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On Local Stability of Equilibrium Profiles of Nonisothermal Axial Dispersion Tubular Reactors ⋆

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Abstract: Exponential (exp.) stability of equilibrium profiles for a nonisothermal axial dispersion tubular reactor is considered. This model is described by nonlinear partial differential equations (PDEs) whose state components are the temperature, the reactant and the product concentrations inside of the reactor. It is shown how to get appropriate local exponential stability of the equilibria for the nonlinear model, on the basis of stability properties of its linearized version and some relaxed Fréchet differentiability conditions of the nonlinear semigroup generated by the dynamics. In the case where the reactor can exhibit only one equilibrium profile, the latter is always locally exponentially stable for the nonlinear system. When three equilibria are highlighted, local bistability is established, i.e. the pattern (locally) "(exp.) stable – unstable – (exp.) stable" holds. The results are illustrated by some numerical simulations. As perspectives, the concept of state feedback is also used in order to show a manner to stabilize exponentially a nonlinear system on the basis of its capacity to stabilize exponentially a linearized version of the nonlinear dynamics and some Fréchet differentiability conditions of the corresponding closed-loop nonlinear semigroup.

Keywords: Nonlinear systems – Equilibrium – Exponential stability – Process models – Feedback stabilization

1. INTRODUCTION

The time evolution of temperature and concentration in nonisothermal axial dispersion tubular reactors is governed by the following nonlinear partial differential equations (PDEs):

$$\begin{cases} \dfrac{\partial x_1}{\partial t} = \dfrac{1}{Pe_h}\dfrac{\partial^2 x_1}{\partial z^2} - \dfrac{\partial x_1}{\partial z} - \gamma(x_1 - x_w) + \delta\alpha(1-x_2)e^{\frac{-\mu}{1+x_1}}, \\ \dfrac{\partial x_2}{\partial t} = \dfrac{1}{Pe_m}\dfrac{\partial^2 x_2}{\partial z^2} - \dfrac{\partial x_2}{\partial z} + \alpha(1-x_2)e^{\frac{-\mu}{1+x_1}}, \\ \dfrac{\partial x_1}{\partial z}(0) = Pe_h x_1(0), \dfrac{\partial x_1}{\partial z}(1) = 0, \\ \dfrac{\partial x_2}{\partial z}(0) = Pe_m x_2(0), \dfrac{\partial x_2}{\partial z}(1) = 0, \end{cases} \tag{1}$$

where x_1 and x_2 denote the dimensionless temperature and the dimensionless reactant concentration inside of the reactor, respectively. Equations (1) are directly deduced from mass and energy balances on a slice of infinitesimal tickness dz during an infinitesimal time dt. These are called convection–diffusion–reaction equations. The variables $t \in [0, \infty)$ and $z \in [0, 1]$ stand for the time and the space variables. The constants $v, \delta, \alpha = k_0 L v^{-1}$ and μ depend on the model parameters, see e.g. (Hastir et al., 2020), (Dochain, 2016). For instance, v is the superficial fluid velocity, L denotes the length of the reactor and k_0 is the kinetic constant. Further analysis of that model relies strongly on the relation between two specific numbers, Pe_h and Pe_m. These represent the mass and energy ratio of convection over diffusion, respectively. The term $\gamma(x_1-x_w)$ is due to a heat exchanger that acts as a distributed control along the reactor, where x_w is the coolant temperature in the heat exchanger. For stability analysis, we shall consider adiabatic conditions, meaning that there is no heat exchange with the environment outside of the reactor, i.e. $\gamma = 0$.

As it is exposed in (Hastir et al., 2020), such systems may exhibit multiple equilibria depending on the parameters, especially the diffusion and conduction ones. When those parameters become large enough, the system may switch from one to three equilibria.

Many control problems for systems like (1) have to handle the stabilization of equilibrium profiles. One could either design a control law that stabilizes an unstable equilibrium or that improves the stability margin of a stable one. For instance, a sliding-mode control approach was developed

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in (Orlov and Dochain, 2002) and applied to the stabilization of both plug-flow and convection – diffusion models. Backstepping was studied in (Boskovic and Krstic, 2002) and extremum-seeking control in (Hudon et al., 2008) where models like (1) are considered without taking the diffusion part into account. LQ-optimal regulation for a plug-flow tubular reactor model was also deeply developed in (Aksikas et al., 2007b).

A preliminary step before envisaging the stabilization of any equilibrium consists in looking at its stability. The main difficulty here comes from the nonlinear term $\alpha(1-x_2)e^{-\mu/(1+x_1)}$ which is due to the Arrhenius' law that models the evolution of the reaction rate as a function of the temperature. Asymptotic stability has already been studied in (Dochain, 2016), (Aksikas et al., 2007a) and also in (Varma and Aris, 1977), (Amundson, 1965) and in (Luss and Amundson, 1967) where bistability is established for a linearized version of (1). More recently, exponential stability of the equilibria of (1) for its Gâteaux linearized version has been studied in (Hastir et al., 2019b).

Deducing stability of the equilibria for a nonlinear system on the basis of stability of a linearization of it is not straightforward. For instance this is studied in (Al Jamal and Morris, 2018) or in (Kato, 1995) where exponential stability of the equilibria for the linearization and also Fréchet differentiability of the nonlinear semigroup generated by the nonlinear dynamics are needed. Checking Fréchet differentiability conditions for nonlinear operators is generally challenging, especially when the operators are unbounded. For instance the theoretical framework that is proposed in (Al Jamal and Morris, 2018) is not directly applicable on their example and a case–by–case study has often to be performed.

