Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

A Consistent Discretisation method for Stable Homogeneous Systems based on Lyapunov Function ⋆

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Abstract: In this paper we propose a discretisation scheme for continuous and asymptotically stable homogeneous systems. This method is based on the dynamics of the system projected on a level surface of a homogeneous Lyapunov function. The discretisation method is explicit and preserves the convergence rate of the continuous-time system.

Keywords: Nonlinear systems, discrete-time systems, homogeneous systems.

1. INTRODUCTION

Homogeneity is a symmetry constraint imposed on the system dynamics that provides useful properties for analysis and design of control systems. For example, this kind of systems can be used to approximate nonlinear models by preserving important nonlinear features of the original dynamics (Zubov, 1964; Hermes, 1986; Andrieu et al., 2008). Some additional interesting characteristics of homogeneous systems are, the existence of homogeneous Lyapunov functions (Rosier, 1992), and homogeneous controllers (Kawski, 1988; Hermes, 1991; Sepulchre and Aeyels, 1996; Grüne, 2000), the direct association of the homogeneity degree with the convergence rates (Haimo, 1986; Hong et al., 1999; Nakamura et al., 2002), and the intrinsic robustness properties to exogenous perturbations and delays (Bernuau et al., 2013),(Efimov et al., 2016). These features have motivated many authors to develop techniques for analysis and design of homogeneous control systems as homogeneous observers for linear systems (Cruz-Zavala and Moreno, 2016), homogeneous polynomial systems (Parrilo, 2000), and several high-order sliding mode controllers (Levant, 2005), to mention a few interesting cases.

On the other hand, with omnipresence of computers and digital devices, discretisation of continuous-time models has become essential to the design of control systems. It is useful, for example, for computer simulation, for implementation in electronic devices, and for the design of digital (or sampled-data) controllers (Nesić and Teel, 2001; Goodwin et al., 2013).

⋆ This work was supported by the Agence Nationale de la Recherche project ANR-18-CE40-0008 "DIGITSLID", by the Ministry of Science and Higher Education of Russian Federation, passport of goszadanie no. 2019-0898, and by Government of Russian Federation (Grant 08-08).

Unfortunately, unlike linear systems, nonlinear ones do not have (in general) an exact discretisation. Nevertheless, it is expected that an approximate discretisation (which is frequently obtained by a method derived for linear dynamics) preserves the most relevant features of the continuous-time system.

Although, there exist several methods to discretise nonlinear systems (Hairer et al., 1993), it has been shown that for nonlinear homogeneous systems these approaches may guarantee convergence of discretizations only locally and loosing important stability properties (Efimov et al., 2017). The standard discretisation techniques become even less applicable in the case of non-smooth dynamics. This has motivated several authors not only to study the properties of standard discretisation methods but also to design new ones to discretise homogeneous systems (Brogliato et al., 2018; Koch and Reichhartinger, 2018; Polyakov et al., 2019; Efimov et al., 2019).

In this paper we propose a procedure to discretise asymptotically stable continuous homogeneous systems. Our method relies on the information provided by a homogeneous Lyapunov function, thus, the discrete-time trajectory is guaranteed to converge to the origin. Moreover, the discretisation is consistent in the sense of (Polyakov et al., 2019), namely, the convergence rate of the continuous-time trajectory is preserved in its discrete-time counterpart. It is also important to emphasise that the proposed method is explicit, therefore, no system of algebraic equations has to be solved at each iteration.

Paper organization: In Section 2 the definition of homogeneity and some properties of homogeneous systems are recalled. In this section we also give two simple motivational examples and state the problem to be solved in the paper. In Section 3 we study the dynamics of a homogeneous system projected on the unitary level set of the Lyapunov function. In Section 4 we describe the proposed discretisation method for homogeneous systems. Some examples of the discretisation technique are presented in Section 5. In Section 6 we state some final remarks. The proofs of the results are not provided due to space restrictions.

Notation: Real and integer numbers are denoted as $\mathbb{R}$ and $\mathbb{Z}$, respectively. $\mathbb{R}_{>0}$ denotes the set $\{x \in \mathbb{R} : x > 0\}$, analogously for the set $\mathbb{Z}$ and the sign $\geq$. For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm. For a continuous positive definite function $V : \mathbb{R}^n \to \mathbb{R}$, we denote the set $\mathcal{S}_V = \{x \in \mathbb{R}^n : V(x) = 1\}$. The set of strictly increasing continuous functions $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\eta(0) = 0$ is denoted by $\mathcal{K}$. For $x \in \mathbb{R}$ and $p \in \mathbb{R}_{\geq 0}$, $\lceil x \rfloor^p := |x|^p \mathrm{sign}(x)$.

