Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Robust Stabilization of Control Affine Systems with Homogeneous Functions ⋆

Konstantin Zimenko *, **Andrey Polyakov** **,*, **Denis Efimov** **,*,

* *Faculty of Control Systems and Robotics, ITMO University, 49 Kronverkskiy av., 197101 Saint Petersburg, Russia. (e-mail: kostyazimenko@gmail.com).*
** *Inria, Univ. Lille, CNRS, UMR 9189 - CRIStAL, F-59000 Lille, France (e-mail: andrey.polyakov@inria.fr, denis.efimov@inria.fr).*

Abstract: The stabilization problem of the affine control system $\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x)$ with homogeneous functions f_0, f_i is studied. This class of systems is of interest due to the robust properties of homogeneity and the fact that many affine systems can be approximated by or transformed to the class under consideration. An advantage of the introduced design method is that the tuning rules are presented in the form of linear matrix inequalities. Performance of the approach is illustrated by a numerical example.

Keywords: Nonlinear control, control affine systems, robust stabilization, homogeneous systems.

1. INTRODUCTION

Homogeneous dynamical systems have a number of very useful properties for system analysis, control design and estimation. In particular, the local and global behaviours are the same; homogeneous systems have certain intrinsic robustness properties with respect to external perturbations, measurement noises and time delays; convergence rate of the system can be assessed by its homogeneity degree, etc. (see, for example, Bacciotti and Rosier, 2001; Bernuau et al., 2013; Ryan, 1995; Hong, 2001; Bhat and Bernstein, 2005; Zimenko et al., 2017; Efimov et al., 2016).

For the stability analysis, a Lyapunov function of a homogeneous system can also be chosen homogeneous (see, e.g. Zubov, 1958; Rosier, 1992; Bacciotti and Rosier, 2001), and its negativeness can be checked only on the unit sphere. The combination of this notion with the implicit Lyapunov function method in some cases allows formalizing control/observer tuning algorithms (even in the case of essentially nonlinear feedbacks) in the form of linear matrix inequalities (LMI). However, in most cases, this effect is achieved only for linear (almost linear) system models (for example, Zimenko et al., 2020; Rios et al., 2017).

The present paper addresses the control design problem for the system

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x), \tag{1}$$

where $x \in \mathbb{R}^n$ are states, $u = (u_1, ..., u_m)^T \in \mathbb{R}^m$ are controls, and f_0, f_i are homogeneous functions. In addition to the useful properties of homogeneity, interest in this class of systems is based on the following reasons:

- the system (1) is of wider class than linear control systems;
- if there is a nonhomogeneous function f_0 or f_i, it can be homogenized by an appropriate control;
- nonhomogeneous functions may be approximated by the homogeneous ones (see, for example, Andrieu et al., 2008; Hermes, 1991; Ménard et al., 2013; Efimov and Perruquetti, 2016).

The proposed homogeneous controller for the system (1) is based on 'universal' control design scheme given in (Sontag, 1989). Despite a significant nonlinearity of the system, the applicability condition of the presented control is in the form of LMI. This condition is obtained following homogeneous function representation in a certain canonical form. The representation procedure substantially repeats the mechanism of embedding nonlinear systems into linear differential inclusions.

The paper is organized in the following way. Notation used in the paper is given in Section 2. Section 3 presents some preliminaries used in the paper. Section 4 introduces the main result on controller design for the system (1). Simulations are shown in Section 5. Finally, concluding remarks are given in Section 6.

2. NOTATION

Through the paper the following notation will be used:

- $\mathbb{N}$ is the set of natural numbers;
- $\mathbb{R}_+ = \{x \in \mathbb{R} \colon x > 0\}$, where $\mathbb{R}$ is the field of real numbers;
- The symbol $\overline{1, m}$ is used to denote a sequence of integers $1, ..., m$;

⋆ This work was partially supported by the Government of Russian Federation (Grant 08-08) and by the Ministry of Science and Higher Education of Russian Federation, passport of goszadanie no. 2019-0898.

- $|\cdot|$ is used to signify the Euclidean norm in $\mathbb{R}^n$, $\|\cdot\|$ denotes a weighted Euclidean norm;
- $\mathbb{S} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ is the unit sphere in $\mathbb{R}^n$;
- $\|A\| = \sup_{x\in\mathbb{R}^n} \frac{\|Ax\|}{\|x\|}$ if $A \in \mathbb{R}^{n\times n}$;
- $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix;
- the inequality $P > 0$ ($P \geq 0$) means that a symmetric matrix $P = P^T \in \mathbb{R}^{n\times n}$ is positive definite (positive semi-definite);
- the eigenvalues of a matrix $G \in \mathbb{R}^{n\times n}$ are denoted by $\lambda_i(G)$, $i = 1, ..., n$, $\lambda_{\min}(G) = \min_{i=\overline{1,n}} \lambda_i(G)$;
- $\mathcal{C}^n(X, Y)$ is the set of continuously differentiable (at least up to the order n) maps $X \to Y$, where X and Y are open subsets of finite-dimensional spaces;
- $\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\right)$;
- $\Re(\lambda)$ denotes the real part of a complex number λ.

