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High-gain Observer-based Output Feedback Control with Sensor Dynamic Governed by Parabolic PDE

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Abstract
In this paper it is proposed to extend the result described in Khalil and Praly (2014) and the references therein, regarding the high-gain observer-based nonlinear control to the case of systems with diffusion sensor dynamic. Based on some usual hypotheses, we provide sufficient conditions involving the high-gain parameter and the length on the PDE sensor. In fact it is brought into light an explicit trade off between them: the larger the observer gain, the smaller the length of the PDE sensor needs to be. The stability analysis of the closed loop is based on a Lyapunov functional.

Keywords: Output feedback, High-gain observers, Cascade ODE-PDE systems.

1. INTRODUCTION

The present paper deals with the design of output feedback control for a class of nonlinear cascade ODE-PDE systems. Throughout the past decades, the high-gain observers have been used extensively for the design of output feedback control of nonlinear systems, see Khalil and Praly (2014) and the references therein. An important advantage of using the high-gain observers is that they can recover the performances of state feedback control in the sense that, for instance, the trajectories of the system under output feedback approach those under state feedback as the observer gain increases.

Moreover, from the work Atassi and Khalil (1999), it is well known that the separation principle holds not only because the observer gain is made high, but also because, by designing the feedback control as a globally bounded function, the state of the plant is protected from peaking phenomenon Khalil (1996) when the high-gain observer estimates are used instead of the true states.

In fact, the design of output feedback control for cascade ODE-PDE systems has not been yet very much studied, one can cite for instance the work Krstic (2009) for a linear ODE or the work Wu (2013) where a nonlinear ODE is considered. However in this later work, the nonlinear ODE is restrictive and the method, based on LMIs, does not provide explicit conditions regarding the length that the PDE must satisfy in order to ensure global exponential stability of the overall system.

In the present paper we propose to extend the design of high-gain observer-based output feedback control for nonlinear systems with sensors described by heat PDEs. More precisely, we will derive explicit sufficient conditions, involving both the high-gain and the length of the PDE, ensuring exponential convergence of the overall closed cascade ODE-PDE. It has also to be noticed that the observer designed here is more simple that the one designed in Ahmed-Ali et al. (2015) for the same cascade ODE-PDE systems which used backstepping technics.

Notations and preliminaries

Throughout the paper the superscript T stands for matrix transposition, $\mathbf{R}^n$ denotes the n-dimensional Euclidean space with vector norm $|.|$, $\mathbf{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathbf{R}^{n\times n}$, means that P is symmetric and positive definite. In matrices, symmetric terms are denoted $*$; $\lambda_{min}(P)$ $(resp.\lambda_{max}(P))$ denotes the smallest (resp. largest) eigenvalue. $L_2(0,D)$ is the Hilbert space of square integrable functions $z(x), x\in[0,D]$ with the corresponding norm $\|z(x)\|_{L_2}=\sqrt{\int_0^D z^2(x)dx}$. $\mathcal{H}^1(0,D)$ is the Sobolev space of absolutely continuous functions $z:(0,D)\to\mathbf{R}$ with the square integrable derivative $\frac{d}{dx}$. $\mathcal{H}^2(0,D)$ is the Sobolev space of absolutely continuous functions $\frac{dz}{dx}:(0,D)\to\mathbf{R}$ and with $\frac{d^2w}{dx^2}\in$

$L_2(0, D)$. Given a two-argument function $u(x, t)$, its partial derivatives are denoted $u_t = \frac{\partial u}{\partial t}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$.

2. PROBLEM STATEMENT

Let us consider the following class of systems:

$$\begin{cases} \dot{X}_i = X_{i+1}, \qquad i = 1, \ldots, n-1 \\ \dot{X}_n = f(X, v) \\ u(D, t) = X_1 \\ u_x(0, t) = 0 \\ u_t = u_{xx}, \quad x \in [0, D] \\ y(t) = u(0, t) \end{cases} \tag{1}$$

where v ad y represent respectively the input and the output of the above system. Throughout the paper, we assume the following hypotheses:

H1: The function f is globally Lipschitz in both X and v with a Lipschitz constant K_0.