2. PRELIMINARIES AND PROBLEM STATEMENT

In this section we recall some properties of homogeneous systems and give the statement of the problem to be solved. We consider the following continuous-time system

$$\dot{x}(t) = f(x(t)), \quad x(t) \in \mathbb{R}^n, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on $\mathbb{R}^n$. We assume that for each $x(0) \in \mathbb{R}^n$, the solution of (1) exists and is unique in forward time for all $t \in \mathbb{R}_{\geq 0}$.

2.1 Homogeneity

Now, let us recall the definition of weighted homogeneity.

Definition 1. (Weighted homogeneity, (Kawski, 1988)). Let $\Lambda_\epsilon^{\mathbf{r}}$ denote the family of dilations given by the square diagonal matrix $\Lambda_\epsilon^{\mathbf{r}} = \mathrm{diag}(\epsilon^{\mathrm{r}_1}, \ldots, \epsilon^{\mathrm{r}_n})$, where $\mathbf{r} = [r_1, \ldots, r_n]^\top$, $r_i \in \mathbb{R}_{>0}$, and $\epsilon \in \mathbb{R}_{>0}$. The components of $\mathbf{r}$ are called the *weights* of the coordinates. Thus:

(a) a function $V : \mathbb{R}^n \to \mathbb{R}$ is $\mathbf{r}$-homogeneous of degree $m \in \mathbb{R}$ if
$$V(\Lambda_\epsilon^{\mathbf{r}} x) = \epsilon^m V(x), \ \forall x \in \mathbb{R}^n, \ \forall \epsilon \in \mathbb{R}_{>0};$$
(b) a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, is $\mathbf{r}$-homogeneous of degree $\mu \in \mathbb{R}$ if
$$f(\Lambda_\epsilon^{\mathbf{r}} x) = \epsilon^\mu \Lambda_\epsilon^{\mathbf{r}} f(x), \ \forall x \in \mathbb{R}^n, \ \forall \epsilon \in \mathbb{R}_{>0}.$$

System (1) is said to be $\mathbf{r}$-homogeneous of degree $\mu \in \mathbb{R}$ if its vector field f is $\mathbf{r}$-homogeneous of degree μ.

Now, we recall some properties of homogeneous systems. Suppose that (1) is $\mathbf{r}-$homogeneous of degree μ, and $V : \mathbb{R}^n \to \mathbb{R}$ is a differentiable $\mathbf{r}-$homogeneous function of degree m. Hence,

$$\dot{V} = -W(x), \quad W(x) := -\nabla V(x) f(x), \tag{2}$$

where the function $W : \mathbb{R}^n \to \mathbb{R}$ is $\mathbf{r}-$homogeneous of degree $m + \mu$. Note that if the origin of (1) is asymptotically stable, then there exists a continuously differentiable function which is a strict [1] Lyapunov function, and is $\mathbf{r}-$homogeneous of some degree $m \in \mathbb{R}_{>0}$. Indeed, the existence of such a Lyapunov function is guaranteed if the condition $m > \max_{i=1,\ldots,n} r_i$ hold (Rosier, 1992). In such a case, W is positive definite, continuous for all $x \in \mathbb{R}^n$, and there exists $\alpha \in \mathbb{R}_{>0}$ such that (Hong et al., 1999),

$$\dot{V} \leq -\alpha V^{\frac{m+\mu}{m}}(x), \tag{3}$$

where the constant α can be given as follows

$$\alpha = \inf_{x \in \mathcal{S}_V} W(x). \tag{4}$$

Note that the degree of homogeneity of W is strictly positive by restricting the degree of V to $m > -\mu$. An interesting consequence of (3) is the estimation of the decreasing rate of V along the solutions of (1).

Lemma 2. (See, e.g. (Haimo, 1986; Hong et al., 1999)). Let (1) be $\mathbf{r}-$homogeneous of degree μ, with a differentiable strict Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ which is $\mathbf{r}-$homogeneous of degree m. For all $x(0) \in \mathbb{R}^n$ and all $t \in \mathbb{R}_{\geq 0}$, the following holds (with α as given in (3), and $V_0 := V(x(0))$):

(1) if $\mu > 0$, then the converge rate is in practical fixed-time, i.e.
$$V(x(t)) \leq V_0 \left(1 + \tfrac{\mu}{m} \alpha V_0^{\frac{\mu}{m}} t\right)^{-\frac{m}{\mu}}; \tag{5}$$
(2) if $\mu = 0$, then the converge rate is exponential, i.e.
$$V(x(t)) \leq V_0 \exp(-\alpha t); \tag{6}$$
(3) if $\mu < 0$, then the converge rate is in finite-time, i.e. $V(x(t)) \leq \bar{V}(x(t))$, where
$$\bar{V}(x(t)) = \begin{cases} \left(V_0^{\frac{-\mu}{m}} - \frac{-\mu\alpha}{m} t\right)^{\frac{m}{-\mu}}, & t < \frac{m V_0^{\frac{-\mu}{m}}}{-\mu\alpha}, \\ 0, & t \geq \frac{m V_0^{\frac{-\mu}{m}}}{-\mu\alpha}. \end{cases} \tag{7}$$

2.2 Problem statement

Let us begin this section with two simple examples on the Euler discretisation of scalar homogeneous systems.