3. PRELIMINARIES

3.1 Stability Notions

Consider the following system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $f \in \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $f(0) = 0$.

Definition 1 (Bhat and Bernstein, 2000; Orlov, 2004) *The origin of (2) is said to be globally finite-time stable if it is globally asymptotically stable and any solution $x(t, x_0)$ of the system (2) reaches the equilibrium point at some finite time moment, i.e. $x(t, x_0) = 0\ \forall t \geq T(x_0)$ and $x(t, x_0) \neq 0\ \forall t \in [0, T(x_0))$, $x_0 \neq 0$, where $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$, $T(0) = 0$ is the settling-time function.*

Definition 2 (Polyakov, 2012) *A set $M \subset \mathbb{R}^n$ is said to be globally finite-time attractive for (2) if any solution $x(t, x_0)$ of (2) reaches M in some finite time moment $t = T(x_0)$ and remains there $\forall t \geq T(x_0)$, $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is the settling-time function. It is fixed-time attractive if in addition the settling-time function $T(x_0)$ is globally bounded by some number $T_{\max} > 0$.*

Theorem 1. (Bhat and Bernstein, 2000). Suppose there exist a positive definite C^1 function V defined on an open neighborhood of the origin $D \subset \mathbb{R}^n$ and real numbers $C > 0$ and $\sigma \geq 0$, such that the following condition is true for the system (2)

$$\dot{V}(x) \leq -CV^\sigma(x), \quad x \in D \setminus \{0\}.$$

Then depending on the value σ the origin is stable with different types of convergence:

- if $\sigma = 1$, the origin is exponentially stable;
- if $0 \leq \sigma < 1$, the origin is finite-time stable and $T(x_0) \leq \frac{1}{C(1-\sigma)} V_0^{1-\sigma}$, where $V_0 = V(x_0)$;
- if $\sigma > 1$ the origin is asymptotically stable and, for every $\varepsilon \in \mathbb{R}_+$, the set $B = \{x \in D : V(x) < \varepsilon\}$ is fixed-time (independent on the initial values) attractive with $T_{\max} = \frac{1}{C(\sigma-1)\varepsilon^{\sigma-1}}$.

If $D = \mathbb{R}^n$ and function V is radially unbounded, then the system (2) admits these properties globally.

3.2 'Universal' Stabilizer for Control Affine Systems

Consider a system in the form (1), where $f_0, f_i \in \mathcal{C}^\infty$, and $f_0(0) = 0$. Let this system admits a control-Lyapunov function V that is, a smooth, proper, and positive definite function $\mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ so that

$$\inf_{u\in\mathbb{R}^m} \left\{ \frac{\partial V(x)}{\partial x} f_0(x) + \frac{\partial V(x)}{\partial x} \sum_{i=1}^m u_i f_i(x) \right\} < 0 \tag{3}$$

for each $x \neq 0$.

Definition 3 (Sontag, 1989) *The control-Lyapunov function V satisfies the small control property if for each $\varepsilon > 0$ there is a $\delta > 0$ such that, if $x \neq 0$ satisfies $\|x\| < \delta$, then there is some u with $\|u\| < \varepsilon$ such that*

$$\frac{\partial V(x)}{\partial x} f_0(x) + \frac{\partial V(x)}{\partial x} \sum_{i=1}^m u_i f_i(x) < 0.$$

Definition 4 (Sontag, 1989) *Let $k : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping, smooth on $\mathbb{R}^n \setminus \{0\}$ and with $k(0) = 0$. This is a smooth feedback stabilizer provided that, with $k = (k_1, ..., k_m)^T$ the closed-loop system*

$$\dot{x} = f_0(x) + \sum_{i=1}^m k_i(x) f_i(x)$$

is globally asymptotically stable.

Theorem 2. (Sontag, 1989). There is a smooth feedback stabilizer k iff there is a smooth control-Lyapunov function V, and k can be chosen continuous at the origin if V satisfied the small control property. Moreover, such k can be chosen in the form

$$k(x) = \begin{cases} \dfrac{a(x) + \sqrt{a^2(x) + |B(x)|^4}}{|B(x)|^2} B(x), & \text{if } |B(x)| \neq 0, \\ 0, & \text{if } |B(x)| = 0, \end{cases}$$

where $a(x) = \frac{\partial V(x)}{\partial x} f_0(x)$, $B(x) = (b_1, ..., b_m)^T$, $b_i(x) = \frac{\partial V(x)}{\partial x} f_i(x)$ for $i = \overline{1, m}$.

