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Control-based Approach to Numerical Integration of Rolling Equations ⋆

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Abstract: In the paper, an approach to numerical integration of equations governing motion of constrained mechanical systems is suggested. In the framework of this approach, unknown reaction forces acting on the system are treated as controls, and the algebraic equations that these reactions satisfy, as control goals. On the basis of the suggested approach, a technique for numerical solving equations of rolling is developed. The discussion is illustrated by the example of application of the algorithm to solving the problem of a heavy wheel with a pendulum (a prototype of a ball-shaped robot) rolling along a curvilinear profile without slippage.

Keywords: Constrained mechanical systems, differential–algebraic equations, feedback linearization, ball-shaped robot, numerical integration.

1. INTRODUCTION

The problem of a wheel rolling on an uneven terrain is of importance in many practical applications. A rising tide of interest to this classical problem (Painlevé (1895)) is due to appearance of robotic systems of a new type—ball-shaped robots—and search for new actuators for such systems (Ylikopri et al. (2014)). To solve control problems for a ball-shaped robot, one needs, in particular, methods for numerical integration of equations governing ball rolling on an uneven surface. A particular case of the above problem is that of a heavy wheel rolling along a curvilinear profile.

The basic difficulty associated with the integration of the rolling equations is that the reaction force and torque acting on the wheel at the touching point are a priori unknown. They are to be determined from certain algebraic equations (constraints) resulting from additional assumptions that the rolling wheel is supposed to satisfy (Painlevé (1895)). These constraints include both holonomic and nonholonomic ones, which makes the direct application of the Lagrange formalism to this problem difficult. In the case of a flat surface, this problem was solved in quadratures by Painlevé (1895). However, the method he used cannot be directly extended to the case of an arbitrary curvilinear profile. On the other hand, it is well known that straightforward integration of equations of motion of mechanical systems subjected to nonholonomic constraints results often in the lack of stability and convergence of the computation process, as well as in the violation of the constraints. The problem of constraint stabilization was first set and solved in Baumgarte (1972). In this work, the author suggested a stable algorithm of numerical integration for the case of one constraint. The proposed method had become very popular; however, its application to a wider circle of problems revealed certain drawbacks inherent in the method (see, e.g., Ascher et al. (1995) and Petzold and Potra (1992) and references therein). It was shown in Ascher (1995) that the direct use of the technique suggested in Baumgarte (1972) does not ensure stability of the process of construction of an approximate solution, which makes it necessary to impose additional restrictions relating stabilization parameters with the step of integration. Additional restrictions were also introduced in the works where attempts were made to extend the approach to the case of an arbitrary number of constraints. For instance, Ascher et al. (1995) consider multiple, but only holonomic, constraints. In Muharlyamov (2011), both holonomic and nonholonomic constraints are considered; however, the latter are supposed to depend linearly on the generalized coordinates. To the best authors' knowledge, there are no works where the general case of multiple constraints is considered, as well as works, where the technique discussed is applied to numerical solving rolling equations.

The paper is organized as follows. In Section 2, we show that finding solution to equations governing motion of a constrained mechanical system reduces to solving a system of differential–algebraic equations (DAE) and outline the proposed technique for numerical integration of this system. In Section 3, we discuss in detail the application of this technique to solving the problem of a heavy wheel with a pendulum rolling along a curvilinear profile. Results

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of numerical experiments demonstrating high accuracy of the proposed method are presented in Section 4. Prospects of future studies are discussed in Section 5.

2. OUTLINE OF THE PROPOSED APPROACH

2.1 Problem Statement

We consider a constrained mechanical system governed by Newton's or Lagrange equations. Let the equations of motion be written in the state-space form as

$$\dot{X} = F(X, N),\ X(0) = X_0, t \in [0, t_1], \tag{1}$$

with the right-hand side of the system being linear in N,

$$F(X, N) = B(X)N + \widetilde{F}(X). \tag{2}$$

Here, $X \in R^n$ is the state vector, an $n\times m$ matrix $B(X)$ and vector $\widetilde{F}(X)$ are continuously differentiable functions, and $N \in R^m$ is an unknown vector of constraint reactions. We assume that the constraints are specified as m algebraic equations of the form

$$f_i(X) = 0,\ i = 1, \dots, m, \tag{3}$$

and that the initial conditions in (1) satisfy these constraints. We also assume that functions $f_i(X)$ have continuous partial derivatives up to the order n and that equations (3) determine a smooth manifold $\mathcal{M}$ in R^n, to which the desired trajectory belongs.