Example 3. Consider the following scalar system

$$\dot{x} = -a\lceil x \rfloor^{\frac{3}{4}}, \ a \in \mathbb{R}_{>0}. \tag{8}$$

Observe that, (8) is $\mathbf{r}-$homogeneous of degree $\mu = -1$ with $\mathbf{r} = 4$. Now, for any initial condition $x(0) \in \mathbb{R}$, the solution of (8) is given by

$$x(t) = \begin{cases} \lceil x(0) \rfloor^0 (|x(0)|^{\frac{1}{4}} - at/4)^4, & at < 4|x(0)|^{\frac{1}{4}}, \\ 0, & at \geq 4|x(0)|^{\frac{1}{4}}. \end{cases}$$

Note that $x = 0$ is an asymptotically stable equilibrium of (8) for any $a \in \mathbb{R}_{>0}$, moreover, it is finite-time stable. The explicit Euler discretisation of (8) is given by

$$\bar{x}(k+1) = \bar{x}(k) - ha\lceil \bar{x}(k) \rfloor^{\frac{3}{4}}, \quad k \in \mathbb{Z}_{\geq 0}, \tag{9}$$

where $h \in \mathbb{R}_{>0}$ is the discretisation step. In Fig. 1 we can see the trajectory generated by (9) with the parameters: $a = 5$, $h = 0.05$, and the initial condition $x(0) = 1$. Observe that, the continuous-time solution converges to zero in $t = 0.8$. However, the discrete-time approximation remains oscillating in the steady-state. Hence, we see that (9) is not a consistent discretisation of (8) since the finite-time stability is not preserved.

Example 4. Consider the following scalar system

$$\dot{x} = -ax^3, \ a \in \mathbb{R}_{>0}. \tag{10}$$

Observe that (8) is $\mathbf{r}-$homogeneous of degree $\mu = 2$ with $\mathbf{r} = 1$. Now, for any initial condition $x(0) \in \mathbb{R}$, the solution of (8) is given by

$$x(t) = x(0)[1 + 2ax^2(0)t]^{-\frac{1}{2}}. \tag{11}$$

Note that $x = 0$ is an asymptotically stable equilibrium of (8) for any $a \in \mathbb{R}_{>0}$, moreover, it is practically fixed-time

[1] In this paper we consider only strict Lyapunov functions i.e. those such that $\nabla V(x) f(x) < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

Fig. 1. Solution of (8) and its discrete-time approximation generated by the explicit Euler method.

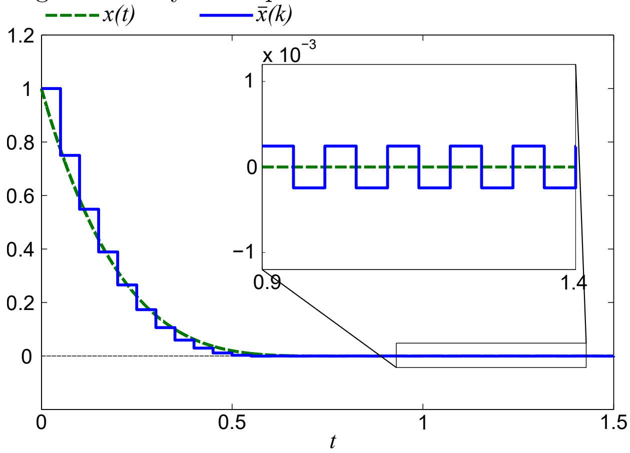
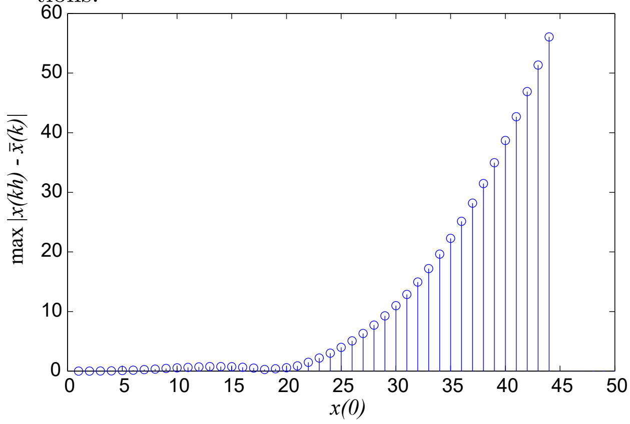


Fig. 2. Maximum error (in ten iterations) of the explicit Euler discretisation of (10) for different initial conditions.



