Robust Control of a Class of Bilinear Systems by Forwarding: Application to Counter Current Heat Exchanger

Bertrand Zitte * **Boussad Hamroun** * **Daniele Astolfi** *
Françoise Couenne *

* *Université Claude Bernard Lyon 1, CNRS, LAGEPP UMR 5007, 43 boulevard du 11 novembre 1918, F-69100, Villeurbanne, France (e-mail:* `(name.surname)@univ-lyon1.fr`*).*

Abstract: In this paper we propose a robust control for the counter-current heat exchanger. By using energy balance equations, we propose a model in structured bilinear system that allows to capture the heat transfer and convection phenomena. We study the problem of regulating the output temperature of the cold (or hot) fluid by controlling the flow rate of the hot (or cold) fluid. Using an integral action and a forwarding based control method, we derive a non linear control which achieves output temperature regulation. Numerical simulations confirm the effectiveness of the proposed control.

Keywords: Control, Bilinear Systems, Heat exchanger, Integral Action, Forwarding, Output Regulation.

1. INTRODUCTION

The heat exchanger (HEX) is a central module in processes where thermal energy is needed and transferred between two (or more) fluid streams. For example, they are intensively used in chemical industries Gu et al. (2015), urban heating and chilling networks Sakawa et al. (2002) and in the thermodynamic machines Wu et al. (2016).

A model for a HEX can be obtained in form of a distributed parameter system by writing energy balance equations, that is a partial differential equations (PDE) where the state variables are space and time dependent. Several authors addressed the control of a HEX based on a PDE model. See, among them, Maidi et al. (2009); Ozorio Cassol et al. (2019); Huhtala and Paunonen (2019). For control purposes, finite dimensional models are also often used in the literature, see, for instance, Varga et al. (1995); Scholten et al. (2017); Chandrashekar and Wong (1982). These models mainly fall into two classes. The first is based on thermodynamic phenomenological (possibly non linear) equations, while the second is based on a linear input output dynamic representation. The control problem of a HEX has been therefore addressed by using different techniques, that depend, in general, on the choice of the model. Among them, we recall the following: linearizing feedback, see Alsop and Edgar (1989); non linear output-based dynamical controller for a simplified one bi-compartmental cell model, see Zavala-Río et al. (2009); model predictive control techniques for a non linear model, see Sridhar et al. (2016). Furthermore, PID controllers are often used for the regulation of the output temperature of HEX Diaz-Mendez et al. (2014).

In this paper, we propose a new approach for the control of counter-current HEX based on a finite dimensional model. The HEX is represented by a cascade of single-phase homogeneous compartments. The dynamical model is obtained by writing the energy balance equation on each compartment in which we considered heat convection, heat transfer phenomena between cold and hot fluids and uniform mass flow rates of the cold and hot streams. If the mass flow rate is considered as the manipulated variable for control, the HEX model turns out to be a bilinear system. Then, we analyze the bilinear dynamical model issued from energy balance equations and we show that it inherits some properties that will be used for control purpose. The control law is obtained by following the forwarding approach as proposed in Praly et al. (2001); Astolfi and Praly (2017). First, the HEX is extended by an integrator processing the desired output error. Then a stabilizing law for the extended system is derived by using a forwarding approach. The resulting bounded control law allows to stabilize asymptotically the system on an operating equilibrium while achieving the output regulation objective, such that the output temperature is regulated to a desired constant set-point. The derived control inherits a robustness property with respect to parameters variations as shown in Praly et al. (2001), Astolfi and Praly (2017).

Furthermore, the proposed saturated control law is given in terms of flow rate and the domain of attraction of the equilibrium is global with respect to the domain of validity of the model.

The paper is organized as follows. In Section 2, the bilinear model of the counter-current HEX is presented. In section 3 we recall the forwarding-based control approach in the context of output regulation on a general class of input affine non linear systems and its application to a class of bilinear systems. Then, in section 4, we give the main results of the application of this control method on the HEX model. We present some numerical simulations in Section 5 and we derive conclusion and perspectives in Section 6.

2. MODELLING OF THE COUNTER-CURRENT HEAT EXCHANGER

We consider a counter-current HEX where single phase hot and cold stream exchange heat. We assume that the pressure P is constant and uniform along the HEX. Moreover we assume that there is no energy accumulation in the wall between the two fluids and there is no heat transfer through the external wall with the environment. We also assume that the convection velocity is uniform along the HEX. This convection velocity is considered as an input for the system, which achieves a spatially uniform steady-state in a neglected time scale compared to the heat transfer dynamics. Thus the dynamical model is derived mainly using an energy balance equation. Naturally, the HEX may be modelled by a distributed parameter system by writing where the state variables are space and time dependent. In this paper we consider a discrete space representation for the HEX. In doing so, the hot and cold side of the HEX are represented by a cascade of homogeneous and uniform compartments. Without loss of generality, and for simplicity of the presentation, we propose a spatial subdivision into 3 homogeneous compartments, see Figure 1. This space resolution may be arbitrarily chosen, while keeping valid the results of this paper.