H2: : There exists a globally Lipschitz function $\alpha(X)$, such that the following dynamical system

$$\begin{cases} \dot{X}_i = X_{i+1} \\ \dot{X}_n = f(X, \alpha(X)) \end{cases}$$

is globally exponentially stable.

Using the converse Lyapunov theorem, we can say that there exists a positive function satisfying $V_0(X) > 0$, $V_0(\infty) = \infty$, and positive parameters $c_i, i = 1, \ldots, 4$ such that:

$$\begin{cases} c_1|X|^2 \leq V_0(X) \leq c_2|X|^2 \\ |\frac{\partial V_0}{\partial X_i}| \leq c_3|X| \\ \sum_{i=1}^{n-1} \frac{\partial V_0}{\partial X_i} X_{i+1} + \frac{\partial V_0}{\partial X_n} f(X, \alpha(X)) \leq -c_4|X|^2 \end{cases} \tag{2}$$

for all X.

3. OUTPUT FEEDBACK DESIGN

Based on the above hypotheses, we propose the following high-gain observer-based output feedback control:

$$\begin{cases} \dot{Z}_i = Z_{i+1} - l_i\theta^i(\hat{u}(0, t) - y), \qquad i = 1, \ldots, n-1 \\ \dot{Z}_n = f(Z, v) - l_n\theta^n(\hat{u}(0, t) - y) \\ v = \alpha(Z) \\ \hat{u}(D, t) = Z_1 \\ \hat{u}_x(0, t) = 0 \\ \hat{u}_t = \hat{u}_{xx} - l_1\theta(\hat{u}(0, t) - y) \quad x \in [0, D] \end{cases} \tag{3}$$

The vector gains $L = (l_1, \ldots, l_n)^T$ is chosen such that the matrix $(A - LC)$ is Hurwitz where

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & 0 & 1 & 0 & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 1 & 0 \\ 0 & 0 & \ldots & \ldots & 0 & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $C = [1, 0, \ldots, 0]$.

Let us consider the dynamical error system $e = Z - X$ and $\tilde{u} = \hat{u} - u$, then we obtain, for $x \in [0, D]$

$$\begin{aligned} \dot{e}_i &= e_{i+1} - l_i\theta^i\tilde{u}(0, t), \qquad i = 1, \ldots, n-1 \\ \dot{e}_n &= f(Z, \alpha(Z)) - f(X, \alpha(Z)) - l_n\theta^n\tilde{u}(0, t) \\ \dot{X}_i &= X_{i+1}, \qquad i = 1, \ldots, n-1 \\ \dot{X}_n &= f(X, \alpha(Z)) \\ \tilde{u}_t &= \tilde{u}_{xx} - l_1\theta\tilde{u}(0, t), \end{aligned}$$

with $\tilde{u}(D, t) = e_1$ and $\tilde{u}_x(0, t) = 0$.

If we now introduce the change of coordinate

$$w(x, t) = \tilde{u}(x, t) - e_1,$$

then we have

$$\begin{aligned} \dot{e}_i &= e_{i+1} - l_i\theta^i e_1 - l_i\theta^i w(0, t) \qquad i = 1, \ldots, n-1 \\ \dot{e}_n &= f(Z, \alpha(Z)) - f(X, \alpha(Z)) - l_n\theta^n e_1 - l_n\theta^n w(0, t) \\ \dot{X}_i &= X_{i+1}, \qquad i = 1, \ldots, n-1 \\ \dot{X}_n &= f(X, \alpha(Z)) \\ w_t &= w_{xx} - e_2, \end{aligned}$$

with $w(D, t) = 0$ and $w_x(0, t) = 0$.