Equations (3) together with (1) form a closed system of $n + m$ equations with respect to $n + m$ unknowns (X, N), with the algebraic equations (3) not depending explicitly on N. For system (1)–(3), we state the problem of finding an approximate solution. Following Filippov (1987), under an approximate solution with accuracy δ in an interval $[0, t_1]$, we mean vector functions $X(t)$ and $N(t)$, $X_i(t), N_i(t) \in C[0, t_1]$, such that $||\dot{X} - F(X, N)|| < \delta$ and $|f_i| < \delta$, $\forall t \in [0, t_1]$, $\forall i = 1, \dots, m$.

Taking into account that the right-hand side of system (1) includes unknown constraint reactions, solution to this problem cannot be obtained by applying standard numerical integration methods, and the fact that the algebraic equations (3) determining these reactions do not explicitly depend on them makes the problem even more complicated. Thus, there is a need in the development of a method for numerical solving the DAE system (1)–(3).

The notation of the equations of motion in the state-space form is convenient in that it makes clear that the approach discussed is applicable to a number of control (stabilization) problems. Indeed, system (1), (2) may be viewed as an affine control system with the vector input N and the left-hand sides of constraints (3), as the outputs $y_i = f_i(X)$ that are to be stabilized at zero. Hence, the problem of finding controls N stabilizing outputs y_i also reduces to solving the DAE system (1)–(3).

Another reason to consider the state-space form of the governing equations rather than the second-order Newton's or Lagrange equations is that the original description of the system under study can include auxiliary coordinates whose dynamics are governed by first-order differential equations (this point is illustrated in the example discussed in Section 3), so that the use of the state-space form makes it possible to consider such systems and those given by second-order differential equations in a unique way.

2.2 Underlying Ideas

To determine unknown reaction forces, we apply a technique similar to that of feedback linearization (Isidori (1995)). For each constraint equation $f_i(X) = 0$, by r_i, we denote the order of the least time derivative by virtue of system (1) that explicitly depends on at least one component of N. Since the right-hand side of (1) linearly depends on N, the r_i-th time derivative of f_i by virtue of (1) also linearly depends on N; i.e.,

$$\frac{d^{r_i} f_i}{d\,t^{r_i}} = A_i(X)N + \varphi_i(X),\ i = 1, \dots, m, \tag{4}$$

where $A_i(X)$ is a row vector of length m and $\varphi_i(X)$ is the sum of all terms in the ith equation not depending on N, whereas, for all $j < r_i$, $d^j f_i/d\,t^j$ does not explicitly depend on N. For a mechanical system, in the majority of cases, either $r_i = 1$ (nonholonomic constraints) or $r_i = 2$ (holonomic constraints). In the general case, however, the algebraic equations (3) are not necessarily mechanical constraints, so that the division of equations (3) into holonomic or nonholonomic ones is not constructive (this is clearly seen in the illustrative example below). For the sake of generality, in the remainder of this section, r_i is considered to be an arbitrary positive integer. Let $A(X)$ denote the $m\times m$ matrix with rows $A_i(X)$. Suppose that $A(X)$ is nonsingular at a point $X = X^*$. Then, by analogy with the affine control systems (Isidori (1995)), we say that the nonlinear system (1)–(3) has a (vector) relative degree $\{r_1, \dots, r_m\}$ at a point X^* . Note that, like in Isidori (1995), the numbers r_i in the vector relative degree can also be defined in terms of the Lie derivatives, since both definitions are clearly equivalent.

Since equations (3) hold identically on the desired trajectories, the left-hand sides in formulas (4) also identically equal to zero:

$$\frac{d^{r_i} f_i}{d\,t^{r_i}} = 0,\ i = 1, ..., m, \tag{5}$$

from which it follows that the desired vector N satisfies the equation

$$A(X)N + \varphi(X) = 0, \tag{6}$$

where $X \in \mathcal{M}$ and $\varphi(X) = [\varphi_1, \dots, \varphi_m]^{\mathrm{T}}$.

However, as noted in Baumgarte (1972) and Ascher et al. (1995), one cannot use equations (5) for numerical finding vector N because of uncontrolled growth of residuals of the algebraic equations (3), which results from errors in initial data or inaccuracies of the method for constructing an approximate solution. This is clearly seen from the form of the general solution of equations (5) given by

$$f_i(t) = f_i(0) + \frac{df_i(0)}{dt} t + \cdots + \frac{1}{(r_i - 1)!} \frac{d^{r_i - 1} f_i(0)}{dt^{r_i - 1}} t^{r_i - 1}.$$

Following the approach suggested in Matrosov (2007), we allow the constraint residuals to be nonzero and replace equations (5) by new differential equations in f_i that

satisfy the following conditions:
1. The order of the ith differential equation of the new system is equal to that of the ith equation in (5).
2. For zero initial conditions, solutions of the new system and system (5) coincide.
3. Any solution of the new system corresponding to nonzero initial conditions exponentially tends to zero.