stable, i.e. given $c \in \mathbb{R}_{>0}$ we have that $|x(t)| \leq c$ for all $t \geq \frac{1}{2ac^2}$ and all $x(0) \in \mathbb{R}$ (this bound is independent of the initial condition). The explicit Euler discretisation of (10) is given by

$$\bar{x}(k+1) = \bar{x}(k) - ha\bar{x}^3(k), \quad k \in \mathbb{Z}_{\geq 0}\,. \tag{12}$$

For the parameters $a = 1$ and $h = 0.001$, we compute ten iterations of (12) with the different initial conditions $x(0) = 1, 2, \ldots, 50$. For each simulation we calculate the maximum error (shown in Fig. 2) between (12) (the Euler discretisation of (10)) and (11) (the exact solution of (10)), i.e. $\max_{k=1,\ldots,10} |x(kh) - \bar{x}(k)|$. Observe that this error grows with the initial condition, and it is unbounded for initial conditions $x(0) \geq 45$. Thus, (12) is not a consistent discretisation of (10) since there exist initial conditions such that the trajectories of (12) diverge.

As we mentioned in the introduction, we cannot have (in general) an exact discretisation for a nonlinear system. However, it is expected that a suitable discretisation preserves important properties of the solutions, for example, the convergence rates described in Lemma 2. It is also expected that if a Lyapunov function is available, then it can be used to improve the discretisation scheme. For the case of homogeneous systems, a homogeneous Lyapunov function provides the information of stability and convergence rates, as stated in Lemma 2. Hence, the problem to be solved in this paper can be described as follows:

For continuous $\mathbf{r}$*−homogeneous systems whose origin is asymptotically stable, to provide a methodology to construct discretisation schemes such that the obtained discrete-time system preserves the converge rate of the continuous-time system.*

This problem is solved by exploiting the information provided by the homogeneous Lyapunov function of the system, and by considering the system's dynamics projected on the unitary level set of the Lyapunov function. Such a projection is described in the following section.

3. PROJECTED DYNAMICS

In this section we develop the fundamentals for the discretisation scheme proposed in Section 4.

Let (1) be $\mathbf{r}$−homogeneous of degree μ, and $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, which is positive definite and $\mathbf{r}$−homogeneous of degree m. Define the following auxiliary variable

$$y = \Lambda^{\mathbf{r}}_{V^{\frac{-1}{m}}(x)} x\,, \quad \forall x \in \mathbb{R}^n \setminus \{0\}\,. \tag{13}$$

Note that (13) is the homogeneous projection of x over the unitary level set of V, thus, $y \in \mathcal{S}_V$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Let us compute the dynamics of the auxiliary variable y. By differentiating (13) along (1) we obtain

$$\dot{y} = \Lambda^{\mathbf{r}}_{V^{\frac{-1}{m}}(x)} \Big(I - \tfrac{1}{m} V^{-1}(x) G x \nabla V(x)\Big) f(x)\,, \tag{14}$$

where I is the identity matrix, and $G := \mathrm{diag}(r_1, \ldots, r_n)$. From (13) we see that $x = \Lambda^{\mathbf{r}}_{V^{\frac{1}{m}}(x)} y$, thus, we rewrite (14) as follows

$$\begin{aligned}
\dot{y} &= \Lambda^{\mathbf{r}}_{V^{\frac{-1}{m}}(x)} f(x) - \tfrac{1}{m} V^{-1}(x) \Lambda^{\mathbf{r}}_{V^{\frac{-1}{m}}(x)} G \Lambda^{\mathbf{r}}_{V^{\frac{1}{m}}(x)} \times \\
&\quad \times y \nabla V(x) f(x)\,, \\
&= V^{\frac{\mu}{m}}(x) f(y) + \tfrac{1}{m} \tfrac{W(x)}{V(x)} G y\,, \\
&= V^{\frac{\mu}{m}}(x) f(y) + \tfrac{1}{m} \tfrac{V^{\frac{m+\mu}{m}}(x) W(y)}{V(x)} G y\,,
\end{aligned}$$

therefore,

$$\dot{y} = V^{\frac{\mu}{m}}(x) \big[f(y) + \tfrac{1}{m} W(y) G y\big]\,. \tag{15}$$

Equation (15) describes the dynamics (1) projected on $\mathcal{S}_V$. However, observe that we cannot recover the trajectory of (1) directly from the trajectory of (15) because (13) is not bijective. To overcome this problem, we proceed to study the dynamics of V, i.e. the derivative of V along (1). Thus, from (2) and (13), we obtain

$$\dot{V} = -W\big(\Lambda^{\mathbf{r}}_{V^{\frac{1}{m}}(x)} y\big) = -V^{\frac{m+\mu}{m}}(x) W(y)\,. \tag{16}$$

Now, define the function $v : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that it satisfies the scalar differential equation (cf. (16))

$$\dot{v}(t) = -v^{\frac{m+\mu}{m}}(t) W(z(t))\,, \tag{17}$$

with the function $z : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ being the solution of the system (cf. (15))

$$\dot{z}(t) = v^{\frac{\mu}{m}}(t) \big[f(z(t)) + \tfrac{1}{m} W(z(t)) G z(t)\big]\,. \tag{18}$$

It can be verified that $\mathcal{S}_V$ is a positively invariant set for the trajectories of (18).