3.3 Generalized Homogeneity

The homogeneity is a dilation symmetry property. For example, if a mathematical object f (e.g., function, vector field, and operator) remains invariant with respect to scaling (dilation operation) of its argument $f(e^s x) = e^{\nu s} f(x)$, $s \in \mathbb{R}$, $x \in \mathbb{R}^n$ then it is called homogeneous, where $\nu \in \mathbb{R}$ is a constant called the degree of homogeneity. In the general case, we can consider a non-uniform scaling $x \to \mathbf{d}(s)x$, where an operator $\mathbf{d}(s) : \mathbb{R}^n \to \mathbb{R}^n$ is called dilation in the space $\mathbb{R}^n$ if it satisfies:

- **group property:** $\mathbf{d}(0) = I_n$ and $\mathbf{d}(t + s) = \mathbf{d}(t)\mathbf{d}(s) = \mathbf{d}(s)\mathbf{d}(t)$ for $t, s \in \mathbb{R}$;
- **continuity property:** $\mathbf{d}$ is a continuous map;
- **limit property:** $\lim_{s\to-\infty} \|\mathbf{d}(s)x\| = 0$ and $\lim_{s\to+\infty} \|\mathbf{d}(s)x\| = +\infty$ uniformly on the unit sphere $\mathbb{S}$.

In (Kawski, 1995; Rosier, 1993; Khomenuk, 1961), the dilation $\mathbf{d}$ is suggested to be generated as a flow of $\mathcal{C}^1$ vector field. Such a dilation is known as geometric dilation. In this paper we deal with generalized homogeneity based on groups of linear dilations, that also can be called linear geometric homogeneity. For this type of homogeneity there exists a generator matrix $G_{\mathbf{d}} = \lim_{s\to 0}\frac{\mathbf{d}(s)-I_n}{s}$ that satisfies the following properties (Pazy, 1983):

$$\frac{\partial}{\partial s}\mathbf{d}(s) = G_{\mathbf{d}}\mathbf{d}(s) = \mathbf{d}(s)G_{\mathbf{d}},$$
$$\mathbf{d}(s) = e^{G_{\mathbf{d}}s} = \sum_{i=0}^{+\infty}\frac{s^iG_{\mathbf{d}}^i}{i!},\quad s\in\mathbb{R}.$$

Definition 5 (Polyakov et al., 2016) *The dilation* $\mathbf{d}$ *is said to be monotone if* $\|\mathbf{d}(s)\|<1$ *as* $s<0$. *It is said to be strictly monotone if* $\exists\beta>0:\|\mathbf{d}(s)\|<e^{\beta s}$ *for* $s\le 0$.

Thus, monotonicity means that $\mathbf{d}(s)$ is a strong contraction for $s<0$ (strong expansion for $s>0$) and implies that for any $x\in\mathbb{R}\backslash\{0\}$ there exists a unique pair $(s_0,x_0)\in\mathbb{R}\times\mathbb{S}$ such that $x=\mathbf{d}(s_0)x_0$.

Note, that monotonicity property may depend on a norm $\|\cdot\|$.

Theorem 3. (Polyakov, 2019). If $\mathbf{d}$ is a dilation in $\mathbb{R}^n$, then

- the generator matrix $G_{\mathbf{d}}$ is anti-Hurwitz (i.e. $\Re(\lambda_i(G_{\mathbf{d}}))>0,\ i=\overline{1,n}$) and there exists a matrix $P\in\mathbb{R}^{n\times n}$ such that
$$PG_{\mathbf{d}}+G_{\mathbf{d}}^TP>0,\quad P>0. \tag{4}$$
- the dilation $\mathbf{d}$ is strictly monotone with respect to the norm $\|x\|=\sqrt{x^TPx}$ for $x\in\mathbb{R}^n$ and P satisfying (4):
$$e^{\alpha s}\le\|\mathbf{d}(s)\|\le e^{\beta s}\quad if\quad s\le 0,$$
$$e^{\beta s}\le\|\mathbf{d}(s)\|\le e^{\alpha s}\quad if\quad s\ge 0,$$
where $\alpha=\frac{1}{2}\lambda_{\max}\left(P^{\frac12}G_{\mathbf{d}}P^{-\frac12}+P^{-\frac12}G_{\mathbf{d}}^TP^{\frac12}\right)$, $\beta=\frac{1}{2}\lambda_{\min}\left(P^{\frac12}G_{\mathbf{d}}P^{-\frac12}+P^{-\frac12}G_{\mathbf{d}}^TP^{\frac12}\right)$.

The dilation $\mathbf{d}$ introduces a sort of norm topology in $\mathbb{R}^n$ by means of the so-called homogeneous norm (Kawski, 1995).