These conditions are satisfied if we take the ith equation of the new system in the form

$$\frac{d^{r_i} f_i}{dt^{r_i}} + c_{ir_i}\frac{d^{r_i-1} f_i}{dt^{r_i-1}} + \cdots + c_{i1} f_i = 0,\ c_{ij} > 0, \tag{7}$$

where the coefficients c_{ij} are selected such that all roots of the characteristic equation have negative real parts. In what follows, for the convenience of the notation and subsequent calculations, we select coefficients in (7) from one-parameter families as $c_{ij}(\lambda_i) = C_{r_i}^{r_i+1-j}\lambda_i^{r_i+1-j}$, $j = 1,\dots,r_i$, $\lambda_i > 0$, which corresponds to the repeated real negative roots $-\lambda_i$ of the characteristic equations (see Pesterev (2016) for detail).

Expressing the r_i-th derivative from (7) and substituting the expression obtained into equation (4), we get

$$A_i(X)N + \varphi_i(X) + c_{ir_i}\frac{d^{r_i-1} f_i}{dt^{r_i-1}} + \cdots + c_{i1} f_i = 0. \tag{8}$$

Denoting the free term in (8) as $\varphi_i(X,\lambda_i)$ and rewriting equations (8) in the matrix form, we get the following linear algebraic system in N:

$$A(X)N + \varphi(X,\lambda) = 0, \tag{9}$$

where $\lambda = [\lambda_1,\dots,\lambda_m]$ and $\varphi(X,\lambda)$ is the vector with the entries $\varphi_i(X,\lambda_i)$, $i = 1,\dots,m$.

Let matrix $A(X)$ be nonsingular for any $X \in \mathcal{M}$. Then, there is a neighborhood of the manifold where this matrix is also nonsingular, and system (9) can be resolved in N:

$$N(X,\lambda) = -A^{-1}(X)\varphi(X,\lambda). \tag{10}$$

Substituting (10) into (1) and solving the resulting ODE system by a standard numerical integration method, such as the Runge–Kutta or Adams method, we get the desired trajectory of system motion.

If the initial conditions in (1) satisfy equations (3), then $f_i(0) = \dot f_i(0) = \cdots = d^{r_i-1}f_i(0)/dt = 0$ and solutions to equations (8) are functions $f_i(t) \equiv 0$; i.e., conditions (3) are fulfilled. If the initial conditions do not satisfy constraints (3), one obtains an approximate solution of the problem, which exponentially converges to the exact one. Taking sufficiently large coefficients λ_i (and, if necessary, reducing the step of integration, see Ascher et al. (1995) for detail), an arbitrarily high accuracy of the solution can be achieved. The errors associated with inaccurate initial values not only are not accumulated but also decrease exponentially with the exponent $-\lambda_i$, which makes it possible to get high-accurate solutions in large time intervals.

For the most frequently met cases of $r_i = 1$ and $r_i = 2$, equation (7) in residuals takes the form

$$\dot f_i + \lambda_i f_i = 0,\ \lambda_i > 0, \tag{11}$$

and

$$\ddot f_i + 2\lambda_i \dot f_i + \lambda_i^2 f_i = 0,\ \lambda_i > 0, \tag{12}$$

respectively. Here and in what follows, we use the dot notation to denote the first and second time derivatives by virtue of system (1).

3. ILLUSTRATIVE EXAMPLE

As an illustration, we apply the method described in the previous section to solving the problem of a heavy wheel of mass M and radius r with a pendulum (which may be viewed as a prototype spherical robot) rolling along a curvilinear gutter with the shape given by an implicit equation $h(x,z) = 0$. We assume that the partial derivative h'_z of the function $h(x,z)$ is not equal to zero (no vertical segments) and that the profile of the gutter is admissible for the given wheel, i.e., the curvature of the curve at any point is less than $1/r$. The pendulum is the point mass m attached to the end of a massless rod of length $l < r$ suspended from the wheel axle (Fig. 1). For the generalized