From these developments we are ready to state the main results of this section. First, we give a useful result about a solution representation of (17).

Lemma 5. (1) For all $t \in \mathbb{R}_{\geq 0}$, and any initial condition $v_0 := v(0) \in \mathbb{R}_{\geq 0}$, the function $v : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$,

$$v(t) = v_0 \exp\big(- \hat{W}_0(t)\big)\,, \qquad \text{for } \mu = 0\,, \tag{19a}$$

$$v(t) = v_0 \Big(1 + \tfrac{\mu}{m} v_0^{\frac{\mu}{m}} \hat{W}_0(t)\Big)^{-\frac{m}{\mu}}\,, \text{ for } \mu > 0\,, \tag{19b}$$

with $\hat{W}_0(t) := \int_0^t W(z(\tau))\,\mathrm{d}\tau$, satisfies (17).

(2) For $\mu < 0$ and any initial condition $v_0 \in \mathbb{R}_{>0}$, there exists $\Theta(v_0) \in \mathbb{R}_{>0}$ such that the function $v : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by

$$v(t) = \begin{cases} \Big(v_0^{\frac{-\mu}{m}} - \frac{-\mu \hat{W}_0(t)}{m}\Big)^{\frac{m}{-\mu}}, & \frac{-\mu \hat{W}_0(t)}{m} < v_0^{\frac{-\mu}{m}} \\ 0, & \frac{-\mu \hat{W}_0(t)}{m} \geq v_0^{\frac{-\mu}{m}} \end{cases}, \tag{19c}$$

satisfies (17) for all $t \in \big[0, \Theta(v_0)\big)$, and $v(t) \to 0$ as $t \to \Theta(v_0)$.

We finalise this section with the statement of the fundamental results for the discretisation method described in Section 4.

Theorem 6. Let (1) be $\mathbf{r}-$homogeneous of degree μ, and $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable Lyapunov function which is $\mathbf{r}-$homogeneous of degree m. Define $\zeta = [v, z^\top]^\top \in Z$, $Z = \mathbb{R}_{>0} \times \mathcal{S}_V$. Consider (1) on $\mathbb{R}^n \setminus \{0\}$ and (17)-(18) on Z. The solutions of (1) and the solutions of (17)-(18) are equivalent with the homeomorphism $\Phi : \mathbb{R}^n \setminus \{0\} \to Z$ given by

$$\Phi(x) = \begin{bmatrix} V(x) \\ \Lambda^{\mathbf{r}}_{V^{-\frac{1}{m}}(x)} x \end{bmatrix}. \tag{20}$$

Corollary 7. Let (1) be $\mathbf{r}-$homogeneous of degree μ, and $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable Lyapunov function which is $\mathbf{r}-$homogeneous of degree m. Let v and z be solutions of (17), (18), respectively, with initial conditions $v(0) = V(x(0))$, $z(0) = \Lambda^{\mathbf{r}}_{v^{\frac{-1}{m}}(0)} x(0)$, for any $x(0) \in \mathbb{R}^n \setminus \{0\}$.

(1) If $\mu \geq 0$, then the function $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ given by

$$x(t) = \Lambda^{\mathbf{r}}_{v^{\frac{1}{m}}(t)} z(t)\,, \tag{21a}$$

is solution of (1) for all $t \in \mathbb{R}_{\geq 0}$.

(2) If $\mu < 0$, the function $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ given by

$$x(t) = \begin{cases} \Lambda^{\mathbf{r}}_{v^{\frac{1}{m}}(t)} z(t), & t < \Theta(v(0)), \\ 0, & t \geq \Theta(v(0)), \end{cases} \tag{21b}$$

(with Θ as given in Lemma 5) is solution of (1) for all $t \in \mathbb{R}_{\geq 0}$.

4. DISCRETISATION SCHEME

The main idea of the discretisation scheme comes from the developments of Section 3, and it can be sketched as follows: since (18) describes the dynamics of (1) projected on $\mathcal{S}_V$, and v (as given in Lemma 5) describes the decreasing behaviour of V along the solutions of (1), the idea is to obtain a numerical solution [2] of (18) on $\mathcal{S}_V$, and expand it to $\mathbb{R}^n$ by using (21) and a discretisation of v.

Remark 8. Although, a numerical solution of (18) can be obtained by different techniques, we restrict ourselves in this paper to the explicit (a.k.a. forward) Euler method.

[2] We mean by numerical solution a sequence $\{\bar{z}(k)\}_{k \in \mathbb{Z}_{\geq 0}}$ such that $\bar{z}(0) = z(0)$, and for some $h \in \mathbb{R}_{>0}$, $\bar{z}(k)$ approximates $z(kh)$.