As we can easily see the above system can be written as an interconnection of two sub-systems:

$$\begin{cases} \dot{e}_i = e_{i+1} - l_i\theta^i e_1 - l_i\theta^i w(0, t), \qquad i = 1, \ldots, n-1 \\ \dot{e}_n = f(Z, \alpha(Z)) - f(X, \alpha(Z)) - l_n\theta^n e_1 - l_n\theta^n w(0, t) \\ \dot{X}_i = X_{i+1}, \qquad i = 1, \ldots, n-1 \\ \dot{X}_n = f(X, \alpha(Z)) \end{cases}$$

and

$$\begin{cases} w_t = w_{xx} - e_2 \\ w(D, t) = 0 \\ w_x(0, t) = 0 \end{cases}$$

By using the classical change of coordinates $\xi_i = \theta^{1-i}e_i$, then we derive

$$\begin{cases} \dot{\xi}_i = \theta\xi_{i+1} - \theta l_i \xi_1 - l_i \theta w(0,t), \qquad i = 1,\dots,n-1 \\ \dot{\xi}_n = \theta^{1-n}\left[f(Z,\alpha(Z)) - f(X,\alpha(Z))\right] - \theta l_n \xi_1 - \theta l_n w(0,t) \\ \dot{X}_i = X_{i+1}, \qquad i = 1,\dots,n-1 \\ \dot{X}_n = f(X,\alpha(Z)) \end{cases}$$

and

$$\begin{cases} w_t = w_{xx} - e_2 \\ w(D,t) = 0 \\ w_x(0,t) = 0 \end{cases}$$

Let us now introduce the following augmented vector state

$$\eta = [\xi, X]^T, \quad \text{where} \quad \xi = (\xi_1,\dots,\xi_n)^T,$$

then these two sub-systems can be written as

$$\begin{cases} \dot{\eta} = F_0(\eta, w(0,t)) \\ w_t = w_{xx} - e_2 \\ w(D,t) = 0 \\ w_x(0,t) = 0 \end{cases} \tag{4}$$

where $F_0(\eta, w(0,t))$ is given by

$$\begin{pmatrix} \theta\xi_{i+1} - \theta l_i \xi_1 - l_i \theta w(0,t), \qquad i = 1,\dots,n-1 \\ \\ \theta^{1-n}\left[f(\Delta\xi + X,\alpha(Z)) - f(X,\alpha(Z))\right] - \theta l_n \xi_1 - \theta l_n w(0,t) \\ \\ X_{i+1}, \qquad i = 1,\dots,n-1 \\ \\ f(X,\alpha(\Delta\xi + X)) \end{pmatrix}.$$

where $\Delta = diag(1,\dots,\theta^{n-1})$. Notice that since $w(D,t) = 0$, then $w(0,t) = -\int_0^D w_x(x,t)dx$.

Remark 1. The well-posedness problem of the system (4) can be proven by using the work of Pazy (1983) and by using similar arguments those used in Ahmed-Ali et al. (2015). For instance, it is not difficult to see that the infinite dimensional part of the system (4) can be written in the Hilbert space $L_2(0,D)$ as an ordinary differential equation :

$$\dot{w}(t) = A_0 w(t) + F(w(t),\xi_2)$$

with

$$\mathcal{A}_0 = \frac{\partial^2}{\partial x^2}$$

which is defined the Dense Domain

$$\mathcal{D}(\mathcal{A}_0) = \{w \in \mathcal{H}^2(0,D), w_x(0) = w(D) = 0\}$$

and

$$F(w(t),\xi_2) = -\theta\xi_2$$

It is well know that the operator $\mathcal{A}_0$ generates a strongly continuous exponentially stable semigroup. Furthermore for $w,\bar{w} \in L_2(0,D)$ and $\forall \xi_2$, we have $||F(w(t),\xi_2) - F(\bar{w}(t),\xi_2)||_2 \le L_0||w(t) - \bar{w}||_2$ with a positive constant L_0. Then by using Theorem 6.1.5 of Pazy (1983) we can conclude that if $w(t_0) \in \mathcal{D}(\mathcal{A}_0)$ then $w(t) \in \mathcal{C}^1((0,\infty), L_2(0,D))$ with $w(t) \in \mathcal{D}(\mathcal{A}_0)$. On the other hand , since F_0 defined in system (4) is continuous and globally Lipschitz with respect to all their arguments, then we also conclude that $\eta \in (\mathcal{C}^1(0,\infty), \mathbf{R}^{2n})$.

Theorem 2. Consider the system (2). Then, there exist θ_0 and $D^\star(\theta)$ such that $\forall \theta > \theta_0$ and $\forall D \in (0, D^\star(\theta))$ the following inequality holds:

$$|X| + |e| + \int_0^D \tilde{u}^2(x,t)dx \le M_1 e^{-\sigma_1 t}$$

for some positive constant $M_1 > 0$ and $\sigma_1 > 0$.