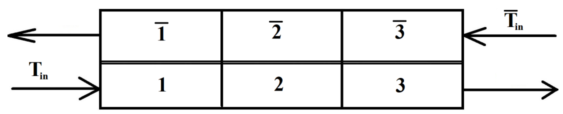

Fig. 1. Counter-current exchanger with inlet and outlet heat flux.

The heat transfer coefficients is denoted as λ $(J/K/s)$ and considered constant. For the hot and the cold fluid, the mass density ρ (Kg/m^3) and heat capacity c_p $(J/Kg.K)$ are assumed to be constant. The volume V (m^3) of the fluid in all the compartments is constant.

To derive the dynamical model, we write the energy balance equation for each compartment by taking into account an accumulation term, exchanged thermal energy through convection and heat transfer terms. Therefore, each compartment energy balance equation may be rewritten as a temperature differential equation, see Zitte et al. (2018) for more details. The dynamical model is given by

$$\begin{cases} \dot{T}_1 = \dfrac{\lambda}{\rho V c_p}(\overline{T}_1 - T_1) + \dfrac{q}{\rho V} T_{\text{in}} - \dfrac{q}{\rho V} T_1 \\ \dot{T}_2 = \dfrac{\lambda}{\rho V c_p}(\overline{T}_2 - T_2) + \dfrac{q}{\rho V} T_1 - \dfrac{q}{\rho V} T_2 \\ \dot{T}_3 = \dfrac{\lambda}{\rho V c_p}(\overline{T}_3 - T_3) + \dfrac{q}{\rho V} T_2 - \dfrac{q}{\rho V} T_3 \\ \dot{\overline{T}}_1 = \dfrac{-\lambda}{\rho V c_p}(\overline{T}_1 - T_1) + \dfrac{\bar{q}}{\rho V} \overline{T}_2 - \dfrac{\bar{q}}{\rho V} \overline{T}_1 \\ \dot{\overline{T}}_2 = \dfrac{-\lambda}{\rho V c_p}(\overline{T}_2 - T_2) + \dfrac{\bar{q}}{\rho V} \overline{T}_3 - \dfrac{\bar{q}}{\rho V} \overline{T}_2 \\ \dot{\overline{T}}_3 = \dfrac{-\lambda}{\rho V c_p}(\overline{T}_3 - T_3) + \dfrac{\bar{q}}{\rho V} \overline{T}_{\text{in}} - \dfrac{\bar{q}}{\rho V} \overline{T}_3 \end{cases} \tag{1}$$

where T_i, $\overline{T}_i \in \mathbb{R}$ represent the temperature of the compartment i of the hot (cold) fluid and of the cold (hot) fluid, respectively, T_{in}, $\overline{T}_{\text{in}} \in \mathbb{R}$ represent the inlet temperatures of the hot (or cold) and cold (or hot) fluid respectively, and $q, \bar{q} \in \mathbb{R}$ (Kg/s) are the mass flow rate of the hot (or cold) fluid and of the cold (or hot) fluid respectively. By using the compact notation $x = (T_1, T_2, T_3, \overline{T}_1, \overline{T}_2, \overline{T}_3)^\top \in \mathbb{R}^6$, $x_{\text{in}} = T_{\text{in}}$, $\bar{x}_{\text{in}} = \overline{T}_{\text{in}}$, we can thus compactly write the dynamical model (1) in the form

$$\dot{x} = Ax + (b_1 x_{\text{in}} + Bx)q + (\bar{b}_1 \bar{x}_{\text{in}} + \overline{B}x)\bar{q} \tag{2}$$

where

$$A = k \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right],$$

$$b_1 = \frac{1}{\rho V}\left[\begin{array}{c} 1 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right], \quad B = \frac{1}{\rho V}\left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$\bar{b}_1 = \frac{1}{\rho V}\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 1 \end{array}\right], \quad \overline{B} = \frac{1}{\rho V}\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right],$$

where $k = \frac{\lambda}{\rho V c_p}$. System (2) represents a multi-variable bilinear system with four degrees of freedom which are mass flow rates $q, \bar{q}$ and inlet temperatures $x_{\text{in}}, \bar{x}_{\text{in}}$ of the two fluids. To complete the system (2), we consider the outlet temperatures of the two fluids as a measured outputs $y, \eta \in \mathbb{R}$ defined as $y = \overline{T}_1$, $\eta = T_3$, that is

$$\begin{pmatrix} y \\ \eta \end{pmatrix} = \begin{pmatrix} C \\ D \end{pmatrix} x, \qquad \begin{array}{l} C = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right], \\ D = \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]. \end{array}$$

In the following section we recall the forwarding based approach for output regulation problem on a class of input affine non linear systems and applied on a class of bilinear systems. Then its application on the HEX will be given in section 4.