To construct the discretisation of v, we see from Lemma 5 that for any $h \in \mathbb{R}_{\geq 0}$,

$$v(t+h) = v(t) \exp\big(- \hat{W}(t)\big)\,, \text{ for } \mu = 0\,, \tag{22a}$$

$$v(t+h) = v(t) \Big(1 + \tfrac{\mu}{m} v^{\frac{\mu}{m}}(t) \hat{W}(t)\Big)^{-\frac{m}{\mu}}\,, \text{ for } \mu > 0, \tag{22b}$$

and for $\mu < 0$,

$$v(t+h) = \begin{cases} \Big(v^{\frac{-\mu}{m}}(t) - \frac{-\mu \hat{W}(t)}{m}\Big)^{\frac{m}{-\mu}}, & \frac{-\mu \hat{W}(t)}{m} < v^{\frac{-\mu}{m}}(t), \\ 0, & \frac{-\mu \hat{W}(t)}{m} \geq v^{\frac{-\mu}{m}}(t), \end{cases} \tag{22c}$$

where $\hat{W}(t) := \int_t^{t+h} W(z(\tau))\,\mathrm{d}\tau$. Thus, for the discretisation of v we only have to define a discrete-time approximation of $\hat{W}(t)$. For example, by using the forward Euler method with an integration step h, the discrete-time approximation $\bar{v} : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ of v is given by

$$\bar{v}^+ = \bar{v} \exp\Big(- W(\bar{z}) h\Big)\,, \text{for } \mu = 0\,, \tag{23a}$$

$$\bar{v}^+ = \bar{v} \Big(1 + \tfrac{\mu}{m} \bar{v}^{\frac{\mu}{m}} W(\bar{z}) h\Big)^{-\frac{m}{\mu}}\,, \text{for } \mu > 0\,, \tag{23b}$$

and for $\mu < 0$,

$$\bar{v}^+ = \begin{cases} \Big(\bar{v}^{\frac{-\mu}{m}} - \frac{-\mu h W(\bar{z})}{m}\Big)^{\frac{m}{-\mu}}, & \frac{-\mu h W(\bar{z})}{m} < \bar{v}^{\frac{-\mu}{m}}, \\ 0, & \frac{-\mu h W(\bar{z})}{m} \geq \bar{v}^{\frac{-\mu}{m}}, \end{cases} \tag{23c}$$

where we denote $\bar{v} = \bar{v}(k)$, $\bar{v}^+ = \bar{v}(k+1)$, and $\bar{z} = \bar{z}(k)$. The function $\bar{z} : \mathbb{Z}_{\geq 0} \to \mathbb{R}^n$ is the discrete-time approximation of z given by

$$\bar{z}^+ = \Lambda^{\mathbf{r}}_{V^{\frac{-1}{m}}(\tilde{z}^+)} \tilde{z}^+\,, \tag{24}$$

$$\tilde{z}^+ = \begin{cases} \bar{z} + h \bar{v}^{\frac{\mu}{m}} \big(f(\bar{z}) + \frac{1}{m} W(\bar{z}) G \bar{z}\big), & \bar{v}^+ > 0, \\ \bar{z}, & \bar{v}^+ = 0. \end{cases}$$

Note that (24) can be seen as the explicit Euler discretisation of (18), including a scaling factor given by $\Lambda^{\mathbf{r}}_{V^{\frac{-1}{m}}(\tilde{z}^+)}$. This scaling is necessary to guarantee that $\bar{z}(k) \in \mathcal{S}_V$ for all $k \in \mathbb{Z}_{\geq 0}$, nonetheless, it is also necessary that $\tilde{z}^+ \neq 0$, thus we require the following assumption.

Assumption 9. For all $z \in \mathcal{S}_V$ and all $\tau \in \mathbb{R}_{>0}$,

$$-z \neq \tau\big(f(z) + \tfrac{1}{m} W(z) G z\big)\,.$$

Observe that, if there exist $\tau \in \mathbb{R}_{>0}$ and $\bar{z} \in \mathcal{S}_V$ such that $\bar{z} + \tau F(\bar{z}) = 0$, with $F(\bar{z}) := f(\bar{z}) + \frac{1}{m} W(\bar{z}) G \bar{z}$, then the vector $F(\bar{z})$ is necessarily collinear to $\bar{z}$. This holds if and only if there exists $w \in \mathbb{R}^n \setminus \{0\}$ which is orthogonal to both $F(\bar{z})$ and $\bar{z}$. Note that $\nabla V(\bar{z})$ is orthogonal to $F(\bar{z})$, thus it is also orthogonal to $\bar{z}$ if and only if $\nabla V(\bar{z})\,\bar{z} = 0$. Hence, we conclude that a sufficient condition to guarantee that Assumption 9 holds is $\nabla V(z)\,z \neq 0$ for all $z \in \mathcal{S}_V$. In simple words, there is no tangent vector to $\mathcal{S}_V$ pointing towards the origin.