The approach that is proposed here is an extension of the works of (Al Jamal and Morris, 2018) for deducing nonlinear local exponential stability of the equilibria of (1). Some relaxed Fréchet differentiability conditions are required on the nonlinear semigroup generated by the dynamics of (1), by considering different spaces and different norms in the definition. This allows more easily checkable Fréchet differentiability conditions, providing local exponential stability of the equilibria for the nonlinear system in a weaker sense, see (Hastir et al., 2019a). This is crucial since the approach of (Al Jamal and Morris, 2018) is also not applicable to our case study.

For control purposes, the theory that is considered here is of primary importance since any exponentially stabilizing state feedback added to the linearized model is still locally (exp.) stabilizing for the nonlinear nominal model once the nonlinear semigroup generated by the closed-loop dynamics is Fréchet differentiable (in a generalized sense, see (Hastir et al., 2019a) and below). The LQ-optimal regulation problem for a linearized model of (1) is also introduced and discussed here.

The paper is organized as follows. In Section 2 some results about the existence, the multiplicity and the exponential stability of a Gâteaux linearized model corresponding to (1) are recalled. Then the new concepts of Fréchet differentiability are tested on the nonlinear semigroup generated by the dynamics (1) in order to deduce stability conclusions for the nonlinear model. In particular it is shown that, when the reactor exhibits only one equilibrium profile, it is locally exponentially stable for the nonlinear model and, when three equilibria are highlighted, bistability is established for the nonlinear model. Section 3 is devoted to the illustration of these results by means of some numerical simulations. The LQ-optimal control problem is introduced in Section 4 where perspectives are given. Note that equal Peclet numbers are considered in what follows, i.e. $Pe_h = Pe_m =: Pe = vL/D$, where D will be called the diffusion coefficient for the sake of simplicity. Moreover L is fixed to one, without loss of generality.

2. LOCAL EXPONENTIAL STABILITY

In this section nonlinear exponential stability of the equilibria of (1) is considered. First we recall some results on the existence and the multiplicity of the equilibrium profiles of (1). Then exponential stability of these equilibria is adressed for a Gâteaux linearized version of (1). The last part of this section is dedicated to the application of the theoretical results presented in (Hastir et al., 2019a) to the nonisothermal axial dispersion tubular reactor in order to deduce stability or instability of the equilibrium profiles for the nonlinear model.

2.1 Linearized stability of the equilibrium profiles

According to (Hastir et al., 2020, Section IV, Lemma 4.1), in the case of equal Peclet numbers the nonisothermal axial dispersion tubular reactor can either exhibit one equilibrium or three equilibria, depending on the parameters of the system. These equilibria are given by

$$x_1^e(z) = a - \frac{k_0 L(\delta - a)e^{\frac{-\mu}{1+a}}}{2D}(1-z)^2 + \mathcal{O}(1/D^2)$$

and $x_2^e(z) = (1/\delta)x_1^e(z)$, see (Hastir et al., 2020, Section IV), where a is the solution of the equation [1]

$$\frac{k_0 L(\delta - a)e^{\frac{-\mu}{1+a}}}{a} = v.$$

From (Laabissi et al., 2001) it is well-known that (1) (without the third component x_3) possesses a unique mild solution, that is the nonlinear operator describing the dynamics is the infinitesimal generator of a C_0−semigroup on $D \cap \mathcal{K}$ where

$$D := \left\{ x = (x_1 \ x_2)^T \in H | \frac{dx_1}{dz}(0) = Pe x_1(0), \right. \\ \left. \frac{dx_2}{dz}(0) = Pe x_2(0), \frac{dx_1}{dz}(1) = 0, \frac{dx_2}{dz}(1) = 0 \right\} \quad (2)$$

and

$$\mathcal{K} := \left\{ x \in \tilde{H} | -1 \leq x_1(z), 0 \leq x_2(z) \leq 1, \text{ a.e. on } [0,1] \right\}, \quad (3)$$

where $H = H^2(0,1) \times H^2(0,1)$ and [2] $\tilde{H} = L^2(0,1) \times L^2(0,1)$.

Moreover, by considering equal Peclet numbers, the change of variables $\hat{\xi}_1 = x_1 - x_1^e, \hat{\xi}_2 = x_2 - x_2^e, \xi_1 = e^{-\frac{Pe}{2}z}\hat{\xi}_1, \xi_2 =$

[1] Depending on the value of the velocity, this equation possesses one or three solutions which characterize the multiplicity of the equilibria.

[2] Here the state space is $\tilde{H}$.

$e^{-\frac{Pe}{2}z}\hat{\xi}_2$ and $\chi = \xi_1 - \delta\xi_2$ yields the following triangular form of (1):

$$\begin{cases} \dfrac{\partial \chi}{\partial t} = \dfrac{1}{Pe}\dfrac{\partial^2 \chi}{\partial z^2} - \dfrac{Pe}{4}\chi, \\ \dfrac{\partial \xi_1}{\partial t} = \dfrac{1}{Pe}\dfrac{\partial^2 \xi_1}{\partial z^2} - \dfrac{Pe}{4}\xi_1 + \delta e^{\frac{-Pe}{2}z} f\left(\xi_1, \dfrac{\xi_1 - \chi}{\delta}\right), \\ \dfrac{\partial \chi}{\partial z}(0) = \dfrac{Pe}{2}\chi(0), \dfrac{\partial \chi}{\partial z}(1) = \dfrac{-Pe}{2}\chi(1), \\ \dfrac{\partial \xi_1}{\partial z}(0) = \dfrac{Pe}{2}\xi_1(0), \dfrac{\partial \xi_1}{\partial z}(1) = \dfrac{-Pe}{2}\xi_1(1), \end{cases} \tag{4}$$

