned}$$

The advantages of this reformulation are that it is straightforward to estimate moments from historical data and that the last two second-order cone constraints do not require any other sampling of ξ . However, since the decisions are robust to the corresponding *worst case* distribution within the ambiguity set \mathcal{P} , DRO OWF will result in more conservative decisions than CVaR OWF. To reduce conservatism, additional assumptions (such as unimodality or knowledge of higher moments) can be made to reduce the size of the ambiguity sets.

In general, two chance constraint convex reformulations, CVaR OWF and DRO OWF, make different assumptions on the underlying uncertainty, and make control decisions robust to this uncertainty.

- CVaR OWF: Conditional Value-at-Risk quantifies the expected severity of constraint violations above a certain quantile. Without a prescribed distribution assumption, the decisions from CVaR OWF will be robust to the empirical distribution supported on the dataset $\hat{\Xi}$.
- DRO OWF: the distributionally robust formulation make the schedules robust to a set of distributions with specified moment information (mean and covariance), which can be estimated directly from data. This approach leads to more conservativeness and tractably incorporates statistical knowledge of water demand variations.

Overall, these two stochastic OWF formulations allow the water distribution operators to account for stochastic water demand from the forecasting dataset $\hat{\Xi}$ or statistical information derived from forecasting and data analysis methods, respectively.

3.4 Stochastic Model Predictive Control Implementation

Our proposed stochastic optimal water flow formulations (CVaR OWF and DRO OWF) can be regarded as a component of a multi-stage closed-loop stochastic model predictive control (MPC) framework. The stochastic MPC control strategy explicitly incorporates information about uncertainty into the feedback controller, and allows the uncertainty description to be updated as more data becomes available over time, Camacho and Alba (2013); Grosso et al. (2014b); Wang et al. (2016). An MPC controller computes optimal open-loop control decisions at the current timestep considering the future system trajectories over a finite time horizon \mathcal{H}_k . Here, the proposed stochastic optimal water flow problems can be utilized as a block for a MPC-based closed-loop control strategy, which includes the current and

future water network states based on the forecasts of the water consumption and the water availability in tanks. The MPC-based control strategy comprises the following steps:

- At current timestep k , we forecast the water demand consumption across the water network over the horizon \mathcal{H}_k , and generate a set of forecast errors samples for CVaR OWF or the moment estimates for DRO OWF.
- Solve (11) or (14) over horizon \mathcal{H}_k .
- Implement the reserve policies and nominal flow manipulation for each actuator at current time step k .
- Move to the next time step $k + 1$, and repeat.

For simplicity, our case studies focus on solving (11) or (14) at one particular time instance; Monte Carlo simulations of the closed-loop stochastic MPC strategies will be performed in future work.

4. CASE STUDIES

To illustrate the proposed framework, we use a portion the Barcelona drinking water network (DWN) Grosso et al. (2014b) to demonstrate its effectiveness for DWN optimization and control. There are 4 demands, 6 water flow control actuators, 2 water sources, 3 tanks and 2 intersection nodes in this portion of DWN, in Fig. 1.

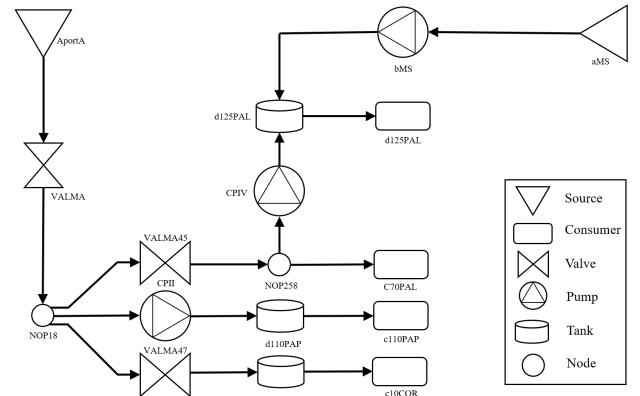


Fig. 1. A three tank drinking water network, Grosso et al. (2014b).

The system parameters from (1) are

$$\bar{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \bar{B} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \Delta t,$$

$$\bar{B}_d = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \Delta t,$$

$$\bar{E}_u = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix}, \bar{E}_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$x_{\min} = [400 \ 300 \ 400]^T, x_{\max} = [3100 \ 1500 \ 3100]^T,$$

$$u_{\min} = [0 \ 0 \ 0 \ 0 \ 0 \ 0]^T,$$

$$u_{\max} = [1.297 \ 0.050 \ 0.120 \ 0.035 \ 0.031 \ 0.040]^T,$$

where the sampling time Δt is 3600s. The water demand pattern is derived from EPANET Rossman et al. (2000) (a standard software package for analysis of drinking water distribution systems). The four nominal exogenous water demands are shown in Fig. 2. For illustrative

purposes, the water demand forecast errors are drawn from a Gaussian probability distribution with zero mean and a standard deviation of 40% of the nominal water demand forecast \bar{d} . In this case study, we only restrict the upper level constraints on three tanks with distributionally robust or CVaR for a clear visualization and demonstration on the tradeoff between efficiency and operation risks. Other constraints including the lower level constraints on tanks are handled by sample average approximation (SAA), Bertsimas et al. (2018). It is straightforward to reformulate these with distributional robustness or CVaR. The simulation takes 60 seconds or less to solve CVaR OWF and DRO OWF with $T = 24$ hours using MOSEK Solver (2017) via the MATLAB interface with CVX (2012) on a laptop with 16GM of memory and a 2.8GHz Intel Core i7.