Definition 6 (Polyakov, 2019) *A continuous function* $p:\mathbb{R}^n\to\mathbb{R}_+\cup\{0\}$ *is said to be a* $\mathbf{d}$*-homogeneous norm if* $p(x)\to 0$ *as* $x\to 0$ *and* $p(\mathbf{d}(s)x)=e^sp(x)>0$ *for* $x\in\mathbb{R}^n\setminus\{0\}$ *and* $s\in\mathbb{R}$.

For monotone dilations we define the canonical homogeneous norm $\|\cdot\|_{\mathbf{d}}:\mathbb{R}^n\to\mathbb{R}_+\cup\{0\}$ as $\|x\|_{\mathbf{d}}=e^{s_x}$ for $x\ne 0$, where $s_x\in\mathbb{R}$ such that $\|\mathbf{d}(-s_x)x\|=1$ and, by continuity, we assign $\|0\|_{\mathbf{d}}=0$. Note that $\|\mathbf{d}(s)x\|_{\mathbf{d}}=e^s\|x\|_{\mathbf{d}}$ and
$$\|\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x\|=1. \tag{5}$$

Definition 7 (Polyakov et al., 2016) *A vector field* $f:\mathbb{R}^n\to\mathbb{R}^n$ *(a function* $h:\mathbb{R}^n\to\mathbb{R}$*) is said to be* $\mathbf{d}$*-homogeneous of degree* $\nu\in\mathbb{R}$ *if*
$$\begin{array}{ll} f(\mathbf{d}(s)x)=e^{\nu s}\mathbf{d}(s)f(x), & \forall x\in\mathbb{R}^n,\quad\forall s\in\mathbb{R}.\\ (resp.\ h(\mathbf{d}(s)x)=e^{\nu s}h(x), & \forall x\in\mathbb{R}^n,\quad\forall s\in\mathbb{R}.)\end{array} \tag{6}$$

Let $\mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ be the set of vector fields $\mathbb{R}^n\to\mathbb{R}^n$ satisfying the identity (6), which are continuous on $\mathbb{R}^n\setminus\{0\}$. Let $\deg_{\mathbb{F}_{\mathbf{d}}}(f)$ denote the homogeneity degree of $f\in\mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$.

Lemma 4. (Polyakov, 2019). The vector field $f\in\mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ is Lipschitz continuous (smooth) on $\mathbb{R}^n\setminus\{0\}$ if and only if it satisfies a Lipschitz condition (it is smooth) on the unit sphere $\mathbb{S}$, provided that $\mathbf{d}$ is strictly monotone on $\mathbb{R}^n$ equipped with a (smooth on $\mathbb{R}^n\setminus\{0\}$) norm $\|\cdot\|$.

If a function (or a vector field) is smooth, then homogeneity is inherited by its derivatives in a certain way.

Claim 5. If $f\in\mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)\cap C^1(\mathbb{R}^n\setminus\{0\},\mathbb{R}^n)$, then
$$e^{\deg_{\mathbb{F}_{\mathbf{d}}}(f)s}\mathbf{d}(s)\frac{\partial f(x)}{\partial x}=\left.\frac{\partial f(z)}{\partial z}\right|_{z=\mathbf{d}(s)x}\mathbf{d}(s) \tag{7}$$
for $x\in\mathbb{R}^n\setminus\{0\}$ and $s\in\mathbb{R}$.

The rate of convergence of homogeneous systems can be assessed by its homogeneity degree (Nakamura et al., 2002; Bhat and Bernstein, 2005; Polyakov, 2019).

Theorem 6. An asymptotically stable $\mathbf{d}$-homogeneous system $\dot{x}=f(x)$, $f:\mathbb{R}^n\to\mathbb{R}^n$ is uniformly finite-time stable if and only if $\deg_{\mathbb{F}_{\mathbf{d}}}(f)<0$.

Analogously, if $\deg_{\mathbb{F}_{\mathbf{d}}}(f)>0$ and the system is stable, then any compact set containing the origin is fixed-time attractive.

The homogeneity theory provides many other advantages to analysis and design of nonlinear control system. For instance, some results about ISS of homogeneous systems can be found in Bernuau et al. (2018, 2013); Ryan (1995). Of particular interest is the use of homogenising control algorithms due to delay robustness properties of homogeneous systems (see, for example, Efimov et al., 2016; Zimenko et al., 2017, 2019).

4. MAIN RESULTS

In this paper we consider the system (1), where $f_i(0)=0$ for $i=\overline{0,m}$, and $f_i\in\mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)\cap\mathcal{C}^1(\mathbb{R}^n,\mathbb{R}^n)$ with strictly monotone dilation $\mathbf{d}$ equipped with the norm $\|x\|=\sqrt{x^TPx}$, $P=P^T>0$.