3. ROBUST CONTROL OF BILINEAR SYSTEMS VIA FORWARDING

Consider a non linear system of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ \dot{z} &= h(x) \end{aligned} \tag{3}$$

where $(x, z) \in \mathbb{R}^n \times \mathbb{R}$ is the state and $u \in \mathbb{R}$ is the control input. A methodology that can be used to design a state-feedback control law ensuring asymptotic stability of the origin of (3) is the forwarding approach developed in Praly et al. (2001), Poulain and Praly (2010). For this, the following assumptions are needed.

Assumption 1. The origin of $\dot{x} = f(x)$ is globally asymptotically stable, that is, there exists a known positive definite Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying

$$\dot{V} = \frac{\partial V}{\partial x} f(x) < 0$$

for all $x \in \mathbb{R}^n \setminus \{0\}$.

Assumption 2. There exists a C^1 *function* $\mathcal{M} : \mathbb{R}^n \to \mathbb{R}$ *such that* $\mathcal{M}(0) = 0$ *and satisfying the following properties*

$$\frac{\partial \mathcal{M}}{\partial x} f(x) = h(x), \qquad \frac{\partial \mathcal{M}}{\partial x} g(0) \neq 0. \tag{4}$$

Then, under previous assumptions, we have the following result, Praly et al. (2001), Poulain and Praly (2010).

Proposition 1. Under Assumptions 1, 2, the origin of system (3) *in closed loop with*

$$u(x,z) = -\left[\frac{\partial V}{\partial x} g(x) - (z - \mathcal{M}(x)) \frac{\partial \mathcal{M}}{\partial x} g(x)\right] \tag{5}$$

is globally asymptotically stable.

Proof: A complete proof can be found in Praly et al. (2001) and references therein. We recall here only the main arguments. Using the following Lyapunov function

$$W(x,z) = V(x) + \tfrac{1}{2}(z - \mathcal{M}(x))^2$$

with (4), we obtain: $\dot{W}(x,z) = \frac{\partial V}{\partial x} f(x) - u(x,z)^2$.

Thus $\dot{W}(x,z) < 0$ for $(x,z) \neq (0,0)$ and $\dot{W}(x,z) = 0$ only if $(x,z) = (0,0)$ since $L_g\mathcal{M}(0) \neq 0$. □

Now, let us specialize this result to bilinear systems:

$$\begin{aligned} \dot{x} &= F_o x + (B_o x + b_o) u \\ \dot{z} &= C_o x \end{aligned} \tag{6}$$

with $(x,z) \in \mathbb{R}^n \times \mathbb{R}$ and $u \in \mathbb{R}$.

Proposition 2. Suppose F_o *is Hurwitz and* $C_o F_o^{-1} b_o \neq 0$. *Then, the origin of system* (6) *in closed loop with*

$$u = -(2x^\top P - (z - Mx)^\top) M)(B_o x + b_o) \tag{7}$$

is globally asymptotically stable, where M *is a matrix satisfying*

$$M F_o = C_o \tag{8}$$

and P *is a positive definite matrix computed as solution to*

$$P F_o + F_o^\top P = -Q \tag{9}$$

where Q *is a positive definite matrix.*

Proof: Since F_o is Hurwitz, Assumption 1 is verified with the Lyapunov function $V = x^\top P x$. Furthermore, with the definition of $M = C_o F_o^{-1}$ in (8), and the fact that $C_o F_o^{-1} b_o \neq 0$, also Assumption 2 is verified with $\mathcal{M}(x) = Mx$. Then, the result follows by Proposition 1 by noting that the control law (5) coincides with (7) since $L_g V = 2x^\top P(B_o x + b_o)$ and $L_g \mathcal{M} = M(B_o x + b_o)$. □

Note that the condition $C_o F_o^{-1} b_o \neq 0$ corresponds to the fact that the transfer function $\mathcal{H}(s) = C_o(sI - F_o)^{-1} b_o$ has no zeros at the origin, see Astolfi and Praly (2017). Equivalently, this condition is verified if the following matrix $\begin{bmatrix} F_o & b_o \\ C_o & 0 \end{bmatrix}$ is full rank.