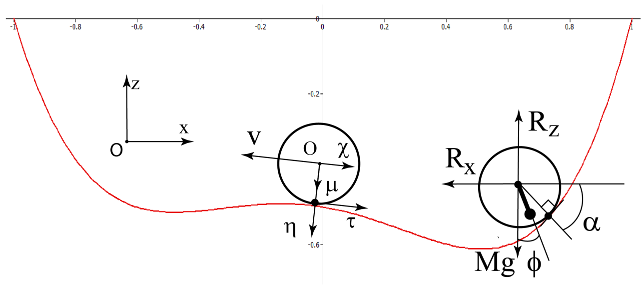

Fig. 1. Kinematic and dynamic sketches of the wheel rolling along a curvilinear gutter.

coordinates, we take coordinates of the wheel center along the horizontal and vertical axes x and z of the inertial frame, the angle θ of the wheel rotation around its axle, and the angular deviation ϕ of the pendulum with respect to the z axis, with $\phi = 0$ corresponding to the lowest position of the mass. Both angles are assumed positive upon counterclockwise rotation.

3.1 Equations Governing Dynamics of the System

The motion of the system is governed by the Lagrange equations

$$\begin{aligned} &\frac{d}{dt}\frac{\partial T}{\partial \dot x} - \frac{\partial T}{\partial x} = R_x, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot z} - \frac{\partial T}{\partial z} = R_z - \frac{\partial U}{\partial z},\\ &\frac{d}{dt}\frac{\partial T}{\partial \dot\theta} - \frac{\partial T}{\partial \theta} = Q, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot\phi} - \frac{\partial T}{\partial \phi} = -\frac{\partial U}{\partial \phi}, \end{aligned} \tag{13}$$

where $T = T_1 + T_2$ and $U = U_1 + U_2$ are kinetic and potential energies of the system,

$$\begin{aligned} T_1 &= \frac{1}{2}M(\dot x^2 + \dot z^2 + r^2\dot\theta^2),\\ T_2 &= \frac{1}{2}m[(\dot x + l\dot\phi\cos\phi)^2 + (\dot z + l\dot\phi\sin\phi)^2],\\ U_1 &= Mgz,\ U_2 = mg(z - l\cos\phi), \end{aligned} \tag{14}$$

g is the acceleration of gravity, $R = (R_x, R_z)$ is the reaction force, Q is the moment of the reaction force,

$$Q = -r\sin\alpha R_x + r\cos\alpha R_z, \tag{15}$$

α is the angle between axis x and the vector directed from the wheel center to the point where the wheel touches the curve (Fig. 1), $-\pi < \alpha < 0$.

Performing differentiation in (13) and resolving the system obtained with respect to the second derivatives, one gets

$$\begin{aligned}(M+m)\ddot{x} &= aR_x + bR_z + ml\dot{\phi}^2\sin\phi,\\ (M+m)\ddot{z} &= bR_x + cR_z - ml\dot{\phi}^2\cos\phi - (M+m)g,\\ Mr^2\ddot{\theta} &= -r\sin\alpha R_x + r\cos\alpha R_z,\\ Ml\ddot{\phi} &= -\cos\phi R_x - \sin\phi R_z,\end{aligned} \tag{16}$$

where

$$a = 1+\gamma\cos^2\phi, b = \gamma\sin\phi\cos\phi, c = 1+\gamma\sin^2\phi, \tag{17}$$

and $\gamma = m/M$. Equations (16) cannot be integrated directly, since the reaction force and angle α are not a priori known. They are determined by constraints imposed on the system. Given that the right-hand sides of (16) depend on three unknown functions (R_x, R_z, α), we need three constraint equations, which can be derived from additional assumptions on the character of rolling.

3.2 Constraint Equations

We assume that the wheel does not lose contact with the gutter, i.e.,

$$h(x_a, z_a) = 0, \tag{18}$$

where (x_a, z_a) is the touching point, $x_a = x + r\cos\alpha$, $z_a = z + r\sin\alpha$, and that there is no slipping, which can be written as