The main goal of the paper is to propose a constructive (i.e. equipped with reliable tuning rules and robustness abilities) stabilizing control algorithm.

4.1 Canonical Representation of Homogeneous Systems

Let us consider the system in the form
$$\dot{x}=f(x), \tag{8}$$
where $f\in\mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)\cap\mathcal{C}^1(\mathbb{R}^n,\mathbb{R}^n)$, $f(0)=0$. For this system let us define the following bounds
$$\overline{g}_{ij}=\sup_{y\in\mathbb{R}^n:\|y\|<1}\left.\frac{\partial f_i(z)}{\partial z_j}\right|_{z=y},$$
$$\underline{g}_{ij}=\inf_{y\in\mathbb{R}^n:\|y\|<1}\left.\frac{\partial f_i(z)}{\partial z_j}\right|_{z=y},$$
that are always exist due to $f\in\mathcal{C}^1(\mathbb{R}^n,\mathbb{R}^n)$. Denote the bounded convex domain
$$\Omega=\left\{\theta=(\theta_{11},...,\theta_{1n},...,\theta_{nn}):\underline{g}_{jl}\le\theta_{jl}\le\overline{g}_{jl},j,l=\overline{1,n}\right\}$$
with the set of vertices defined by
$$\mathcal{V}=\left\{\alpha=(\alpha_{11},...,\alpha_{1n},...,\alpha_{nn}):\alpha_{jl}\in\{\underline{g}_{jl},\overline{g}_{jl}\},j,l=\overline{1,n}\right\}.$$

Proposition 7. Every trajectory of (8) is also a trajectory of
$$\dot{x}=\sum_{i=1}^N\alpha_i(x)\|x\|_{\mathbf{d}}^{\deg_{\mathbb{F}_{\mathbf{d}}}(f)}\mathbf{d}(\ln\|x\|_{\mathbf{d}})A_i\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x, \tag{9}$$
where $A_i\in\mathcal{V}$, $\sum_{i=1}^N\alpha_i(x)=1$, $0\le\alpha_i(x)\le 1$, and $N\in\mathbb{N}$.

Remark 1 Proposition 7 is mostly based on the mechanism of representing a nonlinear systems in the form of linear differential inclusions and convex embedding (see, for example, Boyd et al., 1994; Zemouche et al., 2008; Pyatnitskiy and Rapoport, 1996).

Note that this approach can be used for the case $f(0) \neq 0$ also, where

$$f(x)=\sum_{i=1}^{N}\alpha_i(x)\|x\|_{\mathbf{d}}^{\deg_{\mathbb{F}_\mathbf{d}}(f)}\mathbf{d}(\ln\|x\|_{\mathbf{d}})A_i\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x+f(0).$$

The choice of the form

$$\sum_{i=1}^{N}\alpha_i(x)\|x\|_{\mathbf{d}}^{\deg_{\mathbb{F}_\mathbf{d}}(f)}\mathbf{d}(\ln\|x\|_{\mathbf{d}})A_i\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x$$

for homogeneous functions representation is based on the fact that this form may be effective in control design with a simple tuning in the form of LMI as we will demonstrate below.

It should be noted that the condition $f \in \mathcal{C}^1(\mathbb{R}^n,\mathbb{R}^n)$ of Proposition 7 sufficiently restricts the class of homogeneous functions for representation in the form (9). One of the main directions for the future research is development of such method, that will not require such restrictions on the function f.

4.2 Homogeneous Control Design

Using the result of Proposition 7 the control for (1) can be obtained based on the scheme of a 'universal' stabilizing control design given in the work of Sontag (1989).

Before stating the main contribution of this paper, accordingly to Proposition 7 let us define the matrices $A_{i,j}$, $i=\overline{0,m}$, $j=\overline{1,N}$, and $\alpha_{i,j}(x)$, such that f_i in (1) can be represented as

$$f_i(x)=\sum_{j=1}^{N}\alpha_{i,j}(x)\|x\|_{\mathbf{d}}^{\deg_{\mathbb{F}_\mathbf{d}}(f_i)}\mathbf{d}(\ln\|x\|_{\mathbf{d}})A_{i,j}\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x. \tag{10}$$