4. MAIN RESULTS

4.1 Control objective for heat exchanger

We consider as a manipulated control input the mass flow rate $u(t) = q(t)$ (even hot or cold fluid), and we suppose that the three remaining degrees of freedom $\bar{q}$, x_{in} and $\bar{x}_{\text{in}}$ are fixed at a nominal constant value (fixed operative conditions). The control objective is to regulate the output temperature $y = \overline{T}_1$ at a desired feasible temperature reference $\overline{T}_1^\star$, denoted in the following as r. Towards potential unknown and bounded perturbations of the three fixed operative conditions $\bar{q}$, x_{in} and $\bar{x}_{\text{in}}$, we aim to derive a control law which ensures robust regulation. We assume that the the fluid flow rate $u(t)$ is bounded and satisfies

$$u \in \mathcal{D}_u := [u_m, u_M] = \{u \in \mathbb{R}^+ : u_m \leq u \leq u_M\}, \tag{10}$$

where u_m and u_M are the minimal and the maximal flow rate. In the rest of the paper, given a compact set $\mathcal{A}$, we define $\text{int}(\mathcal{A})$ as its interior, namely the set of all interior points of $\mathcal{A}$. According to this notation, we obtain, for instance,

$$\text{int}(\mathcal{D}_u) = (u_m, u_M) = \{u \in \mathbb{R}^+ : u_m < u < u_M\}.$$

Then system (2) can be formulated as follows:

$$\begin{aligned} \dot{x} &= Fx + (Bx + b)u + G \\ y &= Cx \end{aligned} \tag{11}$$

with $F = A + \bar{q}\overline{B}$, $b = b_1 x_{\text{in}}$ and $G = \bar{b}_1 \bar{x}_{\text{in}} \bar{q}$.

4.2 Analysis of the heat exchanger model

In this subsection we analyze three important properties of the HEX model (11) for fixed operative conditions. First, we show that for bounded inputs u and G, that is, input temperatures T_{in}, $\overline{T}_{\text{in}}$ and flow rates $q, \bar{q}$ in model (1), the trajectories of (11) evolve in an invariant compact set on which $x_i > 0$ for all $i = 1, \ldots, 6$, see Lemma 1 below. Then, we analyze the properties of system (11) at a given equilibrium, showing the stability of a state matrix, see Lemma 2. Finally, we characterize the admissible output regulation set-points for system (12), see Lemma 3.

In the following, we consider, without loss of generality, the case in which T_{in} and $\overline{T}_{\text{in}}$ are the hot inlet temperature and the cold inlet one respectively. We have the result:

Lemma 1. For any $u \in \mathcal{D}_u$ *and fixed operative conditions,* T_{in} *and* $\overline{T}_{\text{in}}$ *(equivalently* x_{in} *and* $\bar{x}_{\text{in}}$*,respectively) the compact domain* $\mathcal{D}_x = \{x \in \mathbb{R}^6 : \overline{T}_{\text{in}} \leq x_i \leq T_{\text{in}}\}$ *is invariant with respect to* (11).

Proof: The sign of the dynamics on the boundary of the domain is analysed and we can show that the dynamics direction remains on the boundary or is such that it enters in the domain. □

Now let x^* be the steady-state of system (11) at a given (constant) input u^*, defined by

$$\begin{aligned} 0 &= Fx^* + (Bx^* + b)u^* + G \\ y^* &= Cx^* \end{aligned} \tag{12}$$

with y^* being the corresponding output. We have the following result concerning the stability of the matrix $F + Bu^*$ in the domain of interest.

Lemma 2. For all $u^* \in \mathcal{D}_u$*, with* $\mathcal{D}_u$ *defined in* (10)*, the matrix* $\hat{F} = F + Bu^*$ *defined as*

$$\hat{F} = A + \bar{B}\ \bar{q} + B\ u^* = \begin{bmatrix} -k-u^* & 0 & 0 & k & 0 & 0 \\ u^* & -k-u^* & 0 & 0 & k & 0 \\ 0 & u^* & -k-u^* & 0 & 0 & k \\ k & 0 & 0 & -k-\bar{q} & \bar{q} & 0 \\ 0 & k & 0 & 0 & -k-\bar{q} & \bar{q} \\ 0 & 0 & k & 0 & 0 & -k-\bar{q} \end{bmatrix}$$

is Hurwitz, namely its eigenvalues have strictly negative real part.

Proof: Thanks to Gershgorin Theorem, see Gershgorin (1931), we know that eigenvalue of the matrix $\hat{F}$ are in the union of the following circles:

$$\Phi_i = \{z \in \mathbb{C} : |z - \hat{F}_{ii}| \leq \sum_{\substack{j=1 \\ j\neq i}}^{6} |\hat{F}_{ij}|\},\ i = 1, 2, ..., 6.$$

The previous expression gives 6 circles with two different centers: $c_1 = -k - u^*$ and $c_2 = -k - \bar{q}$ with radius $r_1 = k + u^*$ and $r_2 = k + \bar{q}$, respectively. This circles are on the left hand side of the complex plane including the origin. Hence all the eigenvalues of $\hat{F}$ have non-positive real part. Furthermore, since the only point of the imaginary axes included in the circle is the origin, $\hat{F}$ cannot have any imaginary eigenvalues with zero real part. Finally, it is possible to show that the origin cannot be an eigenvalue of $\hat{F}$ since the only vector v satisfying $\hat{F}v = 0$ is $v = 0$. We conclude that all the eigenvalues of $\hat{F}$ have strictly negative real parts. □

Finally, we analyze the domain of admissible constant reference outputs for model (12).