where $f(\xi_1, \xi_2) = g(e^{\frac{Pe}{2}z}\xi_1 + x_1^e, e^{\frac{Pe}{2}z}\xi_2 + x_2^e) - g(x_1^e, x_2^e)$ and $g(x, y) = \alpha(1 - y)e^{\frac{-\mu}{1+x}}$ for $(x\ y)^T \in \mathcal{K}$ and $g(-1, y) = 0$. By (Hastir et al., 2019a, Section 4) the component $\chi(t, z)$ of (4) converges exponentially fast to 0 as t tends to $+\infty$. Hence the stability analysis of (1) is based on the following nonlinear PDE

$$\begin{cases} \dfrac{\partial \xi}{\partial t} = \dfrac{1}{Pe}\dfrac{\partial^2 \xi}{\partial z^2} - \dfrac{Pe}{4}\xi + e^{\frac{-Pe}{2}z}[\tilde{g}(e^{\frac{Pe}{2}z}\xi + x^e) - \tilde{g}(x^e)], \\ \dfrac{\partial \xi}{\partial z}(0) = \dfrac{Pe}{2}\xi(0), \dfrac{\partial \xi}{\partial z}(1) = \dfrac{-Pe}{2}\xi(1), \end{cases} \tag{5}$$

where $\tilde{g}(x) = \delta g(x, \frac{1}{\delta}x)$, x^e and ξ stand for x_1^e and ξ_1, respectively. This PDE fits the well-known class of semilinear systems of the form

$$\begin{cases} \dot{\xi} = \mathcal{A}\xi + \mathcal{N}(\xi), \\ \xi(0) = \xi_0 \end{cases} \tag{6}$$

described in (Hastir et al., 2019a). The (unbounded) linear operator $\mathcal{A}$ is defined as

$$\mathcal{A}\xi = \frac{1}{Pe}\frac{d^2\xi}{dz^2} - \frac{Pe}{4}\xi \tag{7}$$

on the domain

$$D(\mathcal{A}) := \left\{\xi \in H^2(0,1) | \frac{d\xi}{dz}(0) = \frac{Pe}{2}\xi(0), \frac{d\xi}{dz}(1) = -\frac{Pe}{2}\xi(1)\right\}. \tag{8}$$

Defining the invariant domain $\tilde{\mathcal{K}}$ by

$$\left\{\xi \in L^2(0,1) | -1 \le \xi(z), 0 \le \frac{\xi - \chi}{\delta} \le 1 \text{ a.e. on } [0,1]\right\},$$

the nonlinear operator $\mathcal{N} : e^{\frac{-Pe}{2}z}(\tilde{\mathcal{K}} - x^e) \to L^2(0,1)$ is expressed as $\mathcal{N}(\xi) = e^{\frac{-Pe}{2}z}\left[\tilde{g}(e^{\frac{Pe}{2}z}\xi + x^e) - \tilde{g}(x^e)\right]$ for $\xi \in e^{-\frac{Pe}{2}z}(\tilde{\mathcal{K}} - x^e)$. It can be shown that the operator $\mathcal{A}$ is dissipative and is the infinitesimal generator of a contraction C_0−semigroup on $L^2(0,1)$. Moreover, the nonlinear operator $\mathcal{N}$ satisfies the Lipschitz condition

$$\|\mathcal{N}(\xi_1) - \mathcal{N}(\xi_2)\|_{L^2(0,1)} \le l_{\mathcal{N}}\|\xi_1 - \xi_2\|_{L^2(0,1)},$$

for every $\xi_1, \xi_2 \in e^{\frac{-Pe}{2}z}(\tilde{\mathcal{K}} - x^e)$, for some positive constant $l_{\mathcal{N}}$. In order to linearize (5) around 0 [3], we introduce now the definitions of Gâteaux and Fréchet derivatives for nonlinear operators, see e.g. (Al Jamal and Morris, 2018).

Definition 1. Let $f : \mathcal{D}(f) \subset X \to X$ be a nonlinear operator defined on the Banach space X. The operator f is Gâteaux differentiable at $x^e \in \mathcal{D}(f)$ if there exists a linear operator $df(x^e) : X \to X$ (the Gâteaux derivative of f at x^e) such that $\lim_{l\to 0} \frac{f(x^e+lh)-f(x^e)}{l} = df(x^e)h$, where $h, x^e + lh \in \mathcal{D}(f)$.

[3] With the changes of variables introduced previously, it is obvious that the equilibrium of (5) is 0.

The operator f is said to be Fréchet differentiable at $x^e \in \mathcal{D}(f)$ if there exists a bounded linear operator $Df(x^e) : X \to X$ such that $\lim_{\|h\|_X\to 0} \frac{\|f(x^e+h)-f(x^e)-Df(x^e)h\|_X}{\|h\|_X} = 0$. That is, for all $h \in X$ such that $x^e + h \in \mathcal{D}(f)$, $f(x^e + h) - f(x^e) = Df(x^e)h + w(x^e, h)$, where $\lim_{\|h\|_X\to 0} \frac{\|w(x^e,h)\|_X}{\|h\|_X} = 0$.

Proving Fréchet differentiability for nonlinear operators is often hard to do when these are unbounded. It is even shown in (Hastir et al., 2019b, Appendix) that the considered nonlinear operator $\mathcal{N}$ is not Fréchet differentiable at 0. For this reason we shall linearize (5) around 0 by using a Gâteaux derivative, which yields the linear PDE

$$\begin{cases} \dfrac{\partial \bar{\xi}}{\partial t} = \dfrac{1}{Pe}\dfrac{\partial^2 \bar{\xi}}{\partial z^2} - \dfrac{Pe}{4}\bar{\xi} + (\alpha\dfrac{\mu(\delta - x^e)}{(1 + x^e)^2}e^{\frac{-\mu}{1+x^e}} - \alpha e^{\frac{-\mu}{1+x^e}})\bar{\xi}, \\ \dfrac{\partial \bar{\xi}}{\partial z}(0) = \dfrac{Pe}{2}\bar{\xi}(0), \dfrac{\partial \bar{\xi}}{\partial z}(1) = \dfrac{-Pe}{2}\bar{\xi}(1). \end{cases} \tag{9}$$

Since the Gâteaux derivative is a bounded operator from $L^2(0,1)$ to $L^2(0,1)$, the linear operator describing the dynamics of (9) is still the infinitesimal generator of a C_0−semigroup on $L^2(0,1)$, see e.g. (Engel and Nagel, 2006, Bounded Perturbation Theorem).