Theorem 8. Let $\deg_{\mathbb{F}_\mathbf{d}}(f_0) \geq -1$, and $\deg_{\mathbb{F}_\mathbf{d}}(f_0) \geq \deg_{\mathbb{F}_\mathbf{d}}(f_i)$ for $i=\overline{1,m}$. If the system of LMIs

$$PA_{0,j}+A_{0,j}^TP<\sum_{i=1}^{m}\tau_i(PA_{i,l}+A_{i,l}^TP), \quad j,l=\overline{1,N} \tag{11}$$

is feasible for some $\tau_i \in \mathbb{R}$, then the control in the form

$$u_i(x)=\begin{cases}-\|x\|_{\mathbf{d}}^{\nu_i}b_i(x)\ \dfrac{a(x)+\sqrt{a^2(x)+|B(x)|^4}}{|B(x)|^2}, & \\ \qquad\qquad\qquad\qquad \text{if} \quad |B(x)|\neq 0, & \\ 0, \qquad\qquad\qquad\qquad \text{if} \quad |B(x)|=0, & \end{cases} \tag{12}$$

where $i=\overline{1,m}$, $\nu_i=\deg_{\mathbb{F}_\mathbf{d}}(f_0)-\deg_{\mathbb{F}_\mathbf{d}}(f_i)$, $B(\cdot)=(b_1(\cdot),...,b_m(\cdot))^T$, and

$$a(x)=\frac{2x^T\mathbf{d}^T(-\ln\|x\|_{\mathbf{d}})Pf_0(\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x)}{x^T\mathbf{d}^T(-\ln\|x\|_{\mathbf{d}})(PG_{\mathbf{d}}+G_{\mathbf{d}}^TP)\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x},$$

$$b_i(x)=\frac{2x^T\mathbf{d}^T(-\ln\|x\|_{\mathbf{d}})Pf_i(\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x)}{x^T\mathbf{d}^T(-\ln\|x\|_{\mathbf{d}})(PG_{\mathbf{d}}+G_{\mathbf{d}}^TP)\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x},$$

globally asymptotically stabilizes the system (1) at the origin.

Remark 2 The system of LMIs (11) guarantees that the condition

$$|B(x)|=0 \Rightarrow a(x)<0 \tag{13}$$

is satisfied, which according to (3) follows from the assumption that $V=\|x\|_{\mathbf{d}}$ is a control-Lyapunov function.

Remark 3 If in Theorem 8 the strict inequality $\deg_{\mathbb{F}_\mathbf{d}}(f_0)>\deg_{\mathbb{F}_\mathbf{d}}(f_i)$ is satisfied, then the control-Lyapunov function satisfies the small control property and the control (12) is continuous at 0.

The controller (12) homogenizes the closed-loop system with homogeneity degree $\deg_{\mathbb{F}_\mathbf{d}}(f_0)$. Thus, if $\deg_{\mathbb{F}_\mathbf{d}}(f_0)>0$, then any compact set containing the origin is fixed-time attractive. If $\deg_{\mathbb{F}_\mathbf{d}}(f_0)<0$ the system is finite-time stable. It should be noted that in the case of the use of Proposition 7 for system representation, the condition $f \in \mathcal{C}^1(\mathbb{R}^n,\mathbb{R}^n)$ restricts an application of control (12), especially for the case $\deg_{\mathbb{F}_\mathbf{d}}(f_0)<0$. However, as it shown in the following remark, the condition $f \in \mathcal{C}^1(\mathbb{R}^n,\mathbb{R}^n)$ can be neglected if instead of Proposition 7 we would use the numerical calculation of $a(x)$, $B(x)$ on a proper grid.

Remark 4 The system of LMI (11) is only a sufficient condition to satisfy (13). In general, since

$$a(x)=\frac{2y^TPf_0(y)}{y^T(PG_{\mathbf{d}}+G_{\mathbf{d}}^TP)y}, \quad b_i(x)=\frac{2y^TPf_i(y)}{y^T(PG_{\mathbf{d}}+G_{\mathbf{d}}^TP)y},$$
$$y=\mathbf{d}(-\ln\|x\|_{\mathbf{d}})x\in\mathbb{S}, \ i=\overline{1,m},$$

the condition (13) can be numerically checked using a numerical grid on a sphere. Explicitly calculating $a(x)$ and $B(x)$ we can obtain that

$$\frac{\partial V(x)}{\partial x}f_0(x)+\frac{\partial V(x)}{\partial x}\sum_{i=1}^{m}u_if_i(x)\leq -CV^{1+\deg_{\mathbb{F}_\mathbf{d}}(f_0)},$$

where $C=\min_{y\in\mathbb{S}}\sqrt{a^2(y)+|B(y)|^4}$. Then, utilizing Theorem 1 we can get estimations for settling-time functions. Moreover, in this case we do not apply the representation (10), and the condition $f \in \mathcal{C}^1(\mathbb{R}^n,\mathbb{R}^n)$ can be neglected.