Lemma 3. For any fixed operative condition $\bar{q}$, x_{in} and $\bar{x}_{\text{in}}$ (equivalently, for any given fixed F and G), there exist $y_M > y_m > 0$ such that for any $r \in (y_m, y_M)$ there exists (x^*, u^*), with $u \in \text{int}(\mathcal{D}_u)$, such that $y^* = r$, with y^* given by (12).

Proof: In view of Lemma 2, the matrix $\hat{F} = F + Bu^*$ is invertible for any $u^* \in \mathcal{D}_u$. Hence, for any u^*, there exists a unique equilibrium point satisfying x^* (12). It is computed as $x^* = -(F + Bu^*)^{-1}(bu^* + G)$. Hence, define the function $\phi : \mathbb{R} \to \mathbb{R}$ as $\phi(u^*) = -C(F + Bu^*)^{-1}(bu^* + G)$. It is continuous on the compact set $\mathcal{D}_u$ and it has a maximum and a minimum defined as $y_m = \inf_{u^* \in \mathcal{D}_u} \phi(u^*)$, $y_M = \sup_{u^* \in \mathcal{D}_u} \phi(u^*)$. Hence ϕ is surjective on $[y_m, y_M]$ and the proof is completed. □

4.3 Robust temperature regulation

For the output temperature reference r, we define the output error variable as follows:

$$e = Cx - r\,. \tag{13}$$

Based on this error variable we define an extended system with output error integral dynamics as follows

$$\begin{aligned} \dot{x} &= Fx + (Bx + b)u + G \\ \dot{z} &= Cx - r\,. \end{aligned} \tag{14}$$

In order to design a stabilizing control law for system (14) achieving the temperature regulation objective $y^* = r$, we follow the forwarding design procedure highlighted in Section 3. For this, let P be a symmetric positive definite matrix solution of

$$\hat{F}^\top P + P\hat{F} = (F + Bu^*)^\top P + P(F + Bu^*) = -Q, \tag{15}$$

for some symmetric positive definite matrix Q and for some operative point u^* to be fixed. Then, let M be a matrix solution of

$$M\hat{F} = M(F + Bu^*) = C, \tag{16}$$

again, for some operative point u^* to be fixed. Note that in light of invertibility of $\hat{F}$, M can be computed as $M = C\hat{F}^{-1}$. Finally, let us define the following saturation function $\text{sat}_{\mathcal{D}_u} : \mathbb{R} \to \mathbb{R}$ as

$$\text{sat}_{\mathcal{D}_u}(u) = \begin{cases} u_m & \text{if } 0 < u \leq u_m, \\ u & \text{if } u_m < u \leq u_M, \\ u_M & \text{if } u_M \leq u. \end{cases} \tag{17}$$

The following theorem presents the main result of this paper.

Theorem 1. Let F, G be fixed, and select any desired temperature reference $r \in (y_m, y_M)$, with y_m, y_M given by Lemma 3. Let (x^*, u^*) be a solution to (12) with $y^* = r$, and let P, M computed according to (15) and (16), respectively. Then, for any initial condition $(x(0), z(0)) \in \text{int}(\mathcal{D}_x) \times \{0\}$, with $\mathcal{D}_x$ given by Lemma 1, the corresponding trajectory of system (14) in closed loop with

$$u = \text{sat}_{\mathcal{D}_u}\bigg(u^* - \Big[2(x - x^*)^\top P - (z - M(x - x^*))^\top M\Big](Bx + b)\bigg) \tag{18}$$

satisfies $x(t) \in \mathcal{D}_x$ for all $t \geq 0$ and $\lim_{t\to\infty} y(t) = r$.

Proof: First, by applying the following change of coordinates $x \mapsto \tilde{x} := x - x^*$, we can rewrite system (11) as

$$\begin{aligned} \dot{\tilde{x}} &= \hat{F}\tilde{x} + (B\tilde{x} + \hat{b})(u - u^*) \\ \dot{z} &= C\tilde{x} \end{aligned} \tag{19}$$

with $\hat{F} = A + \bar{q}\bar{B} + u^*B$ and $\hat{b} = b + Bx^*$, in which we used the relation $(Bx + b) = (B\tilde{x} + \hat{b})$. The control law (18) can be rewritten equivalently as follows

$$u = \text{sat}_{\mathcal{D}_u}(u^* - \alpha(\tilde{x}, z))$$

with the compact notation

$$\alpha(\tilde{x}, z) = \big(2\tilde{x}^\top P - (z - M\tilde{x})M\big)(B\tilde{x} + \hat{b}).$$