According to (Hastir et al., 2019b, Section III) the following proposition gives a bound on the $L^2(0,1)$−norm of the state trajectory corresponding to (9).

Proposition 2. The solution to equation (9) satisfies the following estimation:

$$\|\bar{\xi}(t,\cdot)\|_{L^2(0,1)} \le e^{-\left(\frac{\pi^2}{\pi^2+4Pe}+q(c)\right)t}\|\bar{\xi}(0,\cdot)\|_{L^2(0,1)},$$

where $q(z) = \frac{Pe}{4} + \frac{k_0 L}{v}e^{\frac{-\mu}{1+x^e(z)}} - \frac{k_0 L}{v}\frac{\mu(\delta - x^e(z))}{(1+x^e(z))^2}e^{\frac{-\mu}{1+x^e(z)}}$.

The proof of Proposition 2 is based on the following auxiliary result, obtained by exploiting a variation of *Wirtinger's Inequality*, see (Chung-Fen et al., 2004, Corollary 9) and (Hastir et al., 2019b, Lemma 3.1).

Lemma 3. For any continuously differentiable function w on $[0,1]$,

$$-\frac{1}{2}w^2(0) \le -\frac{1}{4\Lambda}\int_0^1 w^2(z)dz + \frac{2}{\pi^2(2\Lambda - 1)}\int_0^1 w_z^2(z)dz, \tag{10}$$

for all $\Lambda > \frac{1}{2}$.

Note that, Lemma 3 holds also with $w(0)$ replaced by $w(1)$.

Proof. (Proposition 2) Let us choose as Lyapunov functional candidate the function $V : L^2(0,1) \to \mathbb{R}$, defined by $V(\bar{\xi}) = \frac{1}{2}\int_0^1 \bar{\xi}^2 dz = \frac{1}{2}\|\bar{\xi}\|_{L^2}^2$. By differentiating V w.r.t. t along the state trajectories corresponding to (9), one gets

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \bar{\xi}^2 dz = \int_0^1 \bar{\xi}\left(\frac{1}{Pe}\frac{d^2\bar{\xi}}{dz^2} - q(z)\bar{\xi}\right)dz,$$

An integration by parts yields the following form for $\dot{V}(\xi)$:

$$-\frac{1}{2}\bar{\xi}^2(1) - \frac{1}{2}\bar{\xi}^2(0) - \frac{1}{Pe}\int_0^1 \left(\frac{d\bar{\xi}}{dz}\right)^2 dz - q(c)\int_0^1 \bar{\xi}^2 dz,$$

where the Generalized Mean Value Theorem has been used on the last term, for some $c \in (0,1)$.

By applying Lemma 3 to $\overline{\xi}$, $\dot{V}$ is bounded by $-(\frac{1}{2\gamma}+q(c))\int_0^1\overline{\xi}^2dz+(\frac{4}{\pi^2(2\gamma-1)}-\frac{1}{Pe})\int_0^1(\frac{d\overline{\xi}}{dz})^2dz$. We shall now choose γ in such a way that $\frac{4}{\pi^2(2\gamma-1)}-\frac{1}{Pe}=0$, which yields $\gamma=\frac{1}{2}+\frac{2Pe}{\pi^2}>\frac{1}{2}$. Consequently $\frac{1}{2}\frac{d}{dt}\|\overline{\xi}\|^2_{L^2}\leq-\left(\frac{\pi^2}{\pi^2+4Pe}+q(c)\right)\int_0^1\overline{\xi}^2dz$. □

It has been shown in (Hastir et al., 2019b, Section III) that the constant $-(\frac{\pi^2}{\pi^2+4Pe}+q(c))$ is always negative in the case where the reactor exhibits only one equilibrium profile. In the case of three equilibria this quantity has been shown to be negative for the first one and the third one. In that way, exponential stability of the equilibria is established in the following proposition, see (Hastir et al., 2019b, Section III).

Proposition 4. In the case where the nonisothermal axial dispersion tubular reactor (1) admits only one equilibrium profile, there exist $D^\star$ and $\tilde{D}$ sufficiently large such that this equilibrium profile is exponentially stable for all $D\geq\max(D^\star,\tilde{D})$. When the reactor exhibits three equilibria, the pattern "exponentially stable – unstable – exponentially stable" holds for all $D\geq\max(D^\star,\tilde{D})$, i.e. bistability holds.

2.2 Nonlinear stability of the equilibrium profiles

On the basis of the results presented in the previous section, here we aim at linking stability of the Gâteaux linearized dynamics with stability of the nonlinear one. To this end, it is stated in (Al Jamal and Morris, 2018) that, as soon as the nonlinear semigroup generated by the nonlinear dynamics is Fréchet differentiable at the equilibrium, conclusions of (exp.) stability of the equilibria for the nonlinear system can be derived locally (around the equilibrium) from the linearization. Checking the Fréchet differentiability of the nonlinear semigroup relies often on the same property of the generator for which a counter–example is proposed in (Hastir et al., 2019b, Appendix) in order to prove that it is not Fréchet differentiable.

This is the reason why we proposed a new framework in (Hastir et al., 2019a). It is based on a relaxed Fréchet differentiability condition that is in general easier to check since it allows more freedom in the manipulation of norm inequalities. This is called $(Y,X)-$Fréchet differentiability and is defined hereafter, see (Hastir et al., 2019a, Section 2).