Now, we can state the main result of this section.

Theorem 10. Let (1) be $\mathbf{r}-$homogeneous of degree μ with a strict Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$, which is continuously differentiable and $\mathbf{r}-$homogeneous of degree m. Suppose that Assumption 9 holds. Consider the discrete-time approximation $\bar{x}(k)$ of $x(kh)$, given by

$$\bar{x}(k+1) = \Lambda^{\mathbf{r}}_{\bar{v}^{\frac{1}{m}}(k+1)} \bar{z}(k+1)\,, \quad k \in \mathbb{Z}_{\geq 0}\,, \tag{25}$$

where $\bar{v}$ and $\bar{z}$ are given by (23) and (24), respectively, with the initial conditions $\bar{v}(0) = V(x(0))$, $\bar{z}(0) =$

$\Lambda^{\mathbf{r}}_{V^{\frac{-1}{m}}(x(0))} x(0)$. Then, for all $h \in \mathbb{R}_{>0}$ and all $x(0) \in \mathbb{R}^n \setminus \{0\}$, $\bar{x}(k) \to 0$ as $k \to \infty$. Moreover (with α as in (4)):

(1) if $\mu = 0$, then the convergence rate is exponential, i.e.
$$V(\bar{x}(k)) \leq V(\bar{x}(0)) \exp(-\alpha h k)\,;$$
(2) if $\mu > 0$, then the convergence rate is practically in fixed-time, i.e.
$$V(\bar{x}(k)) \leq V(\bar{x}(0))\left(1 + \tfrac{\mu}{m} V^{\frac{\mu}{m}}(\bar{x}(0))\alpha h k\right)^{-\frac{m}{\mu}}\,;$$
(3) if $\mu < 0$, then the convergence rate is in finite-time, i.e. $V(\bar{x}(k)) \leq \bar{V}(k)$, where
$$\bar{V}(k) = \begin{cases} \left(V^{\frac{-\mu}{m}}(\bar{x}(0)) - \frac{-\mu\alpha h}{m} k\right)^{\frac{m}{-\mu}}, & k < \frac{mV^{\frac{-\mu}{m}}(\bar{x}(0))}{-\mu\alpha h}, \\ 0, & k \geq \frac{mV^{\frac{-\mu}{m}}(\bar{x}(0))}{-\mu\alpha h}. \end{cases}$$

Theorem 10 requires the availability of a homogeneous Lyapunov function. Unfortunately there is no universal technique to compute it, nevertheless the existence of such a function is guaranteed by some converse Lyapunov theorems (Rosier, 1992; Nakamura et al., 2002), and there are some methods to design homogeneous Lyapunov functions for particular classes of homogeneous systems, e.g. (Polyakov and Poznyak, 2012; Sanchez and Moreno, 2019; Efimov et al., 2018). Note that the conclusions of Theorem 10 are valid for any discretisation step $h \in \mathbb{R}_{>0}$ provided that Assumption 9 holds.

5. EXAMPLES

In this section we illustrate the proposed discretisation method with two different systems. One of them with the property of finite-time stability and the another one with the property of practical fixed-time stability.

5.1 Finite-time stability

In this section we resume the system of Example 3. We have seen that the origin of (8) is finite-time stable. To apply our proposed discretisation scheme, we consider the function $V : \mathbb{R} \to \mathbb{R}$ given by
$$V(x) = \tfrac{4}{5}|x|^{\frac{5}{4}}\,.$$
This function is $\mathbf{r}-$homogeneous of degree $m = 5$, and a Lyapunov function for (8) for any $a \in \mathbb{R}_{>0}$, with $\dot{V} = -W(x)$, $W(x) = a|x|$. Hence, the discrete-time approximation is given by (23), (24), (25), namely, $\bar{x}(k+1) = (\bar{v}^+)^{\frac{4}{5}} \bar{z}^+$ with $\bar{z}^+ = \left(\frac{4}{5}|\bar{z}^+|^{\frac{5}{4}}\right)^{\frac{-4}{5}} \bar{z}^+ = \left(\frac{5}{4}\right)^{\frac{4}{5}} \mathrm{sign}(\bar{z}^+)$,
$$\bar{v}^+ = \begin{cases} \left(\bar{v}^{\frac{-\mu}{m}} - \frac{1}{5}ha|\bar{z}|\right)^5, & ha|\bar{z}| < 5\bar{v}^{\frac{1}{5}}, \\ 0, & ha|\bar{z}| \geq 5\bar{v}^{\frac{1}{5}}, \end{cases}$$
$$\bar{z}^+ = \begin{cases} \bar{z} + ah\bar{v}^{\frac{-1}{5}}\left(\lceil \bar{z} \rfloor^{\frac{3}{4}} + \frac{4}{5}\lceil \bar{z} \rfloor^2\right), & \bar{v}^+ > 0, \\ \bar{z}, & \bar{v}^+ = 0. \end{cases}$$