Now, by following the results of Propositions 1, 2 let us consider the following Lyapunov function for the extended system (19)

$$W(\tilde{x}, z) = \tilde{x}^\top P\tilde{x} + \tfrac{1}{2}(z - M\tilde{x})^2. \tag{20}$$

The derivative of W along the closed loop system is given by

$$\begin{aligned} \dot{W} =& -\tilde{x}^\top Q\tilde{x} + 2\tilde{x}^\top P(B\tilde{x} + \hat{b})(u - u^*) \\ &+ (z - M\tilde{x})^\top(C\tilde{x} - M\hat{F}\ \tilde{x} - M(B\ \tilde{x} + \hat{b})(u - u^*)) \\ =& -\tilde{x}^\top Q\tilde{x} + U(\tilde{x}, z, u) \end{aligned} \tag{21}$$

where we defined

$$U(\tilde{x}, z, u) = \alpha(\tilde{x}, z)\big[\,\text{sat}_{\mathcal{D}_u}(u^* - \alpha(\tilde{x}, z)) - u^*\big].$$

Next, we show that $U(\tilde{x}, z, u) < 0$ for all $\tilde{x}, z, u$. For this we consider the following three cases, in which we recall that $u_m < u^* < u_M$.

1) $\alpha(\tilde{x}, z) \in [u^* - u_M, u^* - u_m]$. In this case,
$$\text{sat}_{\mathcal{D}_u}(u^* - \alpha(\tilde{x}, z)) = u^* - \alpha(\tilde{x}, z)$$
and therefore $U(\tilde{x}, z, u) = -\alpha(\tilde{x}, z)^2$.
2) $\alpha(\tilde{x}, z) \in [u^* - u_m, +\infty)$. In this case, we have $\alpha(\tilde{x}, z) > 0$ and moreover, $-\alpha(\tilde{x}, z) < u_m - u^*$. We compute $\text{sat}_{\mathcal{D}_u}(u^* - \alpha(\tilde{x}, z)) = u_m$, from which we obtain
$$U(\tilde{x}, z, u) = \alpha(\tilde{x}, z)(u_m - u^*) = -\alpha(\tilde{x}, z)(u^* - u_m) < 0$$
since $\alpha(\tilde{x}, z) > 0$ and $(u^* - u_m) > 0$.
3) $\alpha(\tilde{x}, z) \in (-\infty, u^* - u_M]$. In this case, we have $\alpha(\tilde{x}, z) < 0$ and moreover, $-\alpha(\tilde{x}, z) > u_M - u^*$. We compute $\text{sat}_{\mathcal{D}_u}(u^* - \alpha(\tilde{x}, z)) = u_M$ and therefore
$$U(\tilde{x}, z, u) = \alpha(\tilde{x}, z)(u_M - u^*) < 0$$
since $\alpha(\tilde{x}, z) < 0$ and $(u_M - u^*) > 0$.

Therefore, by combining the above three cases, we obtain $U(\tilde{x}, z, u) \leq -|\alpha(\tilde{x}, z)| \min\{|\alpha(\tilde{x}, z)|, (u^* - u_m), (u_M - u^*)\}$ and therefore
$\dot{W} \leq -\tilde{x}^\top Q \tilde{x} - |\alpha(\tilde{x}, z)| \min\{|\alpha(\tilde{x}, z)|, (u^* - u_m), (u_M - u^*)\}$.
Hence, by using LaSalle invariance principle, we conclude that solutions converge to the largest invariant set contained in $\mathcal{I} = \{(\tilde{x}, z) \in \mathbb{R}^n \times \mathbb{R} : \tilde{x} = 0, \alpha(\tilde{x}, z) = 0\}$. In the set $\mathcal{I}$, the z-dynamics satisfy
$$\dot{z} = 0, \qquad zM\hat{b} = 0. \tag{22}$$
By using the expression of M, we have $M\hat{b} = C\hat{F}^{-1}\hat{b}$. We recall that $\hat{F}$ is full rank and that the expressions of $\hat{b}$ and C are given by
$$\hat{b}^\top = \left[\begin{array}{ccc|ccc} (x_{\text{in}} - x_1^*) & (x_1^* - x_2^*) & (x_2^* - x_1^*) & 0 & 0 & 0 \end{array}\right],$$
$$C = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right].$$
Therefore, it can be verified that for all $x^* \in \text{int}(\mathcal{D}_x)$ (excluding by fact the case where $x_{\text{in}} = x_1^* = x_2^* = x_3^*$) the following matrix $\begin{bmatrix} \hat{F} & \hat{b} \\ C & 0 \end{bmatrix}$ is full rank and equivalently $M\hat{b} \neq 0$. We conclude that the largest invariant set contained in $\mathcal{I}$ is the origin $(\tilde{x}, z) = (0, 0)$. This shows that the origin is globally asymptotically stable.