Note that we shall use the notation ξ^e for an equilibrium profile of a generic system of the form (6) in what follows. In the particular case of System (5), it is obviously 0.

Definition 5. Let us consider the nonlinear operator $\mathcal{N}: D(\mathcal{A})\cap\mathcal{D}(\mathcal{N})\subset X\to X$. Let $(Y,\|\cdot\|_Y)$ be an infinite-dimensional space (possibly Banach) such that $D(\mathcal{A})\cap\mathcal{D}(\mathcal{N})\subset Y\subseteq X$ and $\|h\|_X\leq\|h\|_Y$ for all $h\in D(\mathcal{A})\cap\mathcal{D}(\mathcal{N})$. The operator $\mathcal{N}$ is called $(Y,X)-$Fréchet differentiable at ξ^e if there exists a bounded linear operator $d\mathcal{N}(\xi^e):X\to X$ such that for all $h\in D(\mathcal{A})\cap\mathcal{D}(\mathcal{N}),\mathcal{N}(\xi^e+h)-\mathcal{N}(\xi^e)=d\mathcal{N}(\xi^e)h+R(\xi^e,h)$ where $\lim_{\|h\|_Y\to0}\frac{\|R(\xi^e,h)\|_X}{\|h\|_X}=0$, or equivalently,

$$\lim_{\|h\|_Y\to0}\frac{\|\mathcal{N}(\xi^e+h)-\mathcal{N}(\xi^e)-d\mathcal{N}(\xi^e)h\|_X}{\|h\|_X}=0.$$

Note that, when Y is X, the definition becomes the standard definition of Fréchet differentiability, which will be called $Y-$Fréchet differentiability. In this new framework, we give an adapted concept of local exponential stability.

Definition 6. The equilibrium ξ^e of (6) is said to be $(Y,X)-$locally exponentially stable if there exist $\delta,\alpha,\beta>0$ such that for all $\xi_0\in D(\mathcal{A})\cap\mathcal{D}(\mathcal{N})$ with $\|\xi_0-\xi^e\|_Y<\delta$, it holds $\|\xi(t)-\xi^e\|_X\leq\alpha e^{-\beta t}\|\xi_0-\xi^e\|_X, t\geq0$.

With these new definitions, we need some assumptions in order to be able to deduce local exponential stability of the equilibria for the nonlinear system (6). First, we consider standard assumptions on the operators $\mathcal{A},\mathcal{N}$ and $\mathcal{A}+\mathcal{N}$.

Assumption 7.
- The operator $\mathcal{A}$ is dissipative under some appropriate perturbation, i.e. there exists $l_\mathcal{A}>0$ such that the operator $\mathcal{A}-l_\mathcal{A}I$ is dissipative on $D(\mathcal{A})\cap\mathcal{D}(\mathcal{N})$ and the nonlinear operator $\mathcal{N}$ is Lipschitz continuous on $D(\mathcal{A})\cap\mathcal{D}(\mathcal{N})$ with respect to the X and Y norms;
- The operator $\mathcal{A}+\mathcal{N}$ is the infinitesimal generator of a nonlinear C_0-semigroup $(S(t))_{t\geq0}$ on X and Y.

We now introduce assumptions concerning the Fréchet differentiability of $(S(t))_{t\geq0}$.

Assumption 8.
- The Gâteaux derivative of $\mathcal{N}$ at ξ^e, denoted by $d\mathcal{N}(\xi^e)$ is a bounded linear operator on X and Y. The Gâteaux linearized dynamics of (6) are given by
$$\begin{cases}\dot{\overline{\xi}}=(\mathcal{A}+d\mathcal{N}(\xi^e))\overline{\xi},\\ \overline{\xi}(0)=\xi_0-\xi^e=:\hat{\xi}_0\end{cases}\tag{11}$$
- The nonlinear semigroup $(S(t))_{t\geq0}$ is $Y-$Fréchet differentiable and $(Y,X)-$Fréchet differentiable at ξ^e with $(T_{\xi^e}(t))_{t\geq0}$ as Fréchet derivative, the linear semigroup generated by the Gâteaux derivative of $\mathcal{A}+\mathcal{N}$ at ξ^e;
- When the linear semigroup $(T_{\xi^e}(t))_{t\geq0}$ is exponentially stable on X, it is assumed that the following estimate holds:
$$\|T_{\xi^e}(t)\hat{\xi}_0\|_Y\leq\eta\|\hat{\xi}_0\|_Y, t\geq0,\tag{12}$$
for all $\xi_0\in\mathcal{D}(\mathcal{A})$ such that $\|\hat{\xi}_0\|_Y<\delta^*$, for some $\eta>0$ and $\delta^*>0$ that may depend on η.

Taking Assumption 7 into account, it can be shown that $(Y,X)-$Fréchet differentiability of $(S(t))_{t\geq0}$ is obtained by imposing that

$$\lim_{\|\hat{\xi}_0\|_Y\to0}\frac{\|R(\xi^e,\hat{\xi}(t))\|_{L^\infty([0,t_0);X)}}{\|\hat{\xi}_0\|_X}=0,\tag{13}$$

where $R(\xi^e,\hat{\xi}(t))=\mathcal{N}(\hat{\xi}(t)+\xi^e)-\mathcal{N}(\xi^e)-d\mathcal{N}(\xi^e)\hat{\xi}(t), t_0>0$ and $\hat{\xi}(t)$ is the solution of $\dot{\hat{\xi}}=\mathcal{A}\hat{\xi}+\mathcal{N}(\hat{\xi}+\xi^e)-\mathcal{N}(\xi^e),\hat{\xi}(0)=\hat{\xi}_0$, see (Hastir et al., 2019a, Section 3) for further details [4]. In the same reference, it is then shown how to link stability properties of the linearized system with those of the nonlinear system, see (Hastir et al., 2019a, Theorem 3.1). This result is recalled here below.