$$-\dot{x}\sin\alpha + \dot{z}\cos\alpha + r\dot{\theta} = 0. \tag{19}$$

Let us introduce the notation $\eta = [h'_x(x_a,z_a), h'_z(x_a,z_a)]^{\mathrm{T}}$ and $\tau = [-h'_z(x_a,z_a), h'_x(x_a,z_a)]^{\mathrm{T}}$ for the normal to the curve and the tangent vector, respectively, at the point where the curve touches the wheel (in what follows, for brevity, we omit arguments of function $h(x_a,z_a)$ and its derivatives); $v = [\dot{x},\dot{z}]^{\mathrm{T}}$ for the vector of velocity of the wheel center; $\mu = [\cos\alpha, \sin\alpha]^{\mathrm{T}}$ for the unit vector lying on the line connecting the wheel center and the touching point; and $\chi = [-\sin\alpha, \cos\alpha]^{\mathrm{T}}$ for the vector orthogonal to μ (Fig. 1). Clearly, when the wheel moves without losing contact with the curve, vectors μ and τ are orthogonal, $(\mu,\tau)\equiv 0$; i.e., α must satisfy the equation

$$h'_x(x_a,z_a)\sin\alpha - h'_z(x_a,z_a)\cos\alpha = 0, \tag{20}$$

which will be taken to be the third constraint. Following the approach discussed in Section 2, we denote the left-hand sides of equations (18)–(20) as f_1, f_2, and f_3, respectively, and differentiate them until the constraint reactions appear.

3.3 Expanding the System of Dynamics Equations

Since the right-hand sides of equations (16) are nonlinear in α, we cannot consider α as a constraint reaction in the framework of the approach discussed (recall that the right-hand side of the dynamic equations should linearly depend on unknown constraint reactions). Therefore, it makes sense to consider α as a generalized coordinate and to supplement system (16) with a differential equation in α. The order of this equation is generally determined by the algebraic equations. Taking into account that two constraints (18) and (20) are holonomic ones and that they explicitly depend on α, an obvious guess is to require α to be a solution of a second-order differential equation. However, as will be shown in the next subsection, by confining our consideration to a neighborhood of the manifold (and we can certainly do this in the given problem), we can also employ the first-order equation

$$\dot{\alpha} = U. \tag{21}$$

The right-hand side of (21) depends linearly on the new unknown U, which, thus, can be included in the set of the reaction forces.

3.4 Algebraic Equations in Constraint Reactions

Differentiating the algebraic equations with respect to time and omitting routine intermediate calculations to save room, we get

$$\begin{aligned}\dot{f}_1 &= h'_x\dot{x} + h'_z\dot{z} - r\dot{\alpha}(h'_x\sin\alpha - h'_z\cos\alpha)\\ &\equiv (v,\eta) - (\mu,\tau)r\dot{\alpha};\end{aligned} \tag{22}$$

$$\dot{f}_2 = -\ddot{x}\sin\alpha + \ddot{z}\cos\alpha - \dot{\alpha}(\dot{x}\cos\alpha + \dot{z}\sin\alpha) + r\ddot{\theta}; \tag{23}$$

$$\dot{f}_3 = (\dot{\mu},\tau) + (\mu,\dot{\tau}) \equiv [(\mu,\eta) - (\chi, H_2\chi)r]\dot{\alpha} - (v, H_2\chi), \tag{24}$$

where

$$H_2 = \begin{pmatrix} h''_{xx} & h''_{xz}\\ h''_{xz} & h''_{zz}\end{pmatrix}.$$

As can be seen, by virtue of equation (20), the coefficient of $\dot{\alpha}$ in (22) is zero on the manifold, and no constraint reactions appeared in $\dot{f}_1$. Taking the second derivative of f_1 by virtue of system (16), (21) and bearing in mind that the coefficient $-(\mu,\tau)r$ of $\ddot{\alpha}$ is equal to zero on the manifold, we get

$$\begin{aligned}\ddot{f}_1 = &\frac{1}{M+m}[(ah'_x + bh'_z)R_x + (bh'_x + ch'_z)R_z)]\\ &+ r(v, H_2\chi)U - gh'_z + (v, H_2 v).\end{aligned} \tag{25}$$

Substituting formulas for the second derivatives of the generalized coordinates from (16) into (23), (24), we find the other two derivatives by virtue of system (16), (21) on the manifold:

$$\begin{aligned}\dot{f}_2 = &\frac{1}{M+m}[(b\cos\alpha - a\sin\alpha)R_x + (c\cos\alpha - b\sin\alpha)R_z)]\\ &-\frac{\sin\alpha R_x}{M} + \frac{\cos\alpha R_z}{M} - \frac{ml\dot{\phi}^2\cos(\phi+\alpha)}{M+m} - g\cos\alpha,\end{aligned} \tag{26}$$

$$\dot{f}_3 = (||\eta|| - (\chi, H_2\chi)r)U - (v, H_2\chi). \tag{27}$$

As can be seen, the third (holonomic) constraint does not need second differentiation, since the "constraint reaction" U appeared in the first derivative $\dot{f}_3$.