5. EXAMPLE

Consider the system (1) with $x \in \mathbb{R}^3$, $m=3$, where

$$f_0=\begin{pmatrix}-x_1x_3\\-x_1\\x_2\end{pmatrix}, \quad f_1=\begin{pmatrix}x_2\\-x_3\\0\end{pmatrix},$$
$$f_2=\begin{pmatrix}x_3^3\\0\\x_3\end{pmatrix} \quad \text{and} \quad f_3=\begin{pmatrix}x_1\\x_2\\0\end{pmatrix}.$$

In this system f_0, f_1, f_2 and f_3 are $\mathbf{d}$-homogeneous of degrees 1, −1, 0 and 0, respectively, where $\mathbf{d}$ is strictly monotone dilation equipped with the norm $\|x\|=\sqrt{x^Tx}$, and $G_{\mathbf{d}}=\begin{pmatrix}3&0&0\\0&2&0\\0&0&1\end{pmatrix}$ is a generator of the dilation. For these functions we have representations in the form (10) with corresponding matrices

$$A_{01}=\begin{pmatrix}1&0&1\\-1&0&0\\0&1&0\end{pmatrix},\ A_{02}=\begin{pmatrix}1&0&-1\\-1&0&0\\0&1&0\end{pmatrix},\ A_{03}=\begin{pmatrix}-1&0&1\\-1&0&0\\0&1&0\end{pmatrix},$$
$$A_{04}=\begin{pmatrix}-1&0&-1\\-1&0&0\\0&1&0\end{pmatrix},\ A_{1}=\begin{pmatrix}0&1&0\\0&0&-1\\0&0&0\end{pmatrix},\ A_{21}=\begin{pmatrix}0&0&3\\0&0&0\\0&0&1\end{pmatrix},$$
$$A_{22}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},\ A_{3}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix},$$

for which the system (11) is feasible. The results of the numerical simulation for the considered controller are depicted in Figures 1 and 2. The system has been discretized by means of the explicit Euler method with the step size 0.01.

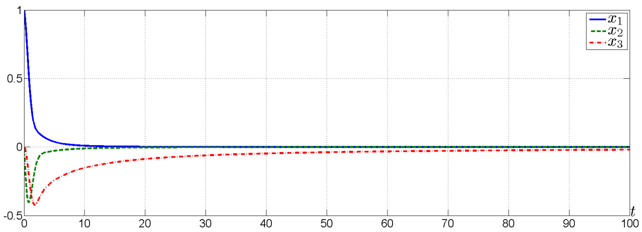

Fig. 1. Evolution of the state of the closed-loop system

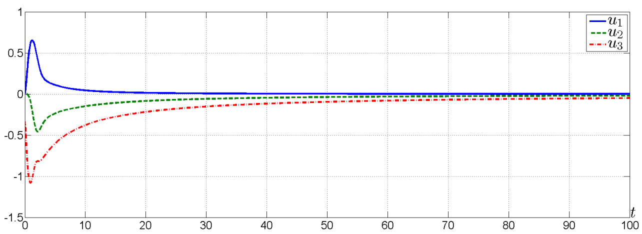

Fig. 2. Evolution of the control signals

6. CONCLUSIONS

The paper presents stabilizing control algorithm for affine control system (1) with homogeneous functions f_i, $i=\overline{0,m}$. Due to homogeneity the closed-loop system has a number of robust properties necessary in practice (e.g. ISS, robustness with respect to delays, etc.). Despite nonlinearity the main condition the system has to satisfy is presented in the form of LMI. This condition is obtained with the use of homogeneous function representation in the form (10).

There are two main directions for future research. In particular, it is expedient to develop a convex embedding method, that allows representing of homogeneous systems in the form (10) for the wider class than in Proposition 7. The second direction is related to transformation and approximation techniques to represent nonhomogeneous systems in the form (1) with homogeneous f_i.

REFERENCES

Andrieu, V., Praly, L., and Astolfi, A. (2008). Homogeneous approximation, recursive observer design, and output feedback. *SIAM Journal on Control and Optimization*, 47, 1814–1850. doi:10.1137/060675861.

Bacciotti, A. and Rosier, L. (2001). *Liapunov Functions and Stability in Control Theory*, volume 267. Berlin: Springer.

Bernuau, E., Efimov, D., and Perruquetti, W. (2018). Robustness of homogeneous and homogeneizable differential inclusions. In S. Li, X. Yu, L. Fridman, Z. Man, and X. Wang (eds.), *Advances in Variable Structure Systems and Sliding Mode Control Theory and Applications. Studies in Systems, Decision and Control*, volume 115, 39–56. Springer, Cham.

Bernuau, E., Polyakov, A., Efimov, D., and Perruquetti, W. (2013). Verification of ISS, iISS and IOSS properties applying weighted homogeneity. *System and Control Letters*, 62, 1159–1167.

Bhat, S. and Bernstein, D. (2000). Finite-time stability of continuous autonomous systems. *SIAM Journal of Control and Optimization*, 38(3), 751–766.