Theorem 9. Let us consider Assumptions 7 and 8. If ξ^e is a (globally) exponentially stable equilibrium of the

[4] For instance, equality (13) can be deduced by imposing the $(Y,X)-$Fréchet differentiability of the nonlinear operator $\mathcal{N}$ together with the continuous dependence of $(S(t))_{t\geq0}$ on the initial condition by using $X-$ and $Y-$norms.

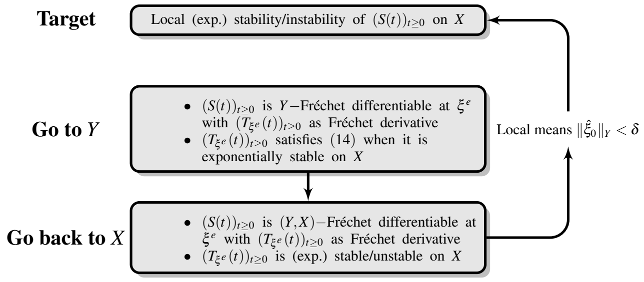

Fig. 1. Summary of the adapted approach.

linearized model (11), then it is a $(Y, X)-$locally exponentially stable equilibrium of (6). Conversely, if ξ^e is an unstable equilibrium of (11), it is $(Y, X)-$locally unstable for the nonlinear system (6).

This is a general theorem valid for semilinear systems of the form (6) satisfying Assumptions 7 and 8. An illustration of the philosophy of the approach is summarized in Figure 1.

In the case of the nonisothermal axial dispersion tubular reactor, the chosen state space X is $L^2(0, 1)$ whereas the "alternative space" Y is $C(0, 1)$ equipped with the $L^\infty(0, 1)-$norm. Since Assumptions 7 and 8 are satisfied for this case study, see (Hastir et al., 2019a, Section 4), the following result concerning the stability of the equilibrium profiles for the tubular reactor considered here holds, see (Hastir et al., 2019a, Theorem 4.1):

Theorem 10. Consider the nonlinear PDE (5) that describes the time evolution of the temperature in a nonisothermal axial dispersion tubular reactor. In the case where the reactor exhibits one equilibrium profile, the latter is $(C(0, 1), L^2(0, 1))-$locally exponentially stable for the nonlinear system (5). In the case of three equilibria the pattern $(C(0, 1), L^2(0, 1))-$"locally exponentially stable − locally unstable − locally exponentially stable" is highlighted, which is called bistability.

3. NUMERICAL SIMULATIONS

This section is devoted to the illustration of Theorem 10 by means of numerical simulations performed on the nonlinear system (5). The chosen approach is the Galerkin Residuals Method, see e.g. (McGowin and Perlmutter, 1970). The method is summarized as follows: the solution $\xi(t, z)$ to (5) is supposed to be of the form

$$\xi(t, z) = \sum_{n=1}^{+\infty} \alpha_n(t)\phi_n(z), \tag{14}$$

where $\{\alpha_n\}_{n\in\mathbb{N}_0}$ is a set of time dependent functions and $\{\phi_n\}_{n\in\mathbb{N}_0}$ are space dependent functions that form an orthonormal basis of $L^2(0, 1)$ and which satisfy the boundary conditions associated to (5). This formulation of $\xi(t, z)$ is well adapted since the PDE we are considering is a parabolic one, see e.g. (McGowin and Perlmutter, 1970) or (Thomée, 2006). In order to compute the solution numerically, the series in (14) is truncated up to order N. Then the resulting residuals error is orthogonalized on the basis $\{\phi_n\}_{n\in\mathbb{N}_0}$ to deduce the dynamics of $\alpha_n(t), n = 1, \ldots, N$.

As orthonormal basis of $L^2(0, 1)$, we choose the eigenfunctions of the operator $\mathcal{A}$ defined in (7) and (8), i.e. $\phi_n(z) = K_n[\beta_n\sqrt{Pe}\cos(\beta_n\sqrt{Pe}z) + \frac{Pe}{2}\sin(\beta_n\sqrt{Pe}z)]$ where $K_n = (\frac{2}{\beta_n^2 Pe+Pe+Pe^2/4})^{\frac{1}{2}}$ and $\{\beta_n\}_{n\in\mathbb{N}_0}$ are solutions of the resolvent equation $\tan(\beta\sqrt{Pe}) = \frac{4\beta\sqrt{Pe}}{4\beta^2-Pe}, \beta > 0$, see e.g. (Delattre et al., 2003). The corresponding eigenvalues are given by $\{-\beta_n^2 - \frac{Pe}{4}\}_{n\in\mathbb{N}_0}$. Plugging the truncated form of ξ into (5) leads to the following residual error

$$\begin{aligned}\Gamma_\xi = &\sum_{n=1}^{N}\frac{d\alpha_n}{dt}\phi_n - \frac{1}{Pe}\sum_{n=1}^{N}\alpha_n\frac{d^2\phi_n}{dz^2} + \frac{Pe}{4}\sum_{n=1}^{N}\alpha_n\phi_n \\ &- e^{-\frac{Pe}{2}z}\left[\tilde{g}\left(e^{\frac{Pe}{2}z}\sum_{n=1}^{N}\alpha_n\phi_n + x^e\right) - \tilde{g}(x^e)\right].\end{aligned}$$

Making Γ_ξ orthogonal to the $i-$th eigenfunction ϕ_i yields

$$\frac{d\alpha_i}{dt} = \left(-\beta_i^2 - \frac{Pe}{4}\right)\alpha_i + \Theta_i(\alpha_1, \ldots, \alpha_N), \tag{15}$$

where Θ_i is a nonlinear function expressed as

$$\begin{aligned}&\Theta_i(\alpha_1, \ldots, \alpha_N) := \\ &\int_0^1 e^{-\frac{Pe}{2}z}\left[\tilde{g}\left(e^{\frac{Pe}{2}z}\sum_{n=1}^{N}\alpha_n\phi_n + x^e\right) - \tilde{g}(x^e)\right]\phi_i dz,\end{aligned}$$

$i = 1, \ldots, N$. The initial condition is given by $\alpha_i(0) = \int_0^1 \xi_0\phi_i dz$, where ξ_0 is the initial condition given to (5). The solutions $\alpha_n, n = 1, \ldots, N$ of equation (15) are then computed via the numerical integration routine `ode23s` of MATLAB©.