Introducing state variables $X = [x,\dot{x},z,\dot{z},\theta,\dot{\theta},\phi,\dot{\phi},\alpha]^{\mathrm{T}}$ and rewriting system (16), (21) in the state-space form (1), we obtain an affine system of nine equations with the vector input $N = [R_x, R_z, U]^{\mathrm{T}}$ and outputs (18)–(20). The vector of the constraint reactions N, as a function of state variables, is found by solving the algebraic system (9), where the rows $A_i(X)$ of matrix $A(X)$ and free terms $\varphi_i(X,\lambda)$ are easily determined from equations (25)–(27). The invertibility of matrix $A(X)$ on the manifold $\mathcal{M}$ is proven in the Appendix. Then, it follows that system (16), (21) with outputs (18)–(20) has vector relative degree $\{2,1,1\}$ at any point $X\in\mathcal{M}$ and, hence, the DAE system (16)–(21) can be solved by the proposed method.

4. NUMERICAL EXAMPLE

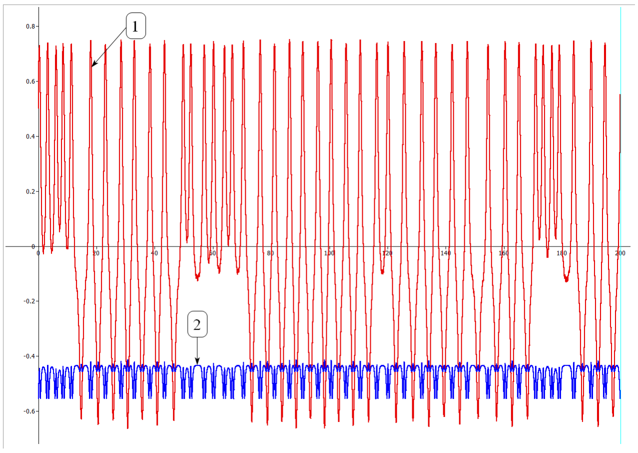

Fig. 2. Plots of $x(t)$ (upper curve) and and $z(t)$ (lower curve).

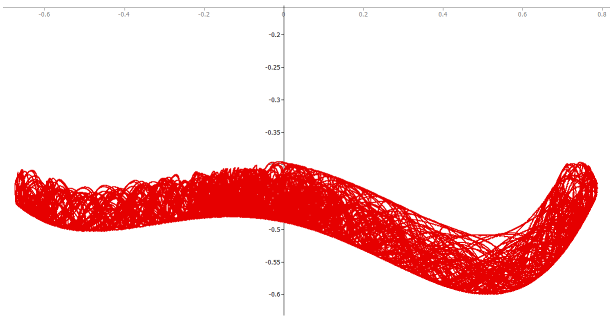

Fig. 3. Trajectory described by the pendulum mass.

Numerical modeling was carried out by means of the computer algebra system DExpert developed by I. Matrosov. This system allows one to automatically derive analytical expressions for unknown constraint reactions, substitute them into the right sides of the equations of system motion, and integrate the equations obtained numerically.

In numerical experiments, we considered the system with the following (dimensionless) parameters: $M = 1$, $r = 7/40$, $m = 0.5$, $l = 0.8r$. The shape of the curve shown in Fig. 1 is given by the equation $h(x, z) = 0$, where $h(x, z) = z-x^4+0.13x^3-0.5x^2-0.13x-0.5$, $-1 \leq x \leq 1$. The system of differential equations was integrated numerically by the Adams' method with the step $h = 10^{-5}$. Our numerical experiments showed that, for the given step, exponents λ_i should be selected from the range $[1, 10^5]$. The results below were obtained for $\lambda_i = 1.5 \cdot 10^3$, $i = 1, 2, 3$. For the given parameters, the maximum residual of the constraints $\max_i \max_t |f_i(t)|$ did not exceed $5 \cdot 10^{-10}$.