Bhat, S. and Bernstein, D. (2005). Geometric homogeneity with applications to finite-time stability. *Mathematics of Control, Signals and Systems*, 17(2), 101–127.

Boyd, S., Ghaoui, L.E., Feron, E., and Balakrishnan, V. (1994). *Linear Matrix Inequalities in System and Control Theory*. SIAM, Philadelphia, PA.

Efimov, D. and Perruquetti, W. (2016). Conditions of oscillations and multi-homogeneity. *Mathematics of Control, Signals, and Systems*, 28(3), 1–37. doi: 10.1007/s00498-015-0157-y.

Efimov, D., Polyakov, A., Perruquetti, W., and Richard, J.P. (2016). Weighted homogeneity for time-delay systems: Finite-time and independent of delay stability. *IEEE Transactions on Automatic Control*, 61(1), 210–215.

Hermes, H. (1991). Nilpotent and high-order approximations of vector field systems. *SIAM Review*, 33(2), 238–264.

Hong, Y. (2001). H_∞ control, stabilization, and input-output stability of nonlinear systems with homogeneous properties. *Automatica*, 37(7), 819–829.

Kawski, M. (1995). Geometric homogeneity and stabilization. *Proc. IFAC Nonlinear Control Symposium, A. Krener and D. Mayne, Eds.*, 164–169.

Khomenuk, V. (1961). On systems of ordinary differential equations with generalized homogeneous right-hand sides. *Izvestia vuzov. Mathematica (in Russian)*, 3(22), 157–164.

Ménard, T., Moulay, E., and Perruquetti, W. (2013). Homogeneous approximations and local observer design. *ESAIM: Control, Optimization and Calculus of Variations*, 19(3), 906–929.

Nakamura, H., Yamashita, Y., and Nishitani, H. (2002). Smooth Lyapunov functions for homogeneous differential inclusions. *Proceedings of the 41st SICE Annual Conference*, 3, 1974–1979.

Orlov, Y. (2004). Finite time stability and robust control synthesis of uncertain switched systems. *SIAM Journal of Control and Optimization*, 43(4), 1253–1271.

Pazy, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer.

Polyakov, A. (2012). Nonlinear feedback design for fixed-time stabilization of linear control systems. *IEEE Transactions on Automatic Control*, 57(8), 2106–2110.

Polyakov, A. (2019). Sliding mode control design using canonical homogeneous norm. *Int. J. Robust. Nonlinear*

Control, 29(3), 682–701. doi:10.1002/rnc.4058.

Polyakov, A., Efimov, D., Fridman, E., and Perruquetti, W. (2016). On homogeneous distributed parameter systems. *IEEE Transactions on Automatic Control*, 61(11), 3657–3662.

Pyatnitskiy, E. and Rapoport, L. (1996). Criteria of asymptotic stability of differential inclusions and periodic motions of time-varying nonlinear control systems. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 43, 219–229.

Rios, H., Mera, M., Efimov, D., and Polyakov, A. (2017). Robust output-control for uncertain linear systems: Homogeneous differentiator-based observer approach. *International Journal of Robust and Nonlinear Control*, 27(11), 1895–1914. doi:10.1002/rnc.3643.

Rosier, L. (1992). Homogeneous Lyapunov function for homogeneous continuous vector field. *System and Control Letters*, 19, 467–473.

Rosier, L. (1993). *Etude de Quelques Problemes de Stabilization [PhD thesis]*. Ecole Normale Superieure de Cachan, Paris, France.

Ryan, E. (1995). Universal stabilization of a class of nonlinear systems with homogeneous vector fields. *System and Control Letters*, 26, 177–184.

Sontag, E. (1989). A 'universal' construction of artstein's theorem on nonlinear stabilization. *Systems and Control Letters*, 13, 117–123.

Zemouche, A., Boutayeb, M., and Bara, G. (2008). Observers for a class of lipschitz systems with extension to H_∞ performance analysis. *Systems and Control Letters*, 57, 18–27.

Zimenko, K., Efimov, D., Polyakov, A., and Kremlev, A. (2019). Independent of delay stabilization using implicit lyapunov function method. *Automatica*, 101, 103–110. doi:10.1016/j.automatica.2018.11.051.

Zimenko, K., Efimov, D., Polyakov, A., and Perruquetti, W. (2017). A note on delay robustness for homogeneous systems with negative degree. *Automatica*, 79, 178–184.

Zimenko, K., Polyakov, A., Efimov, D., and Perruquetti, W. (2020). Robust feedback stabilization of linear MIMO systems using generalized homogenization. *IEEE Transactions on Automatic Control*. doi: 10.1109/TAC.2020.2969718.

Zubov, V. (1958). On systems of ordinary differential equations with generalized homogenous right-hand sides. *Izvestia vuzov. Mathematica (in Russian)*, 1, 80–88.