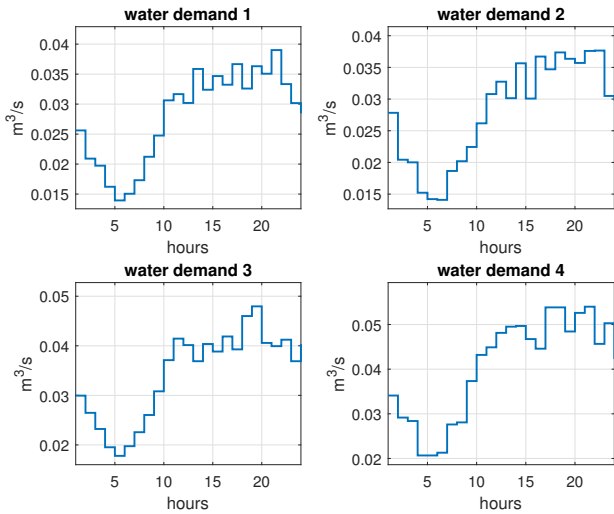


Fig. 2. The nominal water demand forecast.

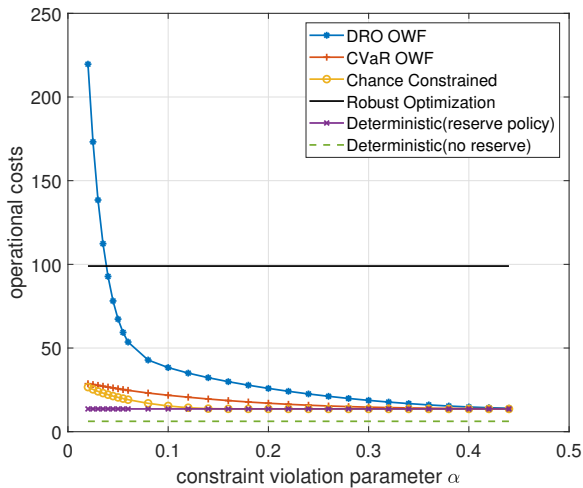


Fig. 3. Optimal operational cost vs. the constraint violation budget α under various stochastic reformulations of the tank volume constraints (including chance-constrained Grosso et al. (2014a), robust optimization Goryashko and Nemirovski (2014)). All cases start from a non-optimal initial state $x_0 = [1700.44, 646.23, 633.89]^T$ in m^3 .

Fig. 3 visualizes the tradeoffs between the operational performance and constraint violation budget α under various stochastic constraint reformulation of maximum

water tank levels: a chance constraint assuming that the uncertainty follow Gaussian distributions (9) Grosso et al. (2014a), CVaR OWF (11), DRO OWF (14), robust optimization Goryashko and Nemirovski (2014) with an assumed support of $\xi \in \Theta := [-\theta\bar{d} \leq \xi \leq \theta\bar{d}, \theta = 0.2]$, and two deterministic cases with and without reserve control policies. The chance-constrained method with Gaussian assumption is always less conservative than other three stochastic approaches (viz., DRO, CVaR and Robust). CVaR OWF is more conservative than chance constraints since it penalizes both frequency and severity of constraint violation. DRO OWF results in more conservative performance and can even be more conservative than the robust approach for small values of α since we do not explicitly bound the uncertainty in this case. However, the CVaR and DRO approaches will result in fewer out-of-sample constraint violations when uncertainties are not Gaussian.

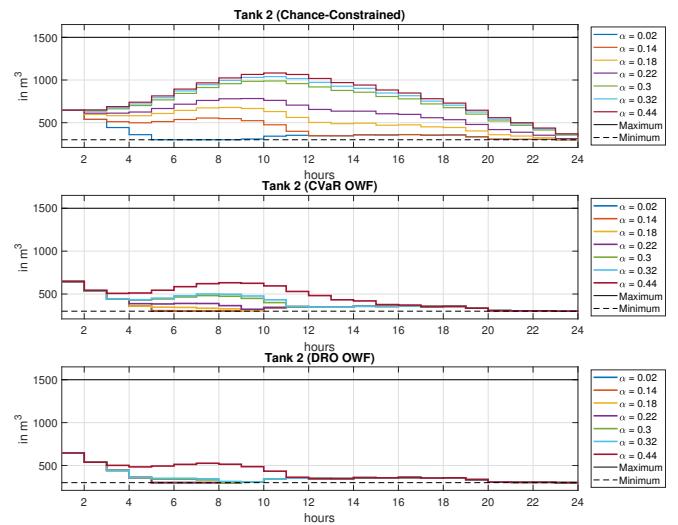


Fig. 4. Optimal state trajectories of tank #2 for varying risk budgets α under three stochastic constraint reformulations.

Fig. 4 illustrates the optimal state trajectories from DRO OWF, CVaR OWF and the chance-constrained method with Gaussian assumption under various risk budgets α . The dashed and solid black indicate the lower and upper volume boundaries of water tank 2, respectively. As expected, the Gaussian chance-constrained approach arrives at less conservative state trajectories, whereas DRO OWF offers tanks levels that are more robust to upper violations.

Note that each stochastic constraint reformulation has different performance and robustness properties based on differing assumptions about the uncertainty. The most appropriate formulation depends on many practical factors and operator risk aversion. It is possible to mix different constraint reformulations in one multi-stage stochastic OWF problem.

5. CONCLUSIONS

In this paper, we proposed two convex reformulations of the stochastic optimal water flow problem to trade off operational efficiency and constraint violation risk. The

CVaR OWF offers improved robustness compared to existing chance-constrained OWF, but requires sample average approximation. The DRO OWF utilizes exact second-order conic constraint reformulations based on moment-based ambiguity sets, which simplifies computation but gives increased conservatism. Ongoing work includes the numerical experiments on larger networks and adapting the proposed framework to handle constraints on head pressure at pumps, valves, and pipes.

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