The results presented in Figs. 2–5 correspond to the following initial data: $x_a(0) = 0.5$, $\dot{x}(0) = 1.05$, $\phi = 3.4$, and $\dot{\phi} = -11.5$. The other initial data were selected such that constraints (18)–(20) are satisfied. The total energy of the system corresponding to these initial values only slightly exceeds the minimal energy needed to overcome the local maximum of the curve at $x \approx -0.132$. This results in a quite nontrivial picture of system behavior shown in Figs. 2–4, which depict results of solving the DAE system (16)–(21) in the time interval $[0, 200]$. The wheel alternately

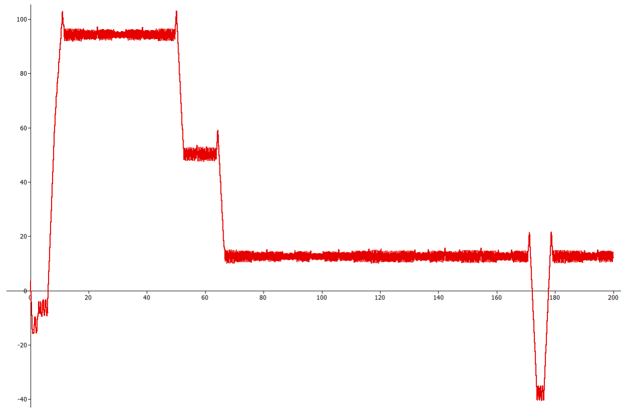

Fig. 4. Plot of the angle $\phi(t)$.

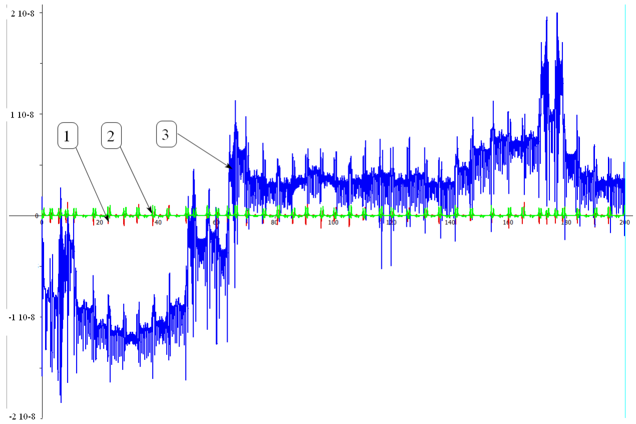

Fig. 5. Deviation of the total energy $\delta E(t)$ on the solution obtained (curve 3) and residuals $f_1(t)$ (curve 1) and $f_2(t)$ (curve 2).

oscillates on the two different segments of the gutter with x varying in the intervals [-0.67, 0.75] and [-0.13, 0.75], respectively, depending on whether the wheel energy when approaching the local maximum from the right is sufficient to overcome it or not, which, in turn, depends on the current distribution of the total energy between the wheel and pendulum. As a result, we observe a quite chaotic behavior of the system, especially in what concerns the trajectory of the pendulum mass depicted in Fig. 3. Figure 4 presents the dependence of the angle $\phi(t)$ on time. As can be seen, the time intervals when the pendulum performs complete turns around the wheel axis (rising or falling segments of the curve) alternate with the periods when the pendulum oscillates about the equilibrium state without making complete turns (horizontal segments). The process of switching from one mode to another also seems chaotic.

The correctness and high accuracy of the solution obtained are substantiated by the plot of the total energy deviation in the course of integration. Figure 5 demonstrates that the total energy error $\delta E(t) = E(t) - E(0)$ (curve 3) for the solution obtained does not exceed $2 \cdot 10^{-8}$. Curves 1 and 2 in the figure depict residuals of the first two constraints, $f_1(t)$ and $f_2(t)$, which are less than $5 \cdot 10^{-10}$ (the residual $f_3(t)$ is much smaller and not shown). The difference in the behavior of $\delta E(t)$ and that of $f_1(t)$ and $f_2(t)$ is easily understood. The residuals $f_1(t)$ and $f_2(t)$ satisfy equations (12) and (11), respectively, and, once appeared, decrease

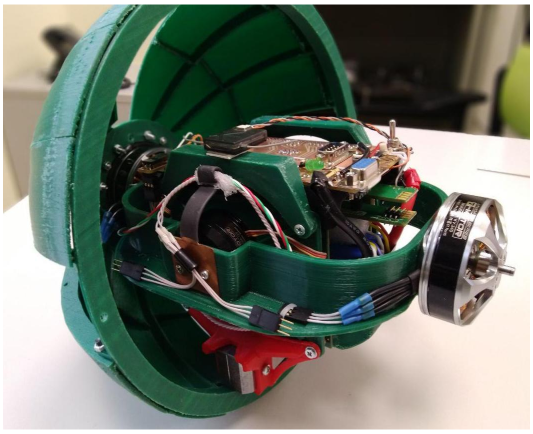

Fig. 6. Prototype ball-shaped robot.

exponentially with the rate 1500, unlike $\delta E(t)$, the law of variation of which is not a priori specified.