Proof. In order to prove the exponential stability of the system (4), we will divide the proof in three parts: for the infinite dimensional sub-system, for the finite dimensional one and finally for the overall error system.

Infinite dimensional sub-system

In this first part we will analyse the sub-system

$$\begin{cases} w_t = w_{xx} - e_2 \\ w(D,t) = 0 \\ w_x(0,t) = 0 \end{cases} \tag{5}$$

In order to do this, we consider the following functional :

$$W = \frac{1}{2}\int_0^D w^2(x,t)dx + \frac{1}{2}\int_0^D w_x^2(x,t)dx$$

Then

$$\dot{W} = \int_0^D w(x,t)w_t(x,t)dx + \int_0^D w_x(x,t)w_{xt}(x,t)dx$$

which gives

$$\dot{W} = \int_0^D w(x,t)(w_{xx}(x,t) - e_2)dx + \int_0^D w_x(x,t)w_{tx}(x,t)dx$$

From the fact that $w(D,t) = 0$, then we have $w_t(D,t) = 0$. From this and by using the integration by parts, we can easily derive that

$$\dot{W} = -\int_0^D w_x^2(x,t)dx - \int_0^D w(x,t)e_2 dx - \int_0^D w_{xx}^2(x,t)dx + \int_0^D w_{xx}(x,t)e_2 dx$$

Using Young inequality leads to

$$\dot{W} \le -\int_0^D w_x^2(x,t)dx - \frac{1}{2}\int_0^D w_{xx}^2(x,t)dx + \frac{1}{2}\int_0^D w^2(x,t)dx + D|e_2|^2$$

Using Wirtinger inequality (as in Lemma 2 of Fridman and Blighovsky (2012)), then we have

$$\dot{W} \le -\left(\frac{\pi^2}{4D^2} - \frac{1}{2}\right)\int_0^D w^2(x,t)dx - \frac{\pi^2}{8D^2}\int_0^D w_x^2(x,t)dx + D|e_2|^2$$

since $|e_2| \le \theta\xi_2$, we obtain the following inequality,

$$\dot{W} \le -\left(\frac{\pi^2}{4D^2} - \frac{1}{2}\right)\int_0^D w^2(x,t)dx - \frac{\pi^2}{8D^2}\int_0^D w_x^2(x,t)dx + D\theta^2|\eta|^2$$

Finite-dimensional system

Let us now analyse the unperturbed sub-system

$$\dot{\eta} = F_0(\eta, 0).$$

By considering the following Lyapunov function,

$$V(\eta) = \frac{1}{\theta^{2(n-1)}} V_0(X) + \xi^T P \xi$$

where P is a positive definite symmetric matrix which satisfies

$$P(A - LC) + (A - LC)^T P = -\mathbf{I}$$

then, we can derive the following inequalities satisfied by V:

$$\begin{cases} c_1'|\eta|^2 \leq V(\eta) \leq c_2'|\eta|^2 \\ |\frac{\partial V}{\partial \eta}| \leq c_3'|\eta| \end{cases}$$

where

$$\begin{cases} c_1' = \min\left\{\frac{c_1}{\theta^{2(n-1)}}, \lambda_{min}(P)\right\} \\ c_2' = \max\left\{\frac{c_2}{\theta^{2(n-1)}}, \lambda_{max}(P)\right\} \\ c_3' = \max\left\{\frac{c_3}{\theta^{2(n-1)}}, 2\lambda_{max}(P)\right\} \end{cases}$$

with c_1, c_2 and c_3 defined in (2).

Now let us compute the derivative of V along the solution of the unperturbed system, we obtain,

$$\dot{V} = \frac{\partial V}{\partial \eta} F_0(\eta, 0).$$

After some computations as in Khalil and Praly (2014), we can easily derive the following inequality:

$$\dot{V} \leq -\frac{c_4}{2\theta^{2(n-1)}}|X|^2 - \left[\theta - 2\lambda_{max}(P)K_0 - \frac{2c_3^2}{c_4}K_0^2\right]|\xi|^2.$$

for some positive constant K_0 independent of θ, and c_4 defined in (2).