Now, recall that the control law u is saturated in $\mathcal{D}_u$. Therefore, in light of Lemma 1, the set $\mathcal{D}_x$ is invariant for solution of the closed-loop system (14), (18). Hence, any solution starting in $\text{int}(\mathcal{D}_x)$ remains in $\mathcal{D}_x$ for all forward times and converges asymptotically to the equilibrium $(\tilde{x}, z) = 0$. Finally, on this equilibrium, $\dot{z} = 0$ and therefore $r = Cx^*$, which concludes the proof. □

It is important to note that that even if $x^\star, u^\star$ are explicitly used in the control law (18), all the results are robust to small variations of F, b and G (due to variations of the operative conditions) thanks to the robustness properties of the forwarding control method shown in Poulain and Praly (2010), Astolfi and Praly (2017),

5. SIMULATIONS

The following simulations are done on Matlab software. Numerical values are listed in Table 1. We choose the hot stream flow rate as the control input variable u and the output temperature of the cold stream $\overline{T}_1$ as the controlled output. Initial state vector corresponds to the steady state for $u = 0.17u_M$.

$\lambda = 10J/K/s$	$\rho = 997Kg/m^3$
$V = 0.002m^3$	$c_p = 4185J/Kg/K$ (for water)
$\bar{q} = 0.02Kg/s$	$u_M = 0.05Kg/s$
$T_{\text{in}} = x_{\text{in}} = 360K$	$\overline{T}_{\text{in}} = \bar{x}_{\text{in}} = 300K$

Table 1. Values of the parameters of the HEX

5.1 Output regulation simulation results

Figure 2 presents the output temperature $\overline{T}_1$ of the HEX (blue) and the regulation temperature reference (green). Figure 3 shows the corresponding closed loop control input expressed as a fraction of the maximal flow rate u_M.

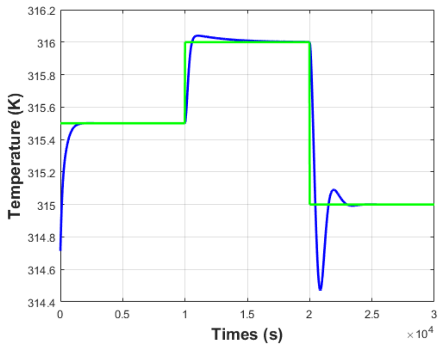

Fig. 2. Output (blue) and reference (green) temperatures

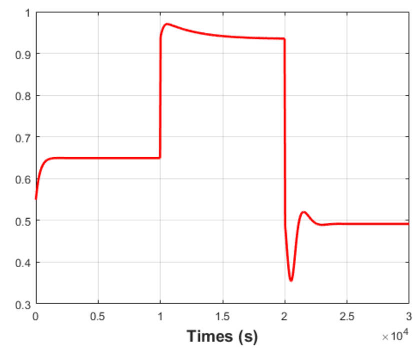

Fig. 3. Closed loop control as a fraction of u_M

The fixed closed loop simulation scenarios start at $t = 0$ with temperature target fixed to $\overline{T}1 = 315.5K$. This reference is switched to $\overline{T}1 = 316K$ at $t = 10000s$, and finally to $\overline{T}1 = 315K$ at $t = 20000s$. We can observe the effectiveness of the the designed control law (18) and the effective asymptotic regulation of the output temperature.

5.2 Robustness: simulation results

Hereafter, we propose one simulation scenario to emphasize some robustness properties of the designed control law towards disturbed inlet temperature T_{in}. The results of the closed loop dynamics for a perturbation on the temperature T_{in} introduced at $t = 10000s$ are depicted in Figures 4 and 5, showing the rejection of this constant perturbation by the controller.

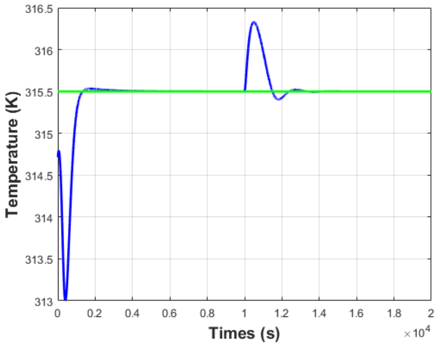

Fig. 4. Output (blue) and reference (green) temperatures

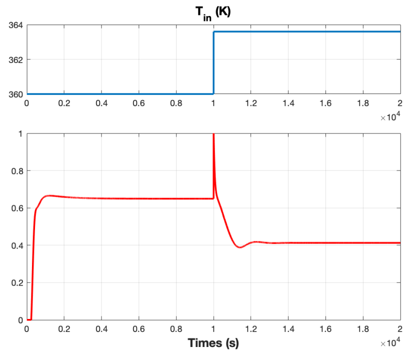

Fig. 5. Disturbed input T_{in} and the closed loop control as a fraction of u_M

6. CONCLUSION AND PERSPECTIVES

We proposed a robust temperature output regulation control based on a bilinear dynamical model of the counter-current HEX. This control law is derived using forwarding approach through error integral extended system. Simulation results confirm the effectiveness and the robustness of the proposed control.