For the simulation we use the parameters: $a = 5$ and $h = 0.05$. We simulate the system 1.5 seconds for the initial condition $x(0) = 2$. In Fig. 3 we show the error signal $\bar{x}(k) - x(kh)$ generated by both the standard Euler discretisation scheme and our proposed method. We can see that, although the errors are comparable along the transient period, the standard Euler scheme shows a permanent deviation in the steady-state. However, the proposed Lyapunov-based scheme provides zero deviation in the steady-state.

Fig. 3. Discretisation error: dotted-line for standard Euler method; solid-line for proposed method.

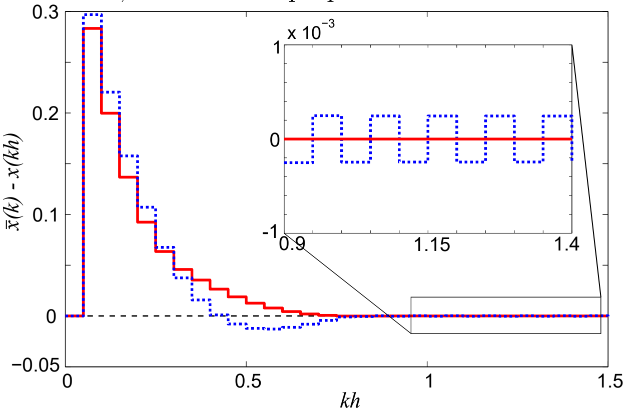
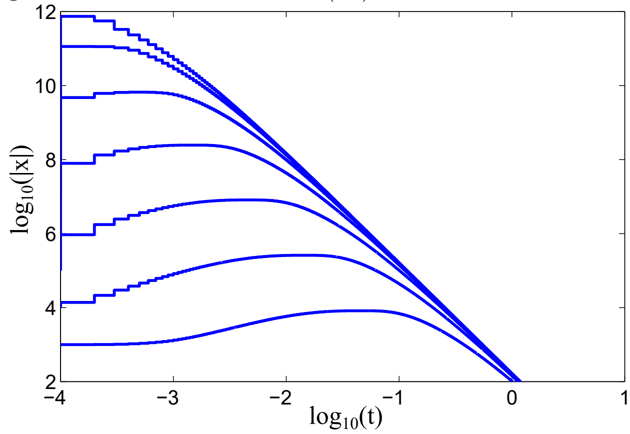


Fig. 4. Norm of the states of (26).



5.2 Practical fixed-time stability

For this example we consider the following system
$$\dot{x}_1 = -k_1\lceil x_1 \rfloor^{\frac{3}{2}} + x_2\,, \quad \dot{x}_2 = -k_2\lceil x_1 \rfloor^2\,, \tag{26}$$
which is $\mathbf{r}-$homogeneous of degree $\mu = 1$ with $\mathbf{r} = [2, 3]^\top$. Consider the function $V : \mathbb{R}^2 \to \mathbb{R}$ given by
$$V(x) = \tfrac{2}{5}k_1|x_1|^{\frac{5}{2}} - x_1x_2 + \tfrac{3}{5}\alpha|x_2|^{\frac{5}{3}}\,.$$
This function is $\mathbf{r}-$homogeneous of degree $m = 5$, and it can be proven that for any $k_1 \in \mathbb{R}_{>0}$ there exist $\alpha, k_2 \in \mathbb{R}_{>0}$ such that V is a Lyapunov function for (26). For the simulation we consider the parameters: $k_1 = 2$, $k_2 = 1$, and $\alpha = 2$. We simulate the system 1.2 seconds by using 12000 steps, i.e. with a step length $h = 0.0001$, for the different initial conditions $x_2(0) = 0$ and $x_1(0) = 10^q$ with $q = 3, \ldots, 9$. The norm of the system's states is shown in the logarithmic plot of Fig. 4, there we can appreciate the fixed-time convergence to the set $\{x \in \mathbb{R}^2 : |x| \leq 100\}$.

6. CONCLUSION

For continuous homogeneous systems with asymptotically stable origin, we developed a methodology to formulate discretisation schemes that preserve the convergence rate of the continuous-time system. The key ingredient of the method is the exploitation of the information provided by the Lyapunov function. It is important to mention that the proposed methodology does not restrict the discretisation of the projected dynamics to the explicit Euler method. Hence, a different technique could be used in such a process to obtain a different discretisation scheme, but preserving the main properties shown in this paper. Some future developments: the extension of the methodology to systems with discontinuities; study of different schemes obtained by modifying the discretisation for the projected dynamics; application of the proposed discretisation method to design sampled data controllers.

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