Choosing $\theta > \theta_0$ such that

$$\theta_0 = \max\left\{2\lambda_{max}(P)K_0 + \frac{2c_3^2}{c_4}K_0^2 + \frac{c_4}{2}, 1\right\}$$

we deduce that

$$\dot{V} \leq -c_4'|\eta|^2$$

with

$$c_4' = \frac{c_4}{2\theta^{2(n-1)}}.$$

From this, we deduce that the unperturbed system is globally uniformly exponentially stable.

Now let us compute the derivative $\dot{V}$ along the sub-system $\dot{\eta} = F_0(\eta, w(0,t))$, we obtain,

$$\dot{V} = \frac{\partial V}{\partial \eta} F_0(\eta, w(0,t))$$

which can be rewritten as follows:

$$\dot{V} = \frac{\partial V}{\partial \eta} F_0(\eta, 0) + \frac{\partial V}{\partial \eta}\left(F_0(\eta, w(0,t)) - F_0(\eta, 0)\right)$$

Using the above inequalities, we easily derive that

$$\dot{V} \leq -c_4'|\eta|^2 + |\frac{\partial V}{\partial \eta}||\left(F_0(\eta, w(0,t)) - F_0(\eta, 0)\right)|$$

Using again Young inequality, this leads to:

$$\dot{V} \leq -c_4'|\eta|^2 + \beta|\frac{\partial V}{\partial \eta}|^2 + \frac{1}{\beta}|\left(F_0(\eta, w(0,t)) - F_0(\eta, 0)\right)|^2$$

for $\beta > 0$.

On the other hand, we can easily see that, from the definition of F_0,

$$|\left(F_0(\eta, w(0,t)) - F_0(\eta, 0)\right)| \leq \theta|L||w(0,t)|$$

which also gives

$$|\left(F_0(\eta, w(0,t)) - F_0(\eta, 0)\right)| \leq \theta|L| \int_0^D |w_x(x,t)|dx$$

Now using Schwartz inequality, we obtain

$$|\left(F_0(\eta, w(0,t)) - F_0(\eta, 0)\right)|^2 \leq \theta^2|L|^2 D \int_0^D w_x^2(x,t)dx$$

and from this we derive that

$$\dot{V} \leq -(c_4' - \beta c_3'^2)|\eta|^2 + \frac{1}{\beta}|\theta^2|L|^2 D \int_0^D w_x^2(x,t)dx$$

Stability of the overall error system

At this stage, we now consider the Lyapunov functionnal $W_1 = W + V$. Then from the two previous parts, its time derivative satisfies the following inequality

$$\dot{W}_1 \leq -\left[c_4' - \beta {c_3'}^2 - D\theta^2\right]|\eta|^2 - \left[\frac{\pi^2}{4D^2} - \frac{1}{2}\right]\int_0^D w^2(x,t)dx$$
$$-\left[\frac{\pi^2}{8D^2} - \frac{1}{\beta}\theta^2|L|^2 D\right]\int_0^D w_x^2(x,t)dx$$

In order to ensure the exponential stability, it is not difficult to see that we need to choose β sufficiently small so that $c_4' - \beta {c_3'}^2 > 0$ and we have to choose D such that

$$D < D^\star(\theta) = \min\left\{\frac{1}{\theta^2}(c_4' - \beta {c_3'}^2), \frac{\pi}{\sqrt{2}}, \left(\frac{\pi^2\beta}{8\theta^2|L|^2}\right)^{1/3}\right\}$$

This ends the proof.

Remark: The expression for $D^*(\theta)$ shows that as θ increases, $D^*(\theta)$ decreases. Hence there is a trade off between θ and D; the larger the observer gain θ, the smaller the length of the PDE sensor.

4. CONCLUSION

An extension of the design of high-gain observer-based output feedback control for nonlinear systems with sensors described by heat PDEs has been proposed by bringing into light explicitly the expected trade-off between the gain of the observer and the length of the PDE. Further works will be undertaken in the same vein by i) considering actuators PDEs and ii) by relaxing the global exponential assumption. This later extension will require careful analysis of the peaking phenomenon.

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