The following set of parameters is chosen in order to highlight three equilibria, that is $\mu = 10, \delta = 1, k_0 = 1, L = 1, v = 0.0011$ and $D = 0.001$ (which entails that $Pe = 1.1$). The chosen initial condition satisfies the boundary conditions associated to (5). Its expression is given by $\xi_0(z) = \omega(\sin(\pi z) + \frac{2\pi}{Pe})$, where ω is a weighting factor that is used to make the $C(0, 1)-$norm of ξ_0 as small as desired. A computation of $\|\xi_0\|_{C(0,1)}$ gives

$$\|\xi_0\|_{C(0,1)} = \sup_{z\in[0,1]}|\xi_0(z)| = \frac{\omega}{Pe}(Pe + 2\pi). \tag{16}$$

In the numerical simulations, $\omega = \epsilon Pe, \epsilon > 0$. Consequently (16) becomes $\epsilon Pe + 2\pi\epsilon$, which can be made as small as desired by imposing that ϵ is small. The state trajectories $\xi(t, z)$ and their $L^2(0, 1)-$norms are represented in Figures 2 to 7 for the different equilibrium profiles. The theoretical conclusions reported in the previous section can be observed on these different Figures.

4. PERSPECTIVES

Here we aim at giving some further considerations about the nonisothermal axial dispersion tubular reactor. A first perspective would be the investigation of non adiabatic conditions inside of the reactor. This leads us to consider (1) with the additional term $-\gamma x_1 + \gamma x_w, \gamma > 0$ in the PDE associated with temperature. Remember that x_1 denotes the temperature and x_w the coolant temperature which is seen as a control variable here. Since the homogeneous

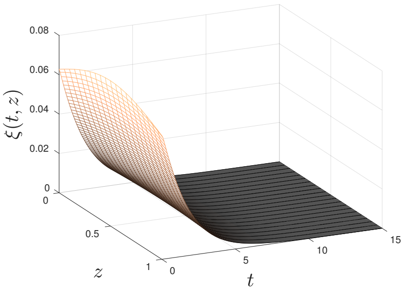

Fig. 2. State trajectory ξ for $\mu = 10, \delta = 1$, first equilibrium.

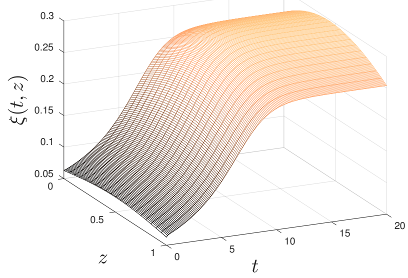

Fig. 3. State trajectory ξ for $\mu = 10, \delta = 1$, second equilibrium.

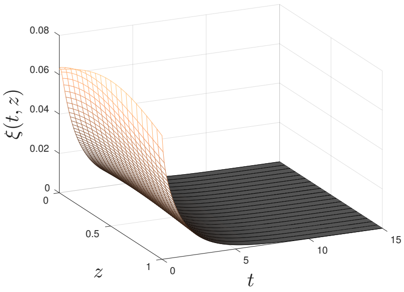

Fig. 4. State trajectory ξ for $\mu = 10, \delta = 1$, third equilibrium.

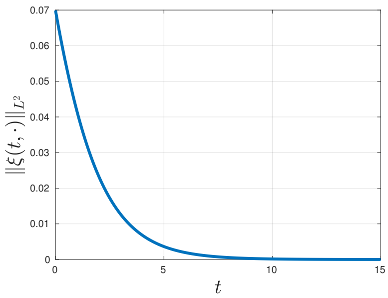

Fig. 5. L^2-norm of the state trajectory ξ for $\mu = 10, \delta = 1$, first equilibrium.

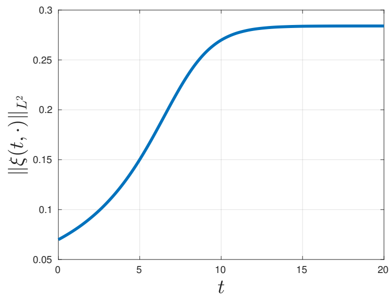

Fig. 6. L^2-norm of the state trajectory ξ for $\mu = 10, \delta = 1$, second equilibrium.

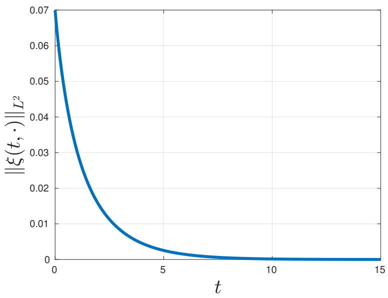

Fig. 7. L^2-norm of the state trajectory ξ for $\mu = 10, \delta = 1$, third equilibrium.