Moreover, since the proposed control law depends on all the state variables of the system, it will be interesting to design a state observer using only the boundary output measurements, so that to obtain an output feedback control law, in the same spirit of Astolfi and Praly (2017). It will be also interesting, from a practical point of view, to consider a HEX model with possible different phase changes along the exchanger (evaporators or condensers). Finally the proposed control design approach can also be studied using an infinite dimensional dynamical model of the HEX.

REFERENCES

Alsop, A.W. and Edgar, T.F. (1989). Nonlinear heat exchanger control through the use of partially linearized control variables. *Chemical Engineering Communications*, 75(1), 155–170.

Astolfi, D. and Praly, L. (2017). Integral action in output feedback for multi-input multi-output nonlinear systems. *IEEE Transactions on Automatic Control*, 62(4), 1559–1574.

Chandrashekar, M. and Wong, F. (1982). Thermodynamic systems analysis - i. a graph-theoretic approach. *Energy*, 7(6), 539 – 566.

Diaz-Mendez, S., Patiño Carachure, C., and Herrera-Castillo, J. (2014). Reducing the energy consumption of an earth-air heat exchanger with a PID control system. *Energy Conversion and Management*, 77, 1 – 6.

Gershgorin, S. (1931). Über die Abgrenzung der Eigenwerte einer Matrix. *Bull. Acad. Sci. URSS*, 1931(6), 749–754.

Gu, W., Chen, X., Liu, K., Zhang, B., Chen, Q., and Hui, C.W. (2015). Retrofitting of the heat exchanger network with steam generation in a crude oil distillation unit. *Chemical Engineering & Technology*, 38(2), 203–214.

Huhtala, K. and Paunonen, L. (2019). Robust output regulation of counter-flow heat exchangers. *IFAC-PapersOnLine*, 52(2), 201 – 206. 3rd IFAC Workshop on Control of Systems Governed by Partial Differential Equations CPDE 2019.

Maidi, A., Diaf, M., and Corriou, J.P. (2009). Boundary geometric control of a counter-current heat exchanger. *Journal of Process Control*, 19(2), 297 – 313.

Ozorio Cassol, G., Ni, D., and Dubljevic, S. (2019). Heat exchanger system boundary regulation. *AIChE Journal*, 65(8), e16623.

Poulain, F. and Praly, L. (2010). Robust asymptotic stabilization of nonlinear systems by state feedback. In *8th IFAC Symposium on Nonlinear Control Systems*, 653–658.

Praly, L., Ortega, R., and Kaliora, G. (2001). Stabilization of nonlinear systems via forwarding mod L/sub g/V. *IEEE Transactions on Automatic Control*, 46(9), 1461–1466.

Sakawa, M., Kato, K., and Ushiro, S. (2002). Operational planning of district heating and cooling plants through genetic algorithms for mixed 0–1 linear programming. *European Journal of Operational Research*, 137(3), 677 – 687.

Scholten, T., de Persis, C., and Tesi, P. (2017). Modeling and control of heat networks with storage: The single-producer multiple-consumer case. *IEEE Transactions on Control Systems Technology*, 25, 414–428.

Sridhar, U.M., Govindarajan, A., and Rhinehart, R.R. (2016). Demonstration of leapfrogging for implementing nonlinear model predictive control on a heat exchanger. *ISA Transactions*, 60, 218 – 227.

Varga, E., Hangos, K., and Szigeti, F. (1995). Controllability and observability of heat exchanger networks in the time-varying parameter case. *Control Engineering Practice*, 3(10), 1409 – 1419.

Wu, J., Tremeac, B., Terrier, M.F., Charni, M., Gagnière, E., Couenne, F., Hamroun, B., and Jallut, C. (2016). Experimental investigation of the dynamic behavior of a large-scale refrigeration – pcm energy storage system. validation of a complete model. *Energy*, 116, 32 – 42.

Zavala-Río, A., Astorga-Zaragoza, C., and Hernández-González, O. (2009). Bounded positive control for double-pipe heat exchangers. *Control Engineering Practice*, 17(1), 136 – 145.

Zitte, B., Hamroun, B., Couenne, F., and Pitault, I. (2018). Representation of heat exchanger networks using graph formalism. *IFAC-PapersOnLine*, 51(3), 44 – 49. 6th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control LHMNC 2018.