5. CONCLUSIONS

In the paper, we presented a technique for numerical integration of equations governing motion of constrained mechanical systems, which is based on the constraint stabilization technique. The application of the proposed method to solving the problem of a heavy wheel rolling along a curvilinear profile is discussed in detail. As noted in Section 1, the application of the method to solving a number of stabilization problems is straightforward. Moreover, numerical integration of equations governing motion of controlled rolling systems (robots) encounters all difficulties inherent upon integrating passive rolling structures, so that the technique discussed can be very helpful in modeling such systems.

The development of control-oriented versions of the technique discussed is underway. In particular, we plan to apply it to controlling a pendulum driven ball-shaped robot shown in Fig. 6, which is currently under construction. The robot design is based on three brushless (BLDC) motors. Electronic components produced by Javad GNSS include a 9D precision IMU, a GNSS receiver, and motor controllers implemented on the base of three STM32 ARM microprocessors. Structural elements, such as the shell and internal frames (one half of the shell and the internal frames are shown in Fig. 6), were made with the help of an FDN 3D printer. Preliminary simulations with a computer model of the robot showed good promise of using the technique discussed for the analysis of complex dynamics of the robot and synthesis of the control law.

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Appendix A. PROOF OF INVERTIBILITY OF MATRIX $A(X)$

Matrix $A(X)$ is easily derived from equations (25)–(27). Multiplying its first and second rows by $M + m$ (which clearly does not affect its invertibility) and leaving the same notation for the matrix obtained, we can write it as

$$A(X)=\begin{bmatrix} ah'_x + bh'_z & bh'_x + ch'_z & e \\ bC\alpha - (a+1+\gamma)S\alpha & (c+1+\gamma)C\alpha - bS\alpha & 0 \\ 0 & 0 & d \end{bmatrix},$$

where $d = ||\eta|| - r(\chi, H_2\chi)$, $e = (M+m)(v, H_2\chi)$, and, for brevity, $S\alpha$ and $C\alpha$ denote $\sin\alpha$ and $\cos\alpha$, respectively.

From (20), it follows that, $\forall X \in \mathcal{M}$, $h'_x/h'_z = \cos\alpha/\sin\alpha$. Denoting this ratio as β, we obtain

$$\det A(X) = d \begin{vmatrix} ah'_x + bh'_z & bh'_x + ch'_z \\ bC\alpha - (a+1+\gamma)S\alpha & (c+1+\gamma)C\alpha - bS\alpha \end{vmatrix}$$

$$= dh'_z \sin\alpha \begin{vmatrix} a\beta + b & b\beta + c \\ b\beta - (a+1+\gamma) & (c+1+\gamma)\beta - b \end{vmatrix}$$

$$= dh'_z \sin\alpha[(ac-b^2)(1+\beta^2) + (1+\gamma)(a\beta^2 + 2b\beta + c)].$$

By virtue of (17), we have $ac - b^2 = 1-\gamma$ and $a\beta^2 + 2b\beta + c = \beta^2 + 1 + \gamma(\beta\cos\phi + \sin\phi)^2$. Substituting these into the formula for the determinant, we finally get

$$\det A(X) = dh'_z \sin\alpha(1+\gamma)[2 + 2\beta^2 + \gamma(\beta\cos\alpha + \sin\alpha)^2].$$

The expression in the square brackets, being the sum of positive numbers, is clearly positive; $h'_z \neq 0$ by the definition of the admissible curve; and $\sin\alpha < 0$ by the definition of the angle α. Let us show that d cannot be equal to zero. With regard to the formula

$$\rho = \frac{((h'_x)^2 + (h'_z)^2)^{3/2}}{(h'_z)^2 h''_{xx} - 2h'_x h'_z h''_{xz} + (h'_x)^2 h''_{zz}} \equiv \frac{||\eta||^3}{(\tau, H_2\tau)}$$

for the curvature radius of a curve given by an implicit equation $h(x,z) = 0$, we have

$$d = ||\eta|| - r(\chi, H_2\chi) = ||\eta|| - \frac{r(\tau, H_2\tau)}{||\eta||^2} = ||\eta|| \left(1 - \frac{r}{\rho}\right).$$

By virtue of the assumption on the admissibility of the curve, $r < \rho$ and, hence, $d > 0$. Thus, we proved that $\det A(X) \neq 0$, and matrix $A(X)$ is invertible $\forall X \in \mathcal{M}$.