system is considered in the analysis, this variable is identically 0. Hence, it suffices to study the system in which $-\gamma x_1$ is incorporated in the PDE associated with x_1. This can be seen as the linear perturbation operator $-\gamma[x_1 \ 0] = -\gamma P[x_1 \ x_2]$, where P is the projection operator defined as $P : L^2(0,1) \times L^2(0,1) \to L^2(0,1) \times L^2(0,1), P[x_1 \ x_2] = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and which norm is obviously less than 1. Note also that P is a positive operator, that is $\langle Px, x\rangle \geq 0$ for all $x \in L^2(0,1) \times L^2(0,1)$. Hence, in the case where the adiabatic reactor exhibits a stable equilibrium, adding the term $-\gamma P[x_1 \ x_2]$ improves its stability margin by a negative constant η which satisfies $-\gamma \leq \eta \leq 0$. This equilibrium remains exponentially stable. Further analysis could be done in the case of an unstable equilibrium by using these "perturbation" arguments. Then, in order to stabilize an equilibrium profile or to improve its stability margin, we shall first model a heat exchanger that acts (in a distributed manner) along the reactor as an additional controlled PDE (dynamic feedback compensator) that produces the coolant temperature x_w. This heat exchanger is described by

$$\frac{\partial x_w}{\partial t} = -\frac{\partial x_w}{\partial z} - \gamma_w(x_w - u_w) + u, x_w(0) = 0, \qquad (17)$$

where $x_w \in L^2(0,1)$ is the coolant temperature, u_w is the heat exchange control variable and u is an additional control that acts on the heat exchanger. The interconnection of (1) and (17) is made by $u_w = x_1$. This leads to the following controlled PDE system

$$\begin{cases} \dfrac{\partial x_1}{\partial t} = \dfrac{1}{Pe_h}\dfrac{\partial^2 x_1}{\partial z^2} - \dfrac{\partial x_1}{\partial z} - \gamma(x_1 - x_w) + \delta\alpha(1-x_2)e^{\frac{-\mu}{1+x_1}}, \\ \dfrac{\partial x_2}{\partial t} = \dfrac{1}{Pe_m}\dfrac{\partial^2 x_2}{\partial z^2} - \dfrac{\partial x_2}{\partial z} + \alpha(1-x_2)e^{\frac{-\mu}{1+x_1}}, \\ \dfrac{\partial x_w}{\partial t} = -\dfrac{\partial x_w}{\partial z} - \gamma_w(x_w - x_1) + u, \\ \dfrac{\partial x_1}{\partial z}(0) = Pe_h x_1(0), \dfrac{\partial x_1}{\partial z}(1) = 0, \\ \dfrac{\partial x_2}{\partial z}(0) = Pe_m x_2(0), \dfrac{\partial x_2}{\partial z}(1) = 0, x_w(0) = 0. \end{cases} \qquad (18)$$

Adding observations on the system above leads us to consider a system of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + N(x(t)) + Bu(t), \\ y(t) = Cx(t), x(0) = x_0, \end{cases} \qquad (19)$$

where $B \in \mathcal{L}(U, L^2(0,1)^3)$ and $C \in \mathcal{L}(L^2(0,1)^3, Y)$ are control and observation operators [5] and $x \in L^2(0,1)^3$ is defined as $x = (x_1 \ x_2 \ x_w)^T$. It can be shown that the homogenous system corresponding to (18) is well-posed. A first perspective now is to show that (18) satisfies Assumptions 7 and 8 of Section 2 by considering state feedbacks $u = K(x_1 \ x_2 \ x_w)^T$ where K is a bounded linear operator from $L^2(0,1)^3$ into $L^2(0,1)$. In that way, the following proposition makes the link between the stabilization of the linearized dynamics and the (local) stabilization of (19).

Proposition 11. Let us consider system (19) in which u is a state feedback, i.e. $u(t) = K(x_1(t) \ x_2(t) \ x_w(t))^T, t \geq 0$ for some $K \in \mathcal{L}(L^2(0,1)^3, U)$. Assume that the closed-loop system dynamics operator $(A + BK) + N$ generates a nonlinear C_0-semigroup $(S_K(t))_{t\geq 0}$ and that its Gâteaux

[5] U and Y are called the control and observation spaces and are supposed to be Hilbert.

linearization around any equilibrium $x^e = (x_1^e\ x_2^e\ x_w^e)^T$ is the infinitesimal generator of a linear C_0-semigroup $(T_{x^e,K}(t))_{t\geq 0}$. Suppose also that Assumptions 7 and 8 hold *mutatis mutandis*. Under these conditions, if the feedback operator K stabilizes exponentially the linearized dynamics (around the equilibrium x^e), then it stabilizes locally exponentially the nonlinear dynamics around that equilibrium.

Thus the design of u could be done on a linear system instead of a nonlinear one. The approach we shall consider first is the LQ-optimal regulation, see e.g. (Callier and Winkin, 1992), (Callier and Winkin, 1990), (Aksikas et al., 2007b). It consists in finding u that minimizes the cost functional

$$J(u, x_0, \infty) = \frac{1}{2}\int_0^{+\infty} \left(\|Cx(t)\|^2 + \langle u(t), Qu(t)\rangle\right) dt,$$

subject to the linearized dynamics of (18), where $Q \in \mathcal{L}(U)$ is a coercive operator. The first step will be the well-posedness of this control problem which is deduced from the exponential stabilizability of the pair (A, B) and the exponential detectability of (C, A). Then the LQ-optimal control can be designed by using numerical techniques. Currently, we are considering an early lumping approach, that is, the system is first discretized and the controller is designed on the corresponding finite-dimensional approximation. Further perspectives aim at considering adaptive (extremum seeking) control techniques, see e.g. (Hudon et al., 2005) and (Hudon et al., 2008).

5. CONCLUSION

Stability analysis techniques for nonlinear infinite – dimensional systems were reported. A new framework was proposed to extend exponential stability of a linearized system to the local exponential stability of the nominal nonlinear system. This new framework relies on an adapted and weakened concept of Fréchet differentiability, the $(Y, X)-$Fréchet differentiability. This new concept seems to be promising and its applicability to different classes of distributed parameter systems is currently under investigation. Some control design perspectives